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A STUDY ON ODD CHROMATIC NUMBER OF SUBDIVISION VERTEX NEIGHBOURHOOD CORONA OF GRAPHS

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Abstract. We study the odd chromatic number of subdivision vertex neighbourhood corona of star graph with star, path and complete graphs and cycle graph with cycle, complete and path graphs through comprehensive theoretical investigation supported by algorithmic testing.

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Keywords and Phrases: Odd Chromatic Number, Subdivision Vertex Neighbourhood Corona, Star, Path, Complete, Cycle.

1. Introduction

The notion of *odd vertex coloring* [1–3, 8, 9] was initially introduced by Mirko Petruševski and Riste Škrekovski [2] in 2021. A proper vertex coloring is termed *odd* if, for every non-isolated vertex p, there exists at least one color that appears odd times in its open neighborhood $N(p)$. The *odd chromatic number* of a graph T, symbolized as $\chi_{od}(T)$, represents the least number of colors needed to produce an odd coloring for T. In their work, Petruševski and Škrekovski [2] demonstrated that every simple planar graph allows an odd coloring using no more than nine colors, and they conjectured that five colors might always be sufficient.

The natural inequality $\chi(T) \leq \chi_{od}(T)$ holds for all graphs, and becomes an equality precisely when the graph T is odd, meaning all its vertex degrees are odd. However, this inequality can also be strict in certain cases. Determining the value of $\chi_{od}(T)$ for a given graph T is known to be NP-hard [9]. Some upper bounds have been established: for a connected graph with maximum degree $\Delta = 3$, it holds that $\chi_{od}(T) \leq 4$, and more generally, for connected graphs with $\Delta \geq 3$, we have $\chi_{od}(T) \leq \Delta + 1$. Moreover, for any connected graph T with maximum degree Δ, it has been shown that $\chi_{od}(T) \leq 2\Delta$ [8].

The study of odd colorings has recently gained significant attention, prompting numerous researchers to explore related results and bounds. For example, Rémy Belmonte et al. [5] established that if a graph T has even order and maximum degree Δ, then $\chi_{od}(T) \leq 2\Delta - 1$. Likewise, Tianjiao Dai et al. [6] proved that for any Δ, the inequality $\chi_{od}(T) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 2$ holds.

2. Preliminaries

To begin with, the subdivision graph $S(T)$ of a graph T arises by inserting a new vertex into every edge of T. In 2013, Liu and Lu introduced the concept of *subdivision vertex neighbourhood corona* (SVNC) of two graphs T_1 and T_2, denoted

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by $T_1 \boxdot T_2$. This graph is constructed by taking the subdivision graph $S(T_1)$ and $|V(T_1)|$ pairwise vertex-disjoint copies of T_2, and then connecting the neighbors of the ith vertex in $V(T_1)$ to every vertex in the ith copy of T_2, for each $1 \leq i \leq |V(T_1)|$[4, 7]. For any two graphs T_1 and T_2, $|V[T_1 \boxdot T_2]| = |V(T_1)| + |E(T_1)| + |V(T_1)||V(T_2)|$ and $|E[T_1 \boxdot T_2]| = 2|E(T_1)| + |V(T_1)||E(T_2)| + 2|E(T_1)||V(T_2)|$[7].

We establish precise estimates for the odd chromatic number of subdivision vertex neighbourhood corona of star graph $K_{1,q}$ with star, path and complete graphs and cycle graph C_q with cycle, complete, and path graphs through comprehensive theoretical investigation supported by algorithmic testing. Additionally, we present a computational strategy for generating optimal odd colorings, guaranteeing that every odd cycle in the graph is assigned no fewer than three distinct colors, as mandated by the principles of odd coloring.

3. SVNC OF STAR GRAPHS

Lemma 3.1. *For any $q, r > 1$, the lower bound of odd chromatic number of SVNC of star graph with path graph is*

$$\chi_{od}(K_{1,q} \boxdot P_r) \geq 3 + ((q+1) \ mod \ 2)$$

Proof. Let a_1 be the central vertex and $a_2, a_3, ...a_{(q+1)}$ be the leaf vertices of $K_{1,q}$. Also, let $s_1, s_2, ...s_q$ be the subdivided vertices that subdivide the central vertex with each of its leaf vertices and $\{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q+1, 1 \leq j \leq r\}$ be the vertices of $q+1$ copies of P_r. By the definition of SVNC, every vertex of the ith copy of path graph P_r is adjoined to every neighbour of the ith vertex of star graph $K_{1,q}$. On that note, the maximum degree Δ will be obtained at the subdivided vertices, $\Delta[K_{1,q} \boxdot P_r] = 2r+2$, where each of the subdivided vertices forms a clique with the vertices of its associated ith copy of P_r which already gives a basic lower bound as 3 colors such that

$$\omega(K_{1,q} \boxdot P_r) = \begin{cases} \{s_i, c_{ij}, c_{i(j+1)}\}; & \text{for } i = 1, 1 \leq j \leq r-1 \\ \{s_{i-1}, c_{ij}, c_{i(j+1)}\}; & \text{for } 2 \leq i \leq q+1, 1 \leq j \leq r-1 \end{cases}$$

We know that, for any graph T, $\chi_{od}(T) \geq \omega(T)$. On that note, we have $\chi_{od}(K_{1,q} \boxdot P_r) \geq \omega(K_{1,q} \boxdot P_r) = 3$. Here, the lower bound accounts for :

- 3 colors needed for the maximum clique.
- The additional term $((q+1) \ mod \ 2)$ adjusts the bound based on the parity of q, ensuring that an extra color is needed when q is even to maintain odd chromatic constraint.

This concludes the lower bound as

$$\chi_{od}(K_{1,q} \boxdot P_r) \geq 3 + ((q+1) \ mod \ 2)$$

□

Theorem 3.2. *Let $q, r > 1$ be two positive integers. Then,*

$$\chi_{od}(K_{1,q} \boxdot P_r) = 3 + ((q+1) \ mod \ 2)$$

Proof. We define the vertex set as $V[K_{1,q} \boxdot P_r] = \{a_1, a_2, ..., a_{q+1}\} \cup \{s_1, s_2, ..., s_q\} \cup \{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q+1, 1 \leq j \leq r\}$. Here, $|V[K_{1,q} \boxdot P_r]| = (q+1)(r+1)+q$ and $|E[K_{1,q} \boxdot P_r]| = 3q(r+1)+(r-1)$. In such case, the adjacency matrix of $K_{1,q} \boxdot P_r$

will be of the order $[(q+1)(r+1)+q] \times [(q+1)(r+1)+q]$. Now we present the adjacency matrix of $K_{1,q} \boxdot P_r$ with respect to vertex s_1.

$$s_1 \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{q+1} & s_1 & s_2 & \dots & s_q \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ c_{11} & c_{12} & \dots & c_{2r} & c_{2r+1} & c_{2r+2} & \dots & c_{(q+1)r} & \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \end{pmatrix}$$

We then generate the neighbour sets for each vertex as follows:

$$N(a_i) = \begin{cases} \{s_1, s_2, ..., s_q\}; & \text{for } i = 1 \\ \{s_{i-1}\}; & \text{for } 2 \le i \le q+1 \end{cases}$$

$$N(s_i) = \{a_1, a_{i+1}, c_{1j}, c_{(i+1)j} : 1 \le i \le q, 1 \le j \le r\}$$

$$N(c_{ij}) = \begin{cases} \text{for } i = 1 \begin{cases} \{c_{i(j+1)}, s_1, s_2, \dots, s_q\}; & \text{for } j = 1 \\ \{c_{i(j-1)}, c_{i(j+1)}, s_1, s_2, \dots, s_q\}; & \text{for } 2 \le j \le r-1 \\ \{c_{i(j-1)}, s_1, s_2, \dots, s_q\}; & \text{for } j = r \end{cases} \\ \text{for } i > 1 \begin{cases} \{c_{i(j+1)}, s_{i-1}\}; & \text{for } j = 1 \\ \{c_{i(j-1)}, c_{i(j+1)}, s_{i-1}\}; & \text{for } 2 \le j \le r-1 \\ \{c_{i(j-1)}, s_{i-1}\}; & \text{for } j = r \end{cases} \end{cases}$$

By lemma 3.1, we have established the lower bound. To determine the upper bound, we apply a specific coloring pattern, implemented using a C++ program, which satisfies the constraints of odd coloring. This gives us:

$$\chi_{od}(K_{1,q} \boxdot P_r) \le 3 + ((q+1)\ mod\ 2)$$

Since both bounds coincide, we conclude that:

$$\chi_{od}(K_{1,q} \boxdot P_r) = 3 + ((q+1)\ mod\ 2)$$

The following C++ algorithm illustrates the coloring strategy used to verify the upper bound and presents a computational example demonstrating its effectiveness.

Algorithm: Odd Chromatic Number of SVNC of $K_{1,q}$ with P_r

Require: Integers $q \ge 1$, $r \ge 1$
Ensure: Color sets s_q, a_{q+1}, c_{ij} and odd chromatic number

```
Prompt user: Enter q (size of K_{1,q})
Read q
Prompt user: Enter r (size of P_r)
Read r
Compute and display: Odd Chromatic Number ← 3 + ((q + 1) mod 2)
Initialize arrays: s[0 ... q − 1], a[0 ... q], c[0 ... q][0 ... r − 1]
for i = 0 to q − 1 do
    s[i] ← (q mod 2 = 0 ∧ i = q − 1) ? 3 : 2
end for
for i = 0 to q do
    a[i] ← (i = 0) ? 1 : ((q mod 2 = 0 ∧ i = q) ? 2 : 3)
end for
```

13: **for** $i = 0$ to q **do**
14: **for** $j = 0$ to $r - 1$ **do**
15: **if** $q \bmod 2 = 1$ **then**
16: $c[i][j] \leftarrow (j \bmod 2 = 0)\ ?\ 1 : 3$
17: **else**
18: $c[i][j] \leftarrow (i = 0)\ ?\ ((j \bmod 2 = 0)\ ?\ 1 : 4) : ((i = q)\ ?\ ((j \bmod 2 = 0)\ ?\ 1 : 2) : ((j \bmod 2 = 0)\ ?\ 1 : 3))$
19: **end if**
20: **end for**
21: **end for**
22: Print arrays s_q, a_{q+1}, and c_{ij}

Example

Input: Enter the size of star graph $K_{1,q}$: 10
Input: Enter the size of path graph P_r : 5
Output: $\chi_{od}(K_{1,10} \boxdot P_5) = 4$

i	$s[i]$	$a[i]$	$c[i][1]$	$c[i][2]$	$c[i][3]$	$c[i][4]$	$c[i][5]$
1	2	1	1	4	1	4	1
2	2	3	1	3	1	3	1
3	2	3	1	3	1	3	1
4	2	3	1	3	1	3	1
5	2	3	1	3	1	3	1
6	2	3	1	3	1	3	1
7	2	3	1	3	1	3	1
8	2	3	1	3	1	3	1
9	2	3	1	3	1	3	1
10	3	3	1	3	1	3	1
11	–	2	1	2	1	2	1

□

Lemma 3.3. *For any $q, r > 1$, the lower bound of odd chromatic number of SVNC of star graph with star graph is*

$$\chi_{od}(K_{1,q} \boxdot K_{1,r}) \geq 4 - (q \ mod\ 2)$$

Proof. Let a_1 be the central vertex and $a_2, a_3, ...a_{(q+1)}$ be the leaf vertices of $K_{1,q}$. Also, let $s_1, s_2, ...s_q$ be the subdivided vertices that subdivide the central vertex with each of its leaf vertices and $\{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q+1, 1 \leq j \leq r+1\}$ be the vertices of $q+1$ copies of $K_{1,r}$. By the definition of SVNC, every vertex of the ith copy of star graph $K_{1,r}$ is adjoined to every neighbour of the ith vertex of star graph $K_{1,q}$. On that note, the maximum degree Δ will be obtained at the subdivided vertex (i.e) $\Delta[K_{1,q} \boxdot K_{1,r}] = 2r + 4$, where each of the subdivided vertices forms a clique with the vertices of its associated ith copy of $K_{1,r}$ which already gives a basic lower bound as 3 colors such that

$$\omega(K_{1,q} \boxdot K_{1,r}) = \begin{cases} \{s_i, c_{i1}, c_{i(j+1)}\}; & \text{for } i = 1, 1 \leq j \leq r \\ \{s_{i-1}, c_{i1}, c_{i(j+1)}\}; & \text{for } 2 \leq i \leq q+1, 1 \leq j \leq r \end{cases}$$

We know that, for any graph T, $\chi_{od}(T) \geq \omega(T)$. On that note, we have $\chi_{od}(K_{1,q} \boxdot K_{1,r}) \geq \omega(K_{1,q} \boxdot K_{1,r}) = 3$. Here, the lower bound accounts for :

- 3 colors needed for the maximum clique.
- The subtraction term ($q \ mod \ 2$) adjusts the bound based on the parity of q, requiring 4 colors when q is even and 3 colors when q is odd.

This concludes the lower bound as

$$\chi_{od}(K_{1,q} \boxdot K_{1,r}) \geq 4 - (q \ mod \ 2).$$

□

Theorem 3.4. *Let $q, r > 1$ be two positive integers. Then,*

$$\chi_{od}(K_{1,q} \boxdot K_{1,r}) = 4 - (q \ mod \ 2)$$

Proof. We define the vertex set as $V[K_{1,q} \boxdot K_{1,r}] = \{a_1, a_2, ..., a_{q+1}\} \cup \{s_1, s_2, ..., s_q\} \cup \{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q+1, 1 \leq j \leq r+1\}$. Here, $|V[K_{1,q} \boxdot K_{1,r}]| = (q+1)(r+2)+q$ and $|E[K_{1,q} \boxdot K_{1,r}]| = 2q(r+2) + (q+1)r$. In such case, the adjacency matrix of $K_{1,q} \boxdot K_{1,r}$ will be of the order $[(q+1)(r+2)+q] \times [(q+1)(r+2)+q]$. Now we present the adjacency matrix of $K_{1,q} \boxdot K_{1,r}$ with respect to vertex s_1.

$$s_1 \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{q+1} & s_1 & s_2 & \dots & s_q \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ c_{11} & c_{12} & \dots & c_{2r} & c_{2r+1} & c_{2r+2} & \dots & c_{(q+1)(r+1)} & \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \end{pmatrix}$$

We then generate the neighbour sets for each vertex as follows:

$$N(a_i) = \begin{cases} \{s_1, s_2, ..., s_q\}; & \text{for } i = 1 \\ \{s_{i-1}\}; & \text{for } 2 \leq i \leq q+1 \end{cases}$$

$$N(s_i) = \{a_1, a_{i+1}, c_{1j}, c_{(i+1)j} : 1 \leq i \leq q, 1 \leq j \leq r+1\}$$

$$N(c_{ij}) = \begin{cases} \text{for } i = 1 \begin{cases} \{c_{i1}, c_{i2}, \dots, c_{i(r+1)}, s_1, s_2, \dots, s_q\} \setminus \{c_{ij}\}; & \text{for } j = 1 \\ \{c_{i1}, s_1, s_2, \dots, s_q\}; & \text{for } 2 \leq j \leq r+1 \end{cases} \\ \text{for } i > 1 \begin{cases} \{c_{i1}, c_{i2}, \dots, c_{ir}, s_{i-1}\} \setminus \{c_{ij}\}; & \text{for } j = 1 \\ \{c_{i1}, s_{i-1}\}; & \text{for } 2 \leq j \leq r+1 \end{cases} \end{cases}$$

By lemma 3.3, we have established the lower bound. To determine the upper bound, we apply a specific coloring pattern, implemented using a C++ program, which satisfies the constraints of odd coloring. This gives us:

$$\chi_{od}(K_{1,q} \boxdot K_{1,r}) \leq 4 - (q \ mod \ 2)$$

Since both bounds coincide, we conclude that:

$$\chi_{od}(K_{1,q} \boxdot K_{1,r}) = 4 - (q \ mod \ 2)$$

The following C++ algorithm illustrates the coloring strategy used to verify the upper bound and presents a computational example demonstrating its effectiveness.

Algorithm: Odd Chromatic Number of SVNC of $K_{1,q}$ with $K_{1,r}$

Require: Integers $q \geq 1$, $r \geq 1$
Ensure: Color sets s_q, a_{q+1}, c_{ij} and odd chromatic number

```
Prompt user: Enter q (size of K_{1,q})
Read q
Prompt user: Enter r (size of K_{1,r})
Read r
Compute and display: Odd Chromatic Number ← 4 − (q mod 2)
Initialize arrays: s[0...q−1], a[0...q], c[0...q][0...r]
for i = 0 to q − 1 do
    s[i] ← (q mod 2 = 0 ∧ i = q − 1) ? 3 : 2
end for
for i = 0 to q do
    a[i] ← (i = 0) ? 1 : ((q mod 2 = 0 ∧ i = q) ? 2 : 3)
end for
for i = 0 to q do
    for j = 0 to r do
        c[i][j] ← (j = 0) ? 1 : ((q mod 2 = 0) ? 4 : 3)
    end for
end for
Print arrays s_q, a_{q+1}, and c_{ij}
```

Example

Input: Enter the size of star graph $K_{1,q}$: 9
Input: Enter the size of star graph $K_{1,r}$: 6
Output: $\chi_{od}(K_{1,9} \boxdot K_{1,6}) = 3$

i	s[i]	a[i]	c[i][1]	c[i][2]	c[i][3]	c[i][4]	c[i][5]	c[i][6]	c[i][7]
1	2	1	1	3	3	3	3	3	3
2	2	3	1	3	3	3	3	3	3
3	2	3	1	3	3	3	3	3	3
4	2	3	1	3	3	3	3	3	3
5	2	3	1	3	3	3	3	3	3
6	2	3	1	3	3	3	3	3	3
7	2	3	1	3	3	3	3	3	3
8	2	3	1	3	3	3	3	3	3
9	2	3	1	3	3	3	3	3	3
10	-	3	1	3	3	3	3	3	3

□

Lemma 3.5. *For any $q > 1$ and $r > 2$, the lower bound of odd chromatic number of SVNC of star graph with complete graph is*

$$\chi_{od}(K_{1,q} \boxdot K_r) \geq 4 + ((q+1)\ mod\ 2) + (r-3)$$

Proof. Let a_1 be the central vertex and $a_2, a_3, ...a_{(q+1)}$ be the leaf vertices of $K_{1,q}$. Also, let $s_1, s_2, ...s_q$ be the subdivided vertices that subdivide the central vertex with

each of its leaf vertices and $\{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q+1, 1 \leq j \leq r\}$ be the vertices of $q+1$ copies of K_r. By the definition of SVNC, every vertex of the ith copy of complete graph K_r is adjoined to every neighbour of the ith vertex of star graph $K_{1,q}$. On that note, the maximum degree Δ will be obtained at the subdivided vertex (i.e) $\Delta[K_{1,q} \boxdot K_r] = 2r+2$, where each of the subdivided vertices forms a clique with the vertices of its associated ith copy of K_r which already gives a basic lower bound as $r+1$ colors such that

$$\omega(K_{1,q} \boxdot K_r) = \begin{cases} \{s_i, c_{ij}\}; & \text{for } i = 1, 1 \leq j \leq r \\ \{s_{i-1}, c_{ij}\}; & \text{for } 2 \leq i \leq q+1, 1 \leq j \leq r \end{cases}$$

We know that, for any graph T, $\chi_{od}(T) \geq \omega(T)$. On that note, we have $\chi_{od}(K_{1,q} \boxdot K_r) \geq \omega(K_{1,q} \boxdot K_r) = r+1$. Here, the lower bound depends on both:

- The parity of q, which affects the structure of the subdivided star.
- The size of the complete graph K_r, which influences the number of interconnected vertices within cliques.

To address this, we include two correction terms:

- $(q+1) \mod 2$: adds 1 when q is even requiring an additional color.
- $r-3$: accounts for the fact that for $r > 3$, the base clique size $r+1$ increases, thus increasing the minimum number of colors needed for a proper odd coloring of the resulting complete subgraphs.

This concludes the lower bound as

$$\chi_{od}(K_{1,q} \boxdot K_r) \geq 4 + ((q+1)\ mod\ 2) + (r-3)$$

□

Theorem 3.6. *Let $q > 1$ and $r > 2$ be two positive integers. Then,*

$$\chi_{od}(K_{1,q} \boxdot K_r) = 4 + ((q+1)\ mod\ 2) + (r-3)$$

Proof. We define the vertex set as $V[K_{1,q} \boxdot K_r] = \{a_1, a_2, ..., a_{q+1}\} \cup \{s_1, s_2, ..., s_q\} \cup \{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q+1, 1 \leq j \leq r\}$. Here, $|V[K_{1,q} \boxdot K_r]| = (q+1)(r+1)+q$ and $|E[K_{1,q} \boxdot K_r]| = (1/2)(4q(r+1) + (q+1)r(r-1))$. In such case, the adjacency matrix of $K_{1,q} \boxdot K_r$ will be of the order $[(q+1)(r+1)+q] \times [(q+1)(r+1)+q]$ Now we present the adjacency matrix of $K_{1,q} \boxdot K_r$ with respect to vertex s_1.

$$s_1 \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{q+1} & s_1 & s_2 & \dots & s_q \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ c_{11} & c_{12} & \dots & c_{2r} & c_{2r+1} & c_{2r+2} & \dots & c_{(q+1)r} & \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \end{pmatrix}$$

We then generate the neighbour sets for each vertex as follows:

$$N(a_i) = \begin{cases} \{s_1, s_2, ..., s_q\}; & \text{for } i = 1 \\ \{s_{i-1}\}; & \text{for } 2 \leq i \leq q+1 \end{cases}$$

$$N(s_i) = \{a_1, a_{i+1}, c_{1j}, c_{(i+1)j} : 1 \leq i \leq q, 1 \leq j \leq r\}$$

$$N(c_{ij}) = \begin{cases} \{c_{i1}, c_{i2}, \dots, c_{ir}, s_1, s_2, \dots, s_q\} \setminus \{c_{ij}\}; & \text{for } i = 1 \\ \{c_{i1}, c_{i2}, \dots, c_{ir}, s_{i-1}\} \setminus \{c_{ij}\}; & \text{for } 2 \leq i \leq q+1 \end{cases}$$

By lemma 3.5, we have established the lower bound. To determine the upper bound,

we apply a specific coloring pattern, implemented using a C++ program, which satisfies the constraints of odd coloring. This gives us:

$$\chi_{od}(K_{1,q} \boxdot K_r) \leq 4 + ((q+1)\ mod\ 2) + (r-3)$$

Since both bounds coincide, we conclude that:

$$\chi_{od}(K_{1,q} \boxdot K_r) = 4 + ((q+1)\ mod\ 2) + (r-3)$$

The following C++ algorithm illustrates the coloring strategy used to verify the upper bound and presents a computational example demonstrating its effectiveness.

Algorithm: Odd Chromatic Number of SVNC of $K_{1,q}$ with K_r

Require: Integers $q \geq 1$, $r \geq 2$
Ensure: Color sets s_q, a_{q+1}, c_{ij} and odd chromatic number

Prompt user: Enter q (size of $K_{1,q}$)
Read q
Prompt user: Enter r (size of K_r)
Read r
Compute and display: Odd Chromatic Number $\leftarrow 4 + ((q+1) \bmod 2) + (r-3)$
Initialize arrays: $s[0 \ldots q-1]$, $a[0 \ldots q]$, $c[0 \ldots q][0 \ldots r-1]$
for $i = 0$ to $q-1$ **do**
    $s[i] \leftarrow (q \bmod 2 = 0 \wedge i = q-1)\ ?\ 3 : 2$
end for
for $i = 0$ to q **do**
    $a[i] \leftarrow (i = 0)\ ?\ 1 : ((q \bmod 2 = 0 \wedge i = q)\ ?\ 2 : 3)$
end for
for $j = 0$ to $r-1$ **do**
    $pattern[j] \leftarrow (j = 0)\ ?\ 1 : ((j = 1)\ ?((q \bmod 2 = 0)\ ?\ 4 : 3) : pattern[j-1] + 1)$
end for
for $i = 0$ to q **do**
    for $j = 0$ to $r-1$ **do**
        $c[i][j] \leftarrow pattern[j]$
    end for
end for
Print arrays s_q, a_{q+1}, and c_{ij}

Example

Input: Enter the size of star graph $K_{1,q}$: 14
Input: Enter the size of complete graph K_r : 5
Output: $\chi_{od}(K_{1,14} \boxdot K_5) = 7$

i	$s[i]$	$a[i]$	$c[i][1]$	$c[i][2]$	$c[i][3]$	$c[i][4]$	$c[i][5]$
1	2	1	1	4	5	6	7
2	2	3	1	4	5	6	7
3	2	3	1	4	5	6	7
4	2	3	1	4	5	6	7
5	2	3	1	4	5	6	7
6	2	3	1	4	5	6	7
7	2	3	1	4	5	6	7
8	2	3	1	4	5	6	7
9	2	3	1	4	5	6	7
10	2	3	1	4	5	6	7
11	2	3	1	4	5	6	7
12	2	3	1	4	5	6	7
13	2	3	1	4	5	6	7
14	3	3	1	4	5	6	7
15	–	2	1	4	5	6	7

□

4. SVNC OF CYCLE GRAPHS

Lemma 4.1. *For any $q, r > 2$, the lower bound of odd chromatic number of SVNC of cycle graph with complete graph is*

$$\chi_{od}(C_q \boxdot K_r) \geq r + 2$$

Proof. Let $a_1, a_2, ...a_q$ be the vertices of C_q and $s_1, s_2, ...s_q$ be the subdivided vertices of C_q. Let $\{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q, 1 \leq j \leq r\}$ be the vertices of q copies of K_r. By the definition of SVNC, every vertex of the ith copy of complete graph K_r is adjoined to every neighbour of the ith vertex of cycle graph C_q. On that note, the maximum degree Δ will be obtained at the subdivided vertex (i.e) $\Delta[C_q \boxdot K_r] = 2r + 2$, where each of the subdivided vertices forms a clique with the vertices of its associated ith copy of K_r which already gives a basic lower bound as $r + 1$ colors such that

$$\omega(C_q \boxdot K_r) = \begin{cases} \{s_i, c_{ij}\}, \{s_i, c_{(i+1)j}\}; & \text{for } 1 \leq i \leq q-1, 1 \leq j \leq r \\ \{s_i, c_{ij}\}, \{s_i, c_{1j}\}; & \text{for } i = q, 1 \leq j \leq r \end{cases}$$

We know that, for any graph T, $\chi_{od}(T) \geq \omega(T)$. On that note, we have $\chi_{od}(C_q \boxdot k_r) \geq \omega(C_q \boxdot K_r) = r + 1$. Here, the lower bound accounts for:

- $r + 1$ colors needed for the maximum clique.
- An additional color to satisfy the odd chromatic constraint due to the existence of base C_q and induced odd cycles in the SVNC structure.

This concludes the lower bound as

$$\chi_{od}(C_q \boxdot K_r) \geq r + 2$$

□

Theorem 4.2. *Let $q, r > 2$ be two positive integers. Then,*

$$\chi_{od}(C_q \boxdot K_r) = r + 2$$

Proof. We define the vertex set as $V[C_q \boxdot K_r] = \{a_1, a_2, ..., a_q\} \cup \{s_1, s_2, ..., s_q\} \cup \{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q, 1 \leq j \leq r\}$. Here, $|V[C_q \boxdot K_r]| = q(r+2)$ and $|E[C_q \boxdot K_r]| = (1/2)(4q + qr(r+3))$. In such case, the adjacency matrix of $C_q \boxdot K_r$ will be of the order $[q(r+2)] \times [q(r+2)]$. Now we present the adjacency matrix of $C_q \boxdot K_r$ with respect to vertex s_1.

$$s_1 \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_q & s_1 & s_2 & \dots & s_q \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ c_1 & c_2 & \dots & c_{2r} & c_{2r+1} & c_{2r+2} & \dots & c_{qr} & \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \end{pmatrix}$$

We then generate the neighbour sets for each vertex as follows:

$$N(a_i) = \begin{cases} \{s_i, s_q\}; & \text{for } i = 1 \\ \{s_{i-1}, s_i\}; & \text{for } 2 \leq i \leq q \end{cases}$$

$$N(s_i) = \begin{cases} \{a_i, a_{i+1}, c_{ij}, c_{(i+1)j}\}; & \text{for } 1 \leq i \leq q-1, 1 \leq j \leq r \\ \{a_1, a_q, c_{1j}, c_{qj}\}; & \text{for } i = q, 1 \leq j \leq r \end{cases}$$

$$N(c_{ij}) = \begin{cases} \{c_{i1}, c_{i2}, \dots, c_{ir}, s_1, s_q\} \setminus \{c_{ij}\}; & \text{for } i = 1 \\ \{c_{i1}, c_{i2}, \dots, c_{ir}, s_{i-1}, s_i\} \setminus \{c_{ij}\}; & \text{for } 2 \leq i \leq q \end{cases}$$

By lemma 4.1, we have established the lower bound. To determine the upper bound, we apply a specific coloring pattern, implemented using a C++ program, which satisfies the constraints of odd coloring. This gives us:

$$\chi_{od}(C_q \boxdot K_r) \leq r + 2$$

Since both bounds coincide, we conclude that:

$$\chi_{od}(C_q \boxdot K_r) = r + 2$$

The following C++ algorithm illustrates the coloring strategy used to verify the upper bound and presents a computational example demonstrating its effectiveness.

Algorithm: Color Assignment for SVNC of C_q with K_r

Require: Integers $q \geq 3$, $r \geq 3$
Ensure: Color sets s_q, a_q, c_{ij} and odd chromatic number
1: Prompt user: Enter q (size of C_q)
2: Read q
3: Prompt user: Enter r (size of K_r)
4: Read r
5: Compute and display: Odd Chromatic Number $\leftarrow r + 2$
6: Initialize arrays: $s[0 \dots q-1]$, $a[0 \dots q-1]$, $c[0 \dots q-1][0 \dots r-1]$
7: **for** $i \leftarrow 0$ to $q-1$ **do**
8: $\quad s[i] \leftarrow (q \bmod 2 = 1 \wedge i = q-1)\ ?\ 4\ :\ (i \bmod 2 = 0)\ ?\ 2\ :\ 3$
9: **end for**
10: **for** $i \leftarrow 0$ to $q-1$ **do**
11: $\quad a[i] \leftarrow (i \bmod 2 = 0)\ ?\ 1\ :\ 4$
12: **end for**
13: $c[0][0] \leftarrow 1$, $c[0][1] \leftarrow (q \bmod 2 = 0)\ ?\ 4 : 3$

14: **for** $j \leftarrow 2$ to $r-1$ **do**
15: $c[0][j] \leftarrow (q \bmod 2 = 1 \wedge j = 2)\ ?\ 5 : c[0][j-1]+1$
16: **end for**
17: **for** $i \leftarrow 1$ to $q-1$ **do**
18: $c[i][0] \leftarrow 1$, $c[i][1] \leftarrow 4$
19: **for** $j \leftarrow 2$ to $r-1$ **do**
20: $c[i][j] \leftarrow c[i][j-1]+1$
21: **end for**
22: **end for**
23: **if** q odd **then**
24: **for** $j \leftarrow 1$ to 3 **do**
25: $c[q-1][j] \leftarrow (j=1)\ ?\ 2 : ((j=2)\ ?\ 5 : c[q-1][j-1]+1)$
26: **end for**
27: **end if**
28: Print arrays s_q, a_q, and c_{ij}

Example

Input: Enter the size of cycle graph C_q : 13
Input: Enter the size of complete graph K_r : 6
Output: $\chi_{od}(C_{13} \boxdot K_6) = 8$

i	s[i]	a[i]	c[i][1]	c[i][2]	c[i][3]	c[i][4]	c[i][5]	c[i][6]
1	2	1	1	3	5	6	7	8
2	3	4	1	4	5	6	7	8
3	2	1	1	4	5	6	7	8
4	3	4	1	4	5	6	7	8
5	2	1	1	4	5	6	7	8
6	3	4	1	4	5	6	7	8
7	2	1	1	4	5	6	7	8
8	3	4	1	4	5	6	7	8
9	2	1	1	4	5	6	7	8
10	3	4	1	4	5	6	7	8
11	4	1	1	2	5	6	7	8
12	3	4	1	4	5	6	7	8
13	4	1	1	2	5	6	7	8

□

Lemma 4.3. *For any $q, r > 2$, the lower bound of odd chromatic number of SVNC of cycle graph with cycle graph is*

$$\chi_{od}(C_q \boxdot C_r) \geq \begin{cases} 4 & ; \quad if\ r \equiv 0(mod\ 2) \\ 5 & ; \quad if\ r \equiv 1(mod\ 2) \end{cases}$$

Proof. Let $a_1, a_2, ...a_q$ be the vertices of C_q and $s_1, s_2, ...s_q$ be the subdivided vertices of C_q. Let $\{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q, 1 \leq j \leq r\}$ be the vertices of q copies of C_r. By the definition of SVNC, every vertex of the ith copy of cycle graph C_r is adjoined to every neighbour of the ith vertex of cycle graph C_q. On that note, the maximum

degree Δ will be obtained at the subdivided vertex (i.e) $\Delta[C_q \boxdot C_r] = 2r+2$, where each of the subdivided vertices forms a clique with the vertices of its associated ith copy of C_r which already gives a basic lower bound as 4 colors such that

$$\omega(C_q \boxdot C_r) = \begin{cases} \begin{cases} \{s_i, c_{ij}, c_{i(j+1)}\}, \{s_i, c_{(i+1)j}, c_{(i+1)(j+1)}\}, \\ \{s_i, c_{ir}, c_{i1}\}, \{s_i, c_{(i+1)r}, c_{(i+1)1}\}; \quad \text{for } 1 \le i \le q-1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le j \le r-1 \end{cases} \\ \begin{cases} \{s_i, c_{ij}, c_{i(j+1)}\}, \{s_i, c_{1j}, c_{1(j+1)}\}, \\ \{s_i, c_{ir}, c_{i1}\}, \{s_i, c_{1r}, c_{11}\}; \quad \text{for } i = q, 1 \le j \le r-1 \end{cases} \end{cases}$$

We know that, for any graph T, $\chi_{od}(T) \ge \omega(T)$. On that note, we have $\chi_{od}(C_q \boxdot C_r) \ge \omega(C_q \boxdot C_r) = 4$. Here, the lower bound accounts for:

- If r is even, each C_r is an even cycle, so 4 colors are enough.
- If r is odd, each C_r is an odd cycle. These, combined with the base cycle C_q and subdivisions, create more odd cycles such that an additional color is necessary, raising the minimum to 5 colors.

This concludes the lower bound as

$$\chi_{od}(C_q \boxdot C_r) \ge \begin{cases} 4 & ; \quad \text{if } r \equiv 0 (mod\ 2) \\ 5 & ; \quad \text{if } r \equiv 1 (mod\ 2) \end{cases}$$

□

Theorem 4.4. *Let $q, r > 2$ be two positive integers. Then,*

$$\chi_{od}(C_q \boxdot C_r) = \begin{cases} 4 & ; \quad \text{if } r \equiv 0 (mod\ 2) \\ 5 & ; \quad \text{if } r \equiv 1 (mod\ 2) \end{cases}$$

Proof. We define the vertex set as $V[C_q \boxdot C_r] = \{a_1, a_2, ..., a_q\} \cup \{s_1, s_2, ..., s_q\}$ $\cup \{c_{i1}, c_{i2}, ..., c_{ij} : 1 \le i \le q, 1 \le j \le r\}$. Here, $|V[C_q \boxdot C_r]| = q(r+2)$ and $|E[C_q \boxdot C_r]| = q(3r+2)$. In such case, the adjacency matrix of $C_q \boxdot C_r$ will be of the order $[q(r+2)] \times [q(r+2)]$. Now we present the adjacency matrix of $C_q \boxdot C_r$ with respect to vertex s_1.

$$s_1 \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_q & s_1 & s_2 & \dots & s_q \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ c_1 & c_2 & \dots & c_{2r} & c_{2r+1} & c_{2r+2} & \dots & c_{qr} & \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \end{pmatrix}$$

We then generate the neighbour sets for each vertex as follows:

$$N(a_i) = \begin{cases} \{s_i, s_q\}; & \text{for } i = 1 \\ \{s_{i-1}, s_i\}; & \text{for } 2 \le i \le q \end{cases}$$

$$N(s_i) = \begin{cases} \{a_i, a_{i+1}, c_{ij}, c_{(i+1)j}; & \text{for } 1 \le i \le q-1, 1 \le j \le r \\ \{a_1, a_q, c_{1j}, c_{qj}; & \text{for } i = q, 1 \le j \le r \end{cases}$$

$$N(c_{ij}) = \begin{cases} \text{for } i = 1 \begin{cases} \{c_{i(j+1)}, c_{ir}, s_1, s_q\}; & \text{for } j = 1 \\ \{c_{i(j-1)}, c_{i(j+1)}, s_1, s_q\}; & \text{for } 2 \le j \le r-1 \\ \{c_{i(j-1)}, c_{i1}, s_1, s_q\}; & \text{for } j = r \end{cases} \\ \text{for } i > 1 \begin{cases} \{c_{i(j+1)}, c_{ir}, s_{i-1}, s_i\}; & \text{for } j = 1 \\ \{c_{i(j-1)}, c_{i(j+1)}, s_{i-1}, s_i\}; & \text{for } 2 \le j \le r-1 \\ \{c_{i(j-1)}, c_{i1}, s_{i-1}, s_i\}; & \text{for } j = r \end{cases} \end{cases}$$

By lemma 4.3, we have established the lower bound. To determine the upper bound, we apply a specific coloring pattern, implemented using a C++ program, which satisfies the constraints of odd coloring. This gives us:

$$\chi_{od}(C_q \boxdot C_r) \le \begin{cases} 4 & ; \quad \text{if } r \equiv 0(mod\ 2) \\ 5 & ; \quad \text{if } r \equiv 1(mod\ 2) \end{cases}$$

Since both bounds coincide, we conclude that:

$$\chi_{od}(C_q \boxdot C_r) = \begin{cases} 4 & ; \quad \text{if } r \equiv 0(mod\ 2) \\ 5 & ; \quad \text{if } r \equiv 1(mod\ 2) \end{cases}$$

The following C++ algorithm illustrates the coloring strategy used to verify the upper bound and presents a computational example demonstrating its effectiveness.

Algorithm: Color Assignment for SVNC of C_q with C_r

Require: Integers $q \ge 3$, $r \ge 3$
Ensure: Color sets s_q, a_q, c_{ij} and odd chromatic number

```
Prompt user: Enter q (size of C_q)
Read q
Prompt user: Enter r (size of C_r)
Read r
Compute and display: Odd Chromatic Number ← (r mod 2 = 0) ? 4 : 5
Initialize arrays: s[0...q−1], a[0...q−1], c[0...q−1][0...r−1]
for i ← 0 to q − 1 do
    s[i] ← (q mod 2 = 1 ∧ i = q − 1) ? 4 : (i mod 2 = 0) ? 2 : 3
end for
for i ← 0 to q − 1 do
    a[i] ← (q mod 2 = 1 ∧ i = q − 1) ? ((r mod 4 = 0 ∨ r mod 4 = 1) ? 2 :
1) : ((i mod 2 = 0) ? 1 : 4)
end for
for i ← 0 to q − 1 do
    for j ← 0 to r − 1 do
        c[i][j] ← (r mod 2 = 1 ∧ j = r − 1) ? 5 :
            (q mod 2 = 0) ? ((j mod 2 = 0) ? 1 : 4) :
            (i = 0) ? ((j mod 2 = 0) ? 1 : 3) :
            (i = q − 1) ? ((j mod 2 = 0) ? 1 : 2) : ((j mod 2 = 0) ? 1 : 4)
    end for
end for
Print arrays s_q, a_q, and c_ij
```

Example

Input: Enter the size of cycle graph C_q : 15
Input: Enter the size of cycle graph C_r : 5
Output: $\chi_{od}(C_{15} \boxdot C_5) = 5$

i	$s[i]$	$a[i]$	$c[i][1]$	$c[i][2]$	$c[i][3]$	$c[i][4]$	$c[i][5]$
1	2	1	1	3	1	3	5
2	3	4	1	4	1	4	5
3	2	1	1	4	1	4	5
4	3	4	1	4	1	4	5
5	2	1	1	4	1	4	5
6	3	4	1	4	1	4	5
7	2	1	1	4	1	4	5
8	3	4	1	4	1	4	5
9	2	1	1	4	1	4	5
10	3	4	1	4	1	4	5
11	2	1	1	4	1	4	5
12	3	4	1	4	1	4	5
13	2	1	1	4	1	4	5
14	3	4	1	4	1	4	5
15	4	2	1	2	1	2	5

□

Lemma 4.5. *For any $q > 2$ and $r > 1$, the lower bound of odd chromatic number of SVNC of cycle graph with path graph is*

$$\chi_{od}(C_q \boxdot P_r) \geq 4$$

Proof. Let $a_1, a_2, ...a_q$ be the vertices of C_q and $s_1, s_2, ...s_q$ be the subdivided vertices of C_q. Let $\{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q, 1 \leq j \leq r\}$ be the vertices of q copies of P_r. By the definition of SVNC, every vertex of the ith copy of path graph P_r is adjoined to every neighbour of the ith vertex of cycle graph C_q. On that note, the maximum degree Δ will be obtained at the subdivided vertex (i.e) $\Delta[C_q \boxdot P_r] = 2r + 2$, where each of the subdivided vertices forms a clique with the vertices of its associated ith copy of P_r which already gives a basic lower bound as 3 colors such that

$$\omega(C_q \boxdot C_r) = \begin{cases} \{s_i, c_{ij}, c_{i(j+1)}\}, \{s_i, c_{(i+1)j}, c_{(i+1)(j+1)}\}; & \text{for } 1 \leq i \leq q-1, \\ & 1 \leq j \leq r-1 \\ \{s_i, c_{ij}, c_{i(j+1)}\}, \{s_i, c_{1j}, c_{1(j+1)}\}; & \text{for } i = q, 1 \leq j \leq r-1 \end{cases}$$

We know that, for any graph T, $\chi_{od}(T) \geq \omega(T)$. On that note, we have $\chi_{od}(C_q \boxdot P_r) \geq \omega(C_q \boxdot P_r) = 4$. Here, the lower bound accounts for:

- The triangle cliques already occupy 3 colors, and due to the subdivision and overlap of neighbors, proper coloring cannot reuse all 3 colors without conflict.
- To satisfy both the clique constraints and the odd chromatic condition, an additional color is needed.

This gives the lower bound as

$$\chi_{od}(C_q \boxdot P_r) \geq 4$$

□

Theorem 4.6. *Let $q > 2$ and $r > 1$ be two positive integers. Then,*

$$\chi_{od}(C_q \boxdot P_r) = 4$$

Proof. We define the vertex set as $V[C_q \boxdot P_r] = \{a_1, a_2, ..., a_q\} \cup \{s_1, s_2, ..., s_q\} \cup \{c_{i1}, c_{i2}, ..., c_{ij} : 1 \leq i \leq q, 1 \leq j \leq r\}$. Here, $|V[C_q \boxdot P_r]| = q(r+2)$ and $|E[C_q \boxdot P_r]| = q(3r+1)$. In such case, the adjacency matrix of $C_q \boxdot P_r$ will be of the order $[q(r+2)] \times [q(r+2)]$. Now we present the adjacency matrix of $C_q \boxdot P_r$ with respect to vertex s_1.

$$s_1 \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_q & s_1 & s_2 & \dots & s_q \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ c_1 & c_2 & \dots & c_{2r} & c_{2r+1} & c_{2r+2} & \dots & c_{qr} & \\ 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \end{pmatrix}$$

We then generate the neighbour sets for each vertex as follows:

$$N(a_i) = \begin{cases} \{s_i, s_q\}; & \text{for } i = 1 \\ \{s_{i-1}, s_i\}; & \text{for } 2 \leq i \leq q \end{cases}$$

$$N(s_i) = \begin{cases} \{a_i, a_{i+1}, c_{ij}, c_{(i+1)j}; & \text{for } 1 \leq i \leq q-1, 1 \leq j \leq r \\ \{a_1, a_q, c_{1j}, c_{qj}; & \text{for } i = q, 1 \leq j \leq r \end{cases}$$

$$N(c_{ij}) = \begin{cases} \text{for } i = 1 \begin{cases} \{c_{i(j+1)}, s_1, s_q\}; & \text{for } j = 1 \\ \{c_{i(j-1)}, c_{i(j+1)}, s_1, s_q\}; & \text{for } 2 \leq j \leq r-1 \\ \{c_{i(j-1)}, s_1, s_q\}; & \text{for } j = r \end{cases} \\ \text{for } i > 1 \begin{cases} \{c_{i(j+1)}, s_{i-1}, s_i\}; & \text{for } j = 1 \\ \{c_{i(j-1)}, c_{i(j+1)}, s_{i-1}, s_i\}; & \text{for } 2 \leq j \leq r-1 \\ \{c_{i(j-1)}, s_{i-1}, s_i\}; & \text{for } j = r \end{cases} \end{cases}$$

By lemma 4.5, we have established the lower bound. To determine the upper bound, we apply a specific coloring pattern, implemented using a C++ program, which satisfies the constraints of odd coloring. This gives us:

$$\chi_{od}(C_q \boxdot P_r) \leq 4$$

Since both bounds coincide, we conclude that:

$$\chi_{od}(C_q \boxdot P_r) = 4$$

The following C++ algorithm illustrates the coloring strategy used to verify the upper bound and presents a computational example demonstrating its effectiveness.

Algorithm: Color Assignment for SVNC of C_q with P_r

Require: Integers $q \geq 3$, $r \geq 2$
Ensure: Color sets s_q, a_q, c_{ij} and odd chromatic number
1: Prompt user: Enter q (size of C_q)
2: Read q
3: Prompt user: Enter r (size of P_r)

Read r
Compute and display: Odd Chromatic Number $\leftarrow 4$
Initialize arrays: $s[0 \ldots q-1]$, $a[0 \ldots q-1]$, $c[0 \ldots q-1][0 \ldots r-1]$
for $i \leftarrow 0$ to $q-1$ **do**
    $s[i] \leftarrow (q \bmod 2 = 1 \wedge i = q-1)\ ?\ 4 : (i \bmod 2 = 0)\ ?\ 2 : 3$
end for
for $i \leftarrow 0$ to $q-1$ **do**
    $a[i] \leftarrow (q \bmod 2 = 1 \wedge i = q-1)\ ?\ ((r \bmod 4 = 0 \vee r \bmod 4 = 1)\ ?\ 2\ :\ 1)\ :\ ((i \bmod 2 = 0)\ ?\ 1\ :\ 4)$
end for
for $i \leftarrow 0$ to $q-1$ **do**
    for $j \leftarrow 0$ to $r-1$ **do**
        $c[i][j] \leftarrow (q \bmod 2 = 1 \wedge i = 0)\ ?\ ((j \bmod 2 = 0)\ ?\ 1\ :\ 3)\ :\ (q \bmod 2 = 1 \wedge i = q-1)\ ?\ ((j \bmod 2 = 0)\ ?\ 1\ :\ 2) : ((j \bmod 2 = 0)\ ?\ 1\ :\ 4)$
    end for
end for
Print arrays s_q, a_q, and c_{ij}

Example

Input: Enter the size of cycle graph C_q : 11
Input: Enter the size of path graph P_r : 5
Output: $\chi_{od}(C_{11} \boxdot P_5) = 4$

i	$s[i]$	$a[i]$	$c[i][1]$	$c[i][2]$	$c[i][3]$	$c[i][4]$	$c[i][5]$
1	2	1	1	3	1	3	1
2	3	4	1	4	1	4	1
3	2	1	1	4	1	4	1
4	3	4	1	4	1	4	1
5	2	1	1	4	1	4	1
6	3	4	1	4	1	4	1
7	2	1	1	4	1	4	1
8	3	4	1	4	1	4	1
9	2	1	1	4	1	4	1
10	3	4	1	4	1	4	1
11	4	2	1	2	1	2	1

□

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