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ON HAMILTONIAN CONNECTEDNESS AND RANDOM HAMILTON LACEABILITY OF TOTAL TRANSFORMATION GRAPHS $\mathcal{Z}^{\alpha\beta\gamma}$

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Abstract. A connected graph $\mathcal{Z} = (V, E)$ is called Hamiltonian if $\mathcal{Z}$ contains a spanning cycle and if a graph $\mathcal{Z}$ contains a spanning path between every pair of its vertices is called Hamilton-connected. A bipartite graph is called Hamilton-laceable if there exist Hamiltonian path between vertices of different partite sets and a graph $\mathcal{Z}$ is random Hamiltonian laceable if there exists a $u - v$ Hamiltonian path between every pair $u, v \in V(\mathcal{Z})$ of vertices at an odd distance. In this paper, we have prove that the transformation graphs $\mathcal{Z}^{+++}$, $\mathcal{Z}^{---}$, $\mathcal{Z}^{++-}$, $\mathcal{Z}^{+--}$ and $\mathcal{Z}^{-+-}$ are Hamiltonian connected and $\mathcal{Z}^{-++}$ is random Hamiltonian laceable for some standard class of graphs.

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1. Introduction

Let $\mathcal{Z}$ be simple and connected graph. A cycle in a graph $\mathcal{Z}$ is called Hamiltonian if it passes through every vertex of $\mathcal{Z}$ exactly once. Likewise, a Hamiltonian path is a path that visits all the vertices without repetition. However, not all graphs include Hamiltonian cycles. For example, trees are acyclic and therefore cannot contain any Hamiltonian cycles, although they may still possess Hamiltonian paths.

A graph $\mathcal{Z}$ is classified as Hamiltonian if it has at least one Hamiltonian cycle. By nature, both cycle graphs and complete graphs $\mathfrak{K}_n$ (also known as cliques) are Hamiltonian. A graph is said to be traceable if it includes at least one Hamiltonian path. While every Hamiltonian graph is necessarily traceable, the reverse is not always true. A classic instance of a traceable but non-Hamiltonian graph is the Petersen graph.

In a graph $\mathcal{Z}$ if every pair of distinct vertices are connected by a spanning path, it is referred to as Hamilton-connected [16, 3].

Bipartite graphs cannot exhibit Hamilton-connectivity, as a Hamiltonian path cannot exist between two vertices in the same partite set. In such cases, if Hamiltonian paths do exist between vertices of opposite partite sets, the graph is termed Hamilton-laceable.

There is a significant amount of research dedicated to the study of Hamiltonicity and Hamilton-connectivity in graphs (see, for example, [17, 13, 27]). If for any two vertices $v, w \in V(\mathcal{Z})$ from different bipartition classes of a bipartite graph $\mathcal{Z}$ there exists a Hamilton-path whose endvertices are v and w then $\mathcal{Z}$ is called Hamilton-laceable, see [1, 4, 5, 6, 7, 8, 10, 18, 20].

Vertex(edge) connectivity of a simple graph G is the minimum number of vertices(edges) in G such that whose removal increases the components of G [9].

Definition 1. *Let $\mathcal{Z} = (V, E)$ be a connected graph. A subgraph $X = (V, E_X)$ is an EPS-subgraph(Extended Persistent Structures Subgraph) of $\mathcal{Z}$ if:*

1. *X is a spanning subgraph ($V(X) = V(\mathcal{Z})$) and connected.*
2. *The edge set E_X can be partitioned into two edge-disjoint subsets:*
   - *An Euler subgraph H (not necessarily connected), where every vertex has even degree.*
   - *A forest P whose components are paths (i.e., each tree in P is a path graph).*

Definition 2. *The vertex set of a line graph $L(\mathcal{Z})$ of a graph $\mathcal{Z} = (V, E)$ is E such that $e_i, e_j \in E(L(\mathcal{Z}))$ if and only if e_i and e_j share a common vertex v in $\mathcal{Z}$.*

Definition 3. *The total graph $T(\mathcal{Z})$ of a graph $\mathcal{Z} = (V, E)$ has $V(T(\mathcal{Z})) = V \cup E$ and any arbitrary vertices $u, v \in V(T(\mathcal{Z}))$ are adjacent if and only if either they are adjacent vertices(edges) or u and v are incident in $\mathcal{Z}$.*

The generalization of total graph put forwarded by Wu and Meng [22].

Definition 4. *Let $\mathcal{Z} = (V, E)$ be a graph and α, β, γ be three variables taking values $+$ or $-$. The total transformation graph is denoted by $\mathcal{Z}^{\alpha\beta\gamma}$ where $V(\mathcal{Z}^{\alpha\beta\gamma}) = V \cup E$ and for $\rho, \mho \in V \cup E$, $\rho \sim \mho$ in $\mathcal{Z}^{\alpha\beta\gamma}$ if and only if*

(1) *$\rho, \mho \in V(\mathcal{Z})$, $\rho \sim \mho$ in $\mathcal{Z}$ if $\alpha = +$ and $\rho \nsim \mho$ if $\alpha = -$.*
(2) *$\rho, \mho \in E(\mathcal{Z})$, $\rho \sim \mho$ in $\mathcal{Z}$ if $\beta = +$ and $\rho \nsim \mho$ if $\beta = -$.*
(3) *$\rho \in V(\mathcal{Z})$ and $\mho \in E(\mathcal{Z})$, $\rho \sim \mho$ in $\mathcal{Z}$ if $\gamma = +$ and $\rho \nsim \mho$ if $\gamma = -$.*

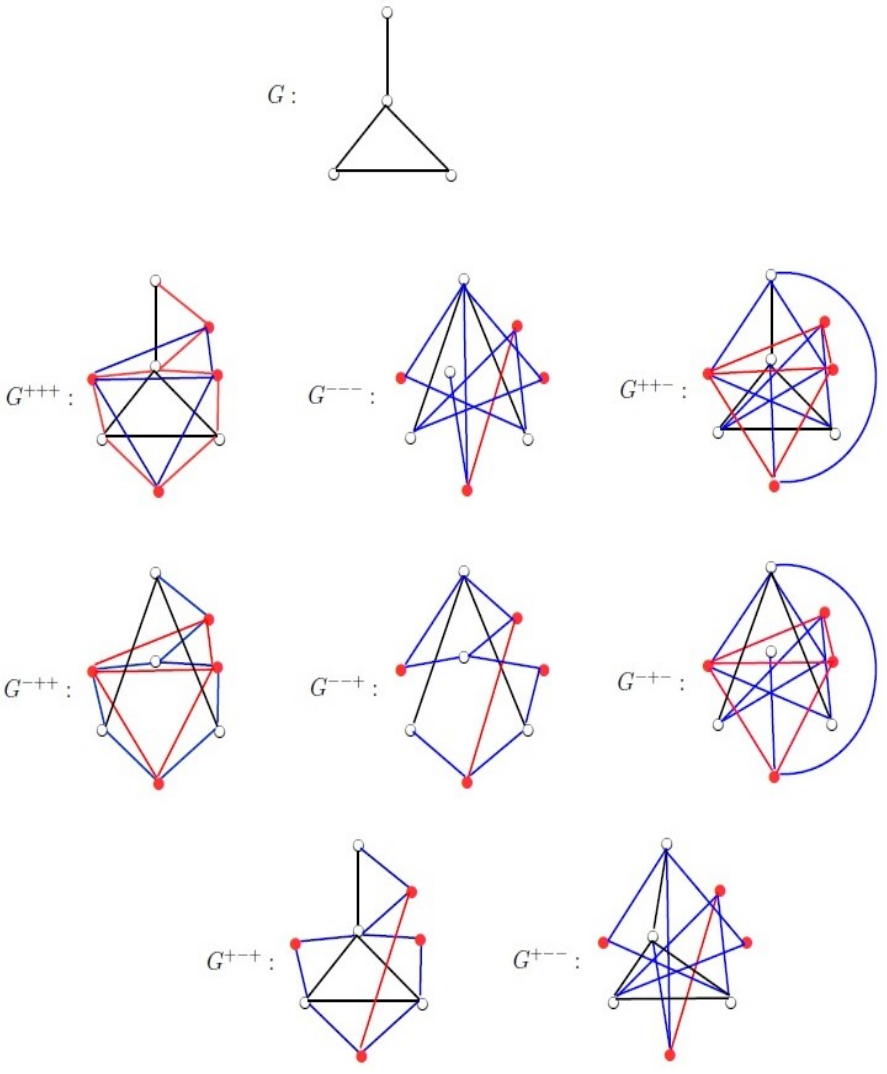


Figure 1. A graph $\mathcal{Z}$ and its total transformation graphs.

2. Auxiliary Results

Theorem 1. [21] Let $\mathcal{Z}$ be a graph with $\kappa = r$ and $\lambda = l$ and $\delta = s$. Then

$$r \leq l \leq s.$$

Theorem 2. [14] Every 4-connected line graph is Hamiltonian if and only if it is Hamilton-connected.

Theorem 3. [9] A graph H is a line graph of some graph if and only if it does not contains the following forbidden subgraphs.

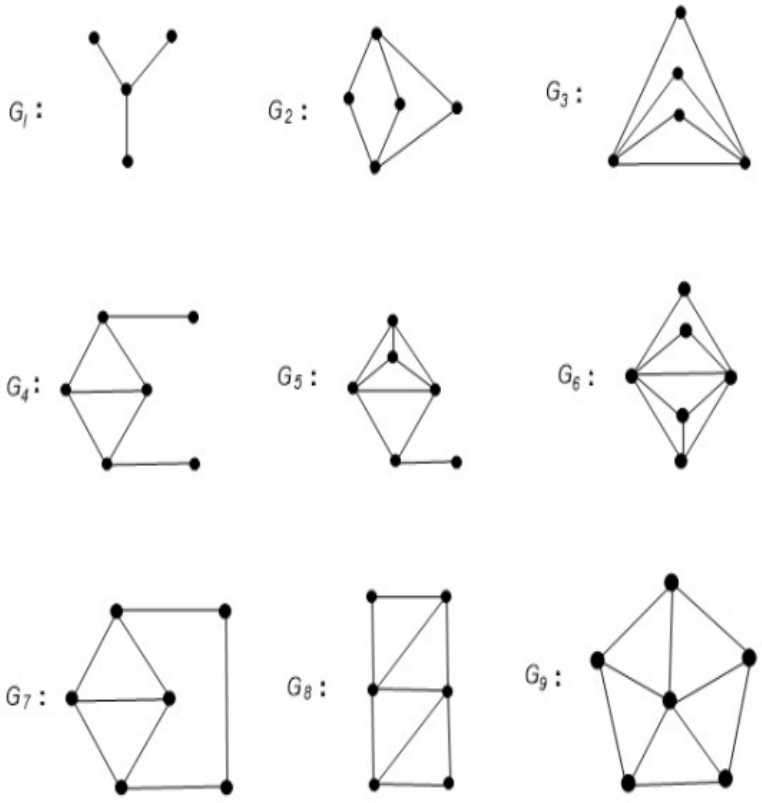


Figure 2. The nine forbidden subgraphs for a line graph.

Theorem 4. [2] The graph $\mathcal{Z}^{+++}$ is Hamiltonian if and only if $\mathcal{Z} = (n, m)$ contains an EPS -subgraph where $(n \geq 2)$.

Theorem 5. [15] The graph $\mathcal{Z}^{---}$ is Hamiltonian if and only if $\mathcal{Z}$ is not isomorphic to any graphs in

$$\{\mathfrak{K}_{1,r} | r \geq 1\} \cup \{\mathfrak{K}_{1,s} + \mathfrak{K}_1 | s \geq 1\} \cup \{\mathfrak{K}_{1,t} + e | t \geq 2\} \cup \{\mathfrak{K}_3, \mathfrak{K}_3 + \mathfrak{K}_1, \mathfrak{K}_3 + 2\mathfrak{K}_1, \mathfrak{K}_4\}$$

Theorem 6. [25] Let $\mathcal{Z} = (n, m)$- graph where $n \geq 6$. If $m \geq \alpha(\mathcal{Z}) + 1$ then $\mathcal{Z}^{++-}$ is Hamiltonian, where $\alpha(\mathcal{Z})$ is the independence number of $\mathcal{Z}$.

Theorem 7. [26] The graph $\mathcal{Z}^{+--}$ is Hamiltonian if and only if $\delta(\mathcal{Z}) \geq 1$ and $\mathcal{Z} \notin \{\mathfrak{K}_{1,n-1}, 2\mathfrak{K}_2, \mathfrak{K}_3\}$.

Theorem 8. [24] The graph $\mathcal{Z}^{-+-}$ is hamiltonian if and only $\mathcal{Z}$ contains at least 4-vertices and

$$\mathcal{Z} \ncong \{\mathfrak{K}_{1,n-1}, \mathfrak{K}_{1,n-1} + e, \mathfrak{K}_{1,n-2} + \mathfrak{K}_1\} \quad \cup \quad \{2\mathfrak{K}_1 + \mathfrak{K}_2, \mathfrak{K}_1 + \mathfrak{K}_3\}.$$

3. Hamiltonian connectedness of total transformation graphs

Let $\mathcal{H}_1 = \{\mathcal{Z}_i | i = 1, 2, ..., 9\}$ be the set containing forbidden subgraphs of line graph (see Figure 2) and let $\mathcal{H}_2 = \{\mathcal{Z}\}$ such that $\overline{\mathcal{Z}} \in \mathcal{H}_1$.

Theorem 9. *Let $\mathcal{Z} \ncong \mathcal{H}_1$ be a graph of order $n \geq 4$ with $\delta(\mathcal{Z}) \geq 2$ and contains an EPS-subgraph then $\mathcal{Z}^{+++}$ is hamiltonian connected.*

Proof. Let $\mathcal{Z}^{+++}$ be a total transformation graph of a graph $\mathcal{Z}$. We divide the proof into the following claims.

Claim 1: $\mathcal{Z}^{+++} = H$ such that $L^{-1}(H) = \mathcal{Z}$.
By the hypothesis of the theorem $\mathcal{Z} \ncong \mathcal{H}_1$. Since $\mathcal{Z}$ is an induced subgraph of $\mathcal{Z}^{+++}$. Therefore, $\mathcal{Z}^{+++}$ does not contain any forbidden subgraphs of line graph. So by Theorem 3, $\mathcal{Z}^{+++}$ is a line graph.

Claim 2: $\kappa(\mathcal{Z}^{+++}) = 4$ for $n \geq 4$.
By the hypothesis of the theorem $n \geq 4$ and $\delta(\mathcal{Z}) \geq 2$. Note that $\mathcal{Z}^{+++}$ has $(m + n)$ vertices and for $u \in V(\mathcal{Z}^{+++}) \cap V(\mathcal{Z})$, $d_{\mathcal{Z}^{+++}}(u) = 2d_{\mathcal{Z}}(u)$ and for $u \in V(\mathcal{Z}^{+++}) \cap E(\mathcal{Z})$, $d_{\mathcal{Z}^{+++}}(u) = d_{\mathcal{Z}}(u) + d_{\mathcal{Z}}(v)$. Hence, $\delta(\mathcal{Z}^{+++}) \geq 4$ and by Theorem 1, $\kappa(\mathcal{Z}^{+++}) \geq 4$. In order to show that $\kappa(\mathcal{Z}^{+++}) \leq 4$ let v_i be minimum degree vertex in $\mathcal{Z}$ such that $d_{\mathcal{Z}}(v_i) = 2$. Then $d_{\mathcal{Z}^{+++}}(v_i) = 4$. Let $N(v_i) = \{v_j, v_k, e_i, e_j\}$ be the vertices at its neighborhood. Note that $V(\mathcal{Z}^{+++}) \setminus N(v_i)$ is disconnected having v_i as an isolated connected component. This implies that $\kappa(\mathcal{Z}^{+++}) \leq 4$. Hence $\mathcal{Z}^{+++}$ is 4-connected.

Claim 3: $\mathcal{Z}^{+++}$ is Hamiltonian.
By hypothesis of the theorem $n \geq 4$ and $\mathcal{Z}$ contains an EPS-subgraph. Therefore, by Theorem 4, $\mathcal{Z}^{+++}$ is Hamiltonian.

Hence $\mathcal{Z}^{+++}$ is a Hamiltonian, $\kappa(\mathcal{Z}^{+++}) = 4$. By Theorem 2, $\mathcal{Z}^{+++}$ is Hamilton-connected. □

Theorem 10. *Let $\mathcal{Z} \ncong \{\mathcal{H}_2, \mathcal{H}_3\}$ with $n \geq 4$ and $m - n + 1 \geq 4$ then $\mathcal{Z}^{---}$ is hamiltonian connected, where*

$$\mathcal{H}_3 = \{\mathfrak{K}_{1,r} | r \geq 1\} \cup \{\mathfrak{K}_{1,s} + \mathfrak{K}_1 | s \geq 1\} \cup \{\mathfrak{K}_{1,t} + e | t \geq 2\} \cup \{\mathfrak{K}_3, \mathfrak{K}_3 + \mathfrak{K}_1, \mathfrak{K}_3 + 2\mathfrak{K}_1, \mathfrak{K}_4\}$$

Proof. Let $\mathcal{Z}^{---}$ be a total transformation graph of a graph $\mathcal{Z}$. We divide the proof into the following claims.

Claim 1: $\mathcal{Z}^{---} = H$ such that $L^{-1}(H) = \mathcal{Z}$.
By the hypothesis of the theorem $\mathcal{Z} \ncong \mathcal{H}_2$. Since $\overline{\mathcal{Z}}$ is an induced subgraph of $\mathcal{Z}^{---}$. Therefore, $\mathcal{Z}^{---}$ does not contain any forbidden subgraphs of line graph. So by Theorem 3, $\mathcal{Z}^{---}$ is a line graph.

Claim 2: $\kappa(\mathcal{Z}^{---}) = 4$.
By the hypothesis of the theorem $n \geq 4$. Note that $\mathcal{Z}^{---}$ has $(m + n)$ vertices and for $u \in V(\mathcal{Z}^{---}) \cap V(\mathcal{Z})$, $d_{\mathcal{Z}^{---}}(u) = m + n - 1 - 2d_{\mathcal{Z}}(u)$ and for $u \in V(\mathcal{Z}^{---}) \cap E(\mathcal{Z})$, $d_{\mathcal{Z}^{---}}(u) = m + n - 1 - (d_{\mathcal{Z}}(u) + d_{\mathcal{Z}}(v))$.

Case 1: Let $v = \Delta(\mathcal{Z})$ then $v = \delta(\mathcal{Z}^{---})$. Suppose $d_{\mathcal{Z}}(v) \leq n-1$. Then $d_{\mathcal{Z}^{---}}(v) \geq m - n + 1 \geq 4$ i.e $\delta(\mathcal{Z}^{---}) \geq 4$.

Case 2: Let $v = \delta(\mathcal{Z})$ then $v = \Delta(\mathcal{Z}^{---})$. Suppose $d_{\mathcal{Z}}(v) \leq 1$. Then $d_{\mathcal{Z}^{---}}(v) \geq m + n - 3 \geq 4$ i.e $\Delta(\mathcal{Z}^{---}) \geq 4$.

Hence, $\delta(\mathcal{Z}^{---}) \geq 4$ and by Theorem 1, $\kappa(\mathcal{Z}^{---}) \geq 4$. In order to show that $\kappa(\mathcal{Z}^{---}) \leq 4$ let v_i be maximum degree vertex in $\mathcal{Z}$ such that $d_{\mathcal{Z}}(v_i) = n - 1$. Then $d_{\mathcal{Z}^{---}}(v_i) = m - n + 1 = 4$. Let $N(v_i) = \{v_j, v_k, e_i, e_j\}$ be the vertices at its neighborhood. Note that $V(\mathcal{Z}^{---}) \setminus N(v_i)$ is disconnected having v_i as an isolated connected component. This implies that $\kappa(\mathcal{Z}^{---}) \leq 4$. Hence $\mathcal{Z}^{---}$ is 4-connected.

Claim 3: $\mathcal{Z}^{---}$ is Hamiltonian.
By hypothesis of the theorem $\mathcal{Z} \ncong \{\mathcal{H}_3\}$ Therefore, by Theorem 5, $\mathcal{Z}^{---}$ is Hamiltonian.

Hence $\mathcal{Z}^{---}$ is a Hamiltonian, $\kappa(\mathcal{Z}^{---}) = 4$. By Theorem 2, $\mathcal{Z}^{---}$ is Hamilton-connected. □

Theorem 11. *Let $\mathcal{Z} \ncong \mathcal{H}_1$ be a graph with of order $n \geq 6$ and size m with $m \geq \alpha(\mathcal{Z}) + 1$ then $\mathcal{Z}^{++-}$ is Hamiltonian connected.*

Proof. Let $\mathcal{Z}^{++-}$ be a total transformation graph of a graph $\mathcal{Z}$. We divide the proof into the following claims.

Claim 1: $\mathcal{Z}^{++-} = H$ such that $L^{-1}(H) = \mathcal{Z}$.
By the hypothesis of the theorem $\mathcal{Z} \ncong \mathcal{H}_1$. Since $\mathcal{Z}$ is an induced subgraph of $\mathcal{Z}^{++-}$. Therefore, $\mathcal{Z}^{++-}$ does not contain any forbidden subgraphs of line graph. So by Theorem 3, $\mathcal{Z}^{++-}$ is a line graph.

Claim 2: $\kappa(\mathcal{Z}^{++-}) = 4$ for $n \geq 4$.
By the hypothesis of the theorem $n \geq 6$. Note that $\mathcal{Z}^{++-}$ has $(m + n)$ vertices and for $u \in V(\mathcal{Z}^{++-}) \cap V(\mathcal{Z})$, $d_{\mathcal{Z}^{++-}}(u) = m$ and for $u \in V(\mathcal{Z}^{++-}) \cap E(\mathcal{Z})$, $d_{\mathcal{Z}^{++-}}(u) = d_{\mathcal{Z}}(u)+d_{\mathcal{Z}}(v)+n-4$. Therefore, Hence, $\delta(\mathcal{Z}^{++-}) \geq 4$ and by Theorem 1, $\kappa(\mathcal{Z}^{++-}) \geq 4$. In order to show that $\kappa(\mathcal{Z}^{++-}) \leq 4$ let v_i be minimum degree vertex in $\mathcal{Z}$ such that $d_{\mathcal{Z}}(v_i) = n - 1$. Then $d_{\mathcal{Z}^{++-}}(v_i) = n + 3 - n + 1 = 4$. Let $N(v_i) = \{v_j, v_k, e_i, e_j\}$ be the vertices at its neighborhood. Note that $V(\mathcal{Z}^{++-}) \setminus N(v_i)$ is disconnected having v_i as an isolated connected component. This implies that $\kappa(\mathcal{Z}^{++-}) \leq 4$. Hence $\mathcal{Z}^{++-}$ is 4-connected.

Claim 3: $\mathcal{Z}^{++-}$ is Hamiltonian.
By hypothesis of the theorem $\mathcal{Z} \ncong \mathcal{H}_1$ and $m \geq \alpha(\mathcal{Z}) + 1$. Therefore, by Theorem 6, $\mathcal{Z}^{++-}$ is Hamiltonian.

Hence $\mathcal{Z}^{++-}$ is a Hamiltonian, $\kappa(\mathcal{Z}^{++-}) = 4$. By Theorem 2, $\mathcal{Z}^{++-}$ is Hamilton-connected. □

Theorem 12. *Let $\mathcal{Z} \ncong \mathcal{H}_1 \cup \{\mathfrak{K}_{1,n-1}, 2\mathfrak{K}_2, \mathfrak{K}_3\}$ be a graph with $\delta(\mathcal{Z}) \geq 3$ then $\mathcal{Z}^{+--}$ is hamiltonian connected.*

Proof. Let $\mathcal{Z}^{+--}$ be a total transformation graph of a graph $\mathcal{Z}$. We divide the proof into the following claims.

Claim 1: $\mathcal{Z}^{+--} = H$ such that $L^{-1}(H) = \mathcal{Z}$.
By the hypothesis of the theorem $\mathcal{Z} \ncong \mathcal{H}_1$. Since $\mathcal{Z}$ is an induced subgraph of $\mathcal{Z}^{+--}$. Therefore, $\mathcal{Z}^{+--}$ does not contain any forbidden subgraphs of line graph. So by Theorem 3, $\mathcal{Z}^{+--}$ is a line graph.

Claim 2: $\kappa(\mathcal{Z}^{+--}) = 4$ for $n \geq 4$.
By the hypothesis of the theorem $n \geq 6$. Note that $\mathcal{Z}^{+--}$ has $(m + n)$ vertices and for $u \in V(\mathcal{Z}^{+--}) \cap V(\mathcal{Z})$, $d_{\mathcal{Z}^{+--}}(u) = m$ and for $u \in V(\mathcal{Z}^{+--}) \cap E(\mathcal{Z})$, $d_{\mathcal{Z}^{+--}}(u) = m + n - (1 + d_{\mathcal{Z}}(u) + d_{\mathcal{Z}}(v))$. Therefore, Hence, $\delta(\mathcal{Z}^{+--}) \geq 4$ and by Theorem 1, $\kappa(\mathcal{Z}^{+--}) \geq 4$. In order to show that $\kappa(\mathcal{Z}^{+--}) \leq 4$ let v_i be minimum degree vertex in $\mathcal{Z}$ such that $d_{\mathcal{Z}}(v_i) = n-1$. Then $d_{\mathcal{Z}^{+--}}(v_i) = n+3-n+1 = 4$. Let $N(v_i) = \{v_j, v_k, e_i, e_j\}$ be the vertices at its neighborhood. Note that $V(\mathcal{Z}^{+--}) \setminus N(v_i)$ is disconnected having v_i as an isolated connected component. This implies that $\kappa(\mathcal{Z}^{+--}) \leq 4$. Hence $\mathcal{Z}^{+--}$ is 4-connected.

Claim 3: $\mathcal{Z}^{+--}$ is Hamiltonian.
By hypothesis of the theorem $\mathcal{Z} \ncong \mathcal{H}_1 \cup \{\mathfrak{K}_{1,n-1}, 2\mathfrak{K}_2, \mathfrak{K}_3\}$. Therefore, by Theorem 7, $\mathcal{Z}^{+--}$ is Hamiltonian.

Hence $\mathcal{Z}^{+--}$ is a Hamiltonian, $\kappa(\mathcal{Z}^{+--}) = 4$. By Theorem 2, $\mathcal{Z}^{+--}$ is Hamilton-connected. □

Theorem 13. *Let $\mathcal{Z} \ncong \{\mathcal{H}_2, \mathcal{H}_4\}$ with $n \geq 4$ then $\mathcal{Z}^{-+-}$ is hamiltonian connected, where*

$$\mathcal{H}_4 = \{\mathfrak{K}_{1,n-1}, \mathfrak{K}_{1,n-1} + e, \mathfrak{K}_{1,n-2} + \mathfrak{K}_1\} \quad \cup \quad \{2\mathfrak{K}_1 + \mathfrak{K}_2, \mathfrak{K}_1 + \mathfrak{K}_3\}.$$

Proof. Let $\mathcal{Z}^{-+-}$ be a total transformation graph of a graph $\mathcal{Z}$. We divide the proof into the following claims.

Claim 1: $\mathcal{Z}^{-+-} = H$ such that $L^{-1}(H) = \mathcal{Z}$.
By the hypothesis of the theorem $\mathcal{Z} \ncong \mathcal{H}_2$. Since $\overline{\mathcal{Z}}$ is an induced subgraph of $\mathcal{Z}^{-+-}$.

Therefore, $\mathcal{Z}^{-+-}$ does not contain any forbidden subgraphs of line graph. So by Theorem 3, $\mathcal{Z}^{-+-}$ is a line graph.

Claim 2: $\kappa(\mathcal{Z}^{-+-}) = 4$ for For $n \geq 4$.

By the hypothesis of the theorem $n \geq 4$. Note that for $u \in V(\mathcal{Z}^{-+-}) \cap V(\mathcal{Z})$, $d_{\mathcal{Z}^{-+-}}(u) = m+n-(1+2d_{\mathcal{Z}}(u))$ and for $u \in V(\mathcal{Z}^{-+-}) \cap E(\mathcal{Z})$, $d_{\mathcal{Z}^{-+-}}(u) = n-4+(d_{\mathcal{Z}}(u)+d_{\mathcal{Z}}(v))$.

Case 1: Let $v = \Delta(\mathcal{Z})$ then $v = \delta(\mathcal{Z}^{-+-})$. Suppose $d_{\mathcal{Z}}(v) \leq n-1$. Then $d_{\mathcal{Z}^{-+-}}(v) \geq m-n+1 \geq 4$ i.e $\delta(\mathcal{Z}^{-+-}) \geq 4$.

Case 2: Let $v = \delta(\mathcal{Z})$ then $v = \Delta(\mathcal{Z}^{-+-})$. Suppose $d_{\mathcal{Z}}(v) \leq 1$. Then $d_{\mathcal{Z}^{-+-}}(v) \geq m+n-3 \geq 4$ i.e $\Delta(\mathcal{Z}^{-+-}) \geq 4$.

Hence, $\delta(\mathcal{Z}^{-+-}) \geq 4$ and by Theorem 1, $\kappa(\mathcal{Z}^{-+-}) \geq 4$. In order to show that $\kappa(\mathcal{Z}^{-+-}) \leq 4$ let v_i be maximum degree vertex in $\mathcal{Z}$ such that $d_{\mathcal{Z}}(v_i) = n-1$. Then $d_{\mathcal{Z}^{-+-}}(v_i) = m-n+1 = 4$. Let $N(v_i) = \{v_j, v_k, e_i, e_j\}$ be the vertices at its neighborhood. Note that $V(\mathcal{Z}^{-+-}) \setminus N(v_i)$ is disconnected having v_i as an isolated connected component. This implies that $\kappa(\mathcal{Z}^{-+-}) \leq 4$. Hence $\mathcal{Z}^{-+-}$ is 4-connected.

Claim 3: $\mathcal{Z}^{-+-}$ is Hamiltonian.

By hypothesis of the theorem $\mathcal{Z} \ncong \{\mathcal{H}_2, \mathcal{H}_4\}$, where $\mathcal{H}_4 = \{\mathfrak{K}_{1,n-1}, \mathfrak{K}_{1,n-1}+e, \mathfrak{K}_{1,n-2}+\mathfrak{K}_1\} \cup \{2\mathfrak{K}_1+\mathfrak{K}_2, \mathfrak{K}_1+\mathfrak{K}_3\}$. Therefore, by Theorem 8, $\mathcal{Z}^{-+-}$ is Hamiltonian.

Hence $\mathcal{Z}^{-+-}$ is a Hamiltonian, $\kappa(\mathcal{Z}^{-+-}) = 4$. By Theorem 2, $\mathcal{Z}^{-+-}$ is Hamilton-connected. □

4. Hamiltonicity of $\mathcal{Z}^{-++}$

Definition 5. *A graph $\mathcal{Z} = (V, E)$ with $|V| \geq 2$, is random Hamiltonian laceable if there exists a $u-v$ Hamiltonian path between every pair $u, v \in V(\mathcal{Z})$ of vertices at an odd distance.*

Theorem 14. [23] $\mathcal{Z}^{-++}$ is connected for any graph $\mathcal{Z}$.

Theorem 15. [23] Let $\mathcal{Z}$ be a graph then $diam(\mathcal{Z}^{-++}) \leq 3$ and the equality holds if and only if $diam(L(\mathcal{Z})) > 2$.

Theorem 16. [23] Let $\mathcal{Z}$ be a graph with of order at least three then $\mathcal{Z}^{-++}$ is Hamiltonian.

Theorem 17. *The transformation graph $\mathcal{Z}^{-++}$ is not Hamiltonian laceable.*

Proof. Consider a graph $\mathcal{Z} = (V, E)$ with $|V| = n$ and $|E| = m$. Assume that $\mathcal{Z}^{-++}$is Hamiltonian laceable. Let $e = uv$ be an edge of a graph $\mathcal{Z}$ then by definition of $\mathcal{Z}^{-++}$is $e \in (\mathcal{Z}^{-++})$ $anduv \in (\mathcal{Z}^{-++})$then
Let $(e_i, e_j) \in E(\mathcal{Z})$ where e_i and e_j are adjacent in $\mathcal{Z}$. Let $v \in V(\mathcal{Z})$ be a common vertex between e_iande_j now by the definition $\mathcal{Z}^{-++}, (e_i, e_j, v) \in V(\mathcal{Z}^{-++})$ further (e_i, e_j) are adjacent edges in $\mathcal{Z}$.
$\therefore$ (e_i, e_j) are adjacent vertices in $\mathcal{Z}^{-++}$also it can be easily verified that (e_i, e_j, v) will form cycle graph of length 3 which is a contradiction to our assumption that $\mathcal{Z}^{-++}$is Hamiltonian laceable due to fact that $\mathcal{Z}^{-++}$ is a non-bipartite graph. □

Theorem 18. *The Transformation graph $\mathcal{Z}^{-++}$is Hamiltonian connected if and only if $\overline{\mathcal{Z}}$ and $L(\mathcal{Z})$ are Hamiltonian connected.*

Proof. Let $\mathcal{Z} = (V, E)$ be a graph such that $\overline{\mathcal{Z}}$ and its $L(\mathcal{Z})$ are Hamiltonian connected, i.e. a Hamiltonian path exists between every $(v_i, v_j) \in \overline{\mathcal{Z}}$ and $(e_i, e_j) \in L(\mathcal{Z})$ respectively. Now we have to show that $\mathcal{Z}^{-++}$ is Hamiltonian connected.

We prove this result by considering the following cases.

- **Case (1):** $(v_i, v_j) \in V(\mathcal{Z}^{-++})$ such that $(v_i, v_j) \in V(\mathcal{Z})$. Since $\overline{\mathcal{Z}}$ and $L(\mathcal{Z})$ are subgraphs of $\mathcal{Z}^{-++}$ which are Hamiltonian connected, we have the following Hamiltonian path between $(v_i, v_j) \in \mathcal{Z}^{-++}$:

$$P_H(v_i, v_j) = v_i \sim e_i \sim \bigcup_{2 \leq j \leq m} e_j \sim v_k \sim \bigcup_{2 \leq s \leq n-2} v_s \sim v_j.$$

- **Case (2):** $(e_i, e_j) \in V(\mathcal{Z}^{-++})$ such that $(e_i, e_j) \in V(\mathcal{Z})$. Since $\overline{\mathcal{Z}}$ and $L(\mathcal{Z})$ are subgraphs of $\mathcal{Z}^{-++}$ which are Hamiltonian connected, we have the following Hamiltonian path between $(e_i, e_j) \in \mathcal{Z}^{-++}$:

$$P_H(e_i, e_j) = e_i \sim v_i \sim \bigcup_{2\leq j\leq n} v_j \sim e_k \sim \bigcup_{2\leq s\leq m-2} e_s \sim e_j.$$

- **Case (3):** $(v_i, e_j) \in V(\mathcal{Z}^{-++})$ such that $v_i \in V(\mathcal{Z})$ and $e_j \in E(\mathcal{Z})$. Since $\overline{\mathcal{Z}}$ and $L(\mathcal{Z})$ are subgraphs of $\mathcal{Z}^{-++}$ which are Hamiltonian connected, we have the following Hamiltonian path between $(v_i, e_j) \in \mathcal{Z}^{-++}$:

$$P_H(v_i, e_j) = v_i \sim \bigcup_{2\leq j\leq n} v_j \sim e_k \sim \bigcup_{2\leq s\leq m-1} e_s \sim e_j.$$

Conversely, assume that $\mathcal{Z}^{-++}$ is Hamiltonian connected and $\overline{\mathcal{Z}}$, $L(\mathcal{Z})$ are not Hamiltonian connected. Let (v_i, v_j) be any two vertices of $\overline{\mathcal{Z}}$ such that v_i, v_j are not traceable. Since $\overline{\mathcal{Z}}$ is a subgraph of $\mathcal{Z}^{-++}$, therefore (v_i, v_j) are not traceable in $\mathcal{Z}^{-++}$ also, which is a contradiction to our assumption that $\mathcal{Z}^{-++}$ is Hamiltonian connected. □

By Theorem 4 it is clear that $\mathcal{Z}^{-++}$is non-Hamiltonian laceable therefore we focus random Hamiltonian laceable property of $\mathcal{Z}^{-++}$in the next few results:

Proposition 19. *Let $\mathcal{Z} = P_n, n \geq 3$ then ${P_n}^{-++}$ is random Hamiltonian laceable for $1 \leq t \leq diam({P_n}^{-++})$.*

Proof. we prove the above proposition by considering the following cases
Case (1) : Let $(v_i, v_j) \in V({P_n}^{-++})$ by Theorem 15 $diam({P_n}^{-++}) \leq 3$. We need to show that there exists a Hamiltonian Path between (v_i, v_j) at a distance 1 and 3. Further it can be easily verify that $d(v_i, v_j)_{\overline{\mathcal{Z}}} \leq 2$. Because if $d_{\mathcal{Z}}(u, v) \geq 3$ then $d_{\overline{\mathcal{Z}}}(u, v) \leq 2$. Therefore we need to show that for every pairs of adjacent vertices $(v_i, v_j) \in V({P_n}^{-++})$ there exists a Hamiltonian path,
$P_{H(v_i,v_j)} = (v_i \sim v_{i+2} \sim v_{i+4} \sim, \ldots., v_{i+n-1}) \sim (v_{i+1} \sim v_{i+3} \sim v_{i+5} \sim, \ldots, v_{i+n})$
$\sim (e_m \sim e_{m-1} \sim, \ldots, \sim e_1 \sim v_j)$
Case (2) : Let $(e_i, e_j) \in V({P_n}^{-++})$
Subcase (1) : $d(e_i, e_j) = 1 \in V(\mathcal{Z}^{-++})$since $L(P_n) = P_{n-1}$ therefore we need to show that for every pair of adjacent vertices in P_{n-1} there exists a Hamiltonian path,
$P_{H(v_i,v_j)} = e_i \sim (v_i \sim v_{i+2} \sim v_{i+4} \sim, \ldots., v_{i+(n-1)}) \sim (v_{i+1} \sim v_{I+3} \sim v_{i+5} \sim, \ldots, v_{i+n}) \sim (e_m \sim e_{m-1} \sim, \ldots, \sim e_j)$
Subcase (2) : $d(e_i, e_j) = 3 \in V(\mathcal{Z}^{-++})$. Then $P_{H(e_i,e_j)} = e_i \sim (v_i \sim v_{i+2} \sim v_{i+4} \sim , \ldots., v_{i+(n-1)}) \sim (v_{I+3} \sim v_{i+5} \sim, \ldots, v_{i+n}) \sim v_{i+1} \sim (e_{i+1} \sim e_{i+2} \sim, \ldots, \sim e_j)$
Case (3) : $d(v_i, e_j) = 1 \in \mathcal{Z}^{-++}$, since maximum distance between $(v_i, e_j) \in \mathcal{Z}^{-++}$is 2 Therefore we consider every adjacent pairs:
$P_{H(v_i,e_j)} = (v_i \sim v_{i+2} \sim v_{i+4} \sim, \ldots., v_{i+(n-1)}) \sim (v_{i+1} \sim v_{I+3} \sim v_{i+5} \sim, \ldots, v_{i+n})$
$\sim (e_m \sim e_{m-1} \sim, \ldots, \sim e_j)$. □

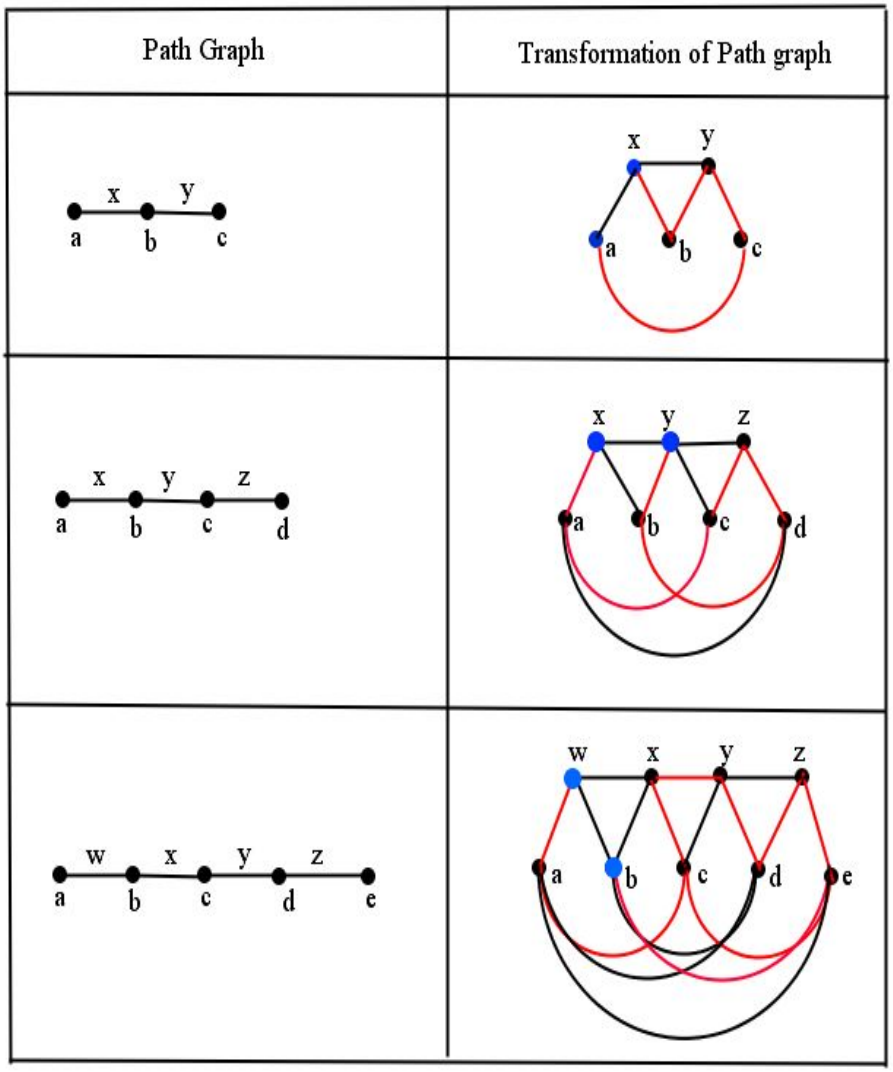


Figure 2. Total transformation graphs of P_n.

Proposition 20. *Let $\mathcal{Z} = C_n; n \geq 3$, then C_n^{-++} is random Hamiltonian laceable for $1 \leq t \leq diam(C_n^{-++})$.*

Proof. Let $\mathcal{Z} = C_n$ be a cycle with $n \geq 3$. Let $V(C_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(C_n) = \{e_1, e_2, e_3, \cdots, e_m\}$. Clearly by Theorem 14, C_n^{-++} is connected and by Theorem 15, $diam(C_n^{-++}) \leq 3$. Also from Theorem 16, C_n^{-++} Hamiltonian. Since $\overline{\mathcal{Z}}$ and $L(\mathcal{Z})$ is a subgraphs of $\mathcal{Z}^{-++}$. Therefore $diam(C_n^{-++}) \leq 2$ due to the fact that if $diam(\mathcal{Z}) \geq 3$ then $diam(\overline{\mathcal{Z}}) \leq 2$ and $L(C_n) \cong C_n$. Hence we need to show that for every pair of adjacent vertices in C_n^{-++} there exists a Hamiltonian path. To prove this, consider the following cases:

Case 1: Suppose $v_i, v_j \in V(C_n) \setminus V(C_n^{-++})$. Let v_i and v_j be any two adjacent vertices in C_n, then by the definition of C_n^{-++} they will be nonadjacent vertices in C_n^{-++} and vice-versa. Therefore assume that v_i and v_j are non-adjacent vertices in C_n. Then the

following Hamiltonian path exists between v_i and v_j:
$P_H(v_i, v_j) : v_i \sim e_i \sim v_{i+1} \sim e_{i+1} \sim \cup_{2\leq k\leq j} e_{i+k}v_{i+k} \sim e_m \sim v_n \sim v_j$.

Case 2: Suppose $v_i \in V(C_n) \setminus V(C_n^{-++})$ and $e_i \in E(C_n) \setminus V(C_n^{-++})$. Let $v_i \in V(C_n)$ and $e_i \in E(C_n)$ be any two adjacent vertices in C_n^{-++}. Then the following Hamiltonian path exists between v_i and e_i:
$P_H(v_i, v_i) : v_i \sim v_n \sim e_m \sim v_{n-1} \sim e_{m-1} \sim \cup_{1\leq i\leq n-1} v_{n-i}e_{m-i} \sim e_i$.

Case 3: Suppose $e_i, e_j \in E(C_n) \setminus V(C_n^{-++})$. Let e_i and e_j be any two adjacent edges in C_n. Since $L(C_n)$ is a subgraph of C_n^{-++} therefore by the definition of C_n^{-++} e_i and e_j will be adjacent vertices in C_n^{-++}. Then the following Hamiltonian path exists between e_i and e_j:
$P_H(e_i, e_j) : e_i \sim v_i \sim e_{i+m-1} \sim v_{i+n-1} \sim v_{i+n-3} \sim e_{i+m-3} \sim e_{i+m-2} \sim v_{i+m-2} \sim \cup_{4\leq i\leq k} e_{i+m-k}v_{i+m-k} \sim e_j$.

Hence, C_n^{-++} contains Hamiltonian path between every pair of adjacent vertices. Therefore, C_n^{-++} random Hamiltonian laceable. □

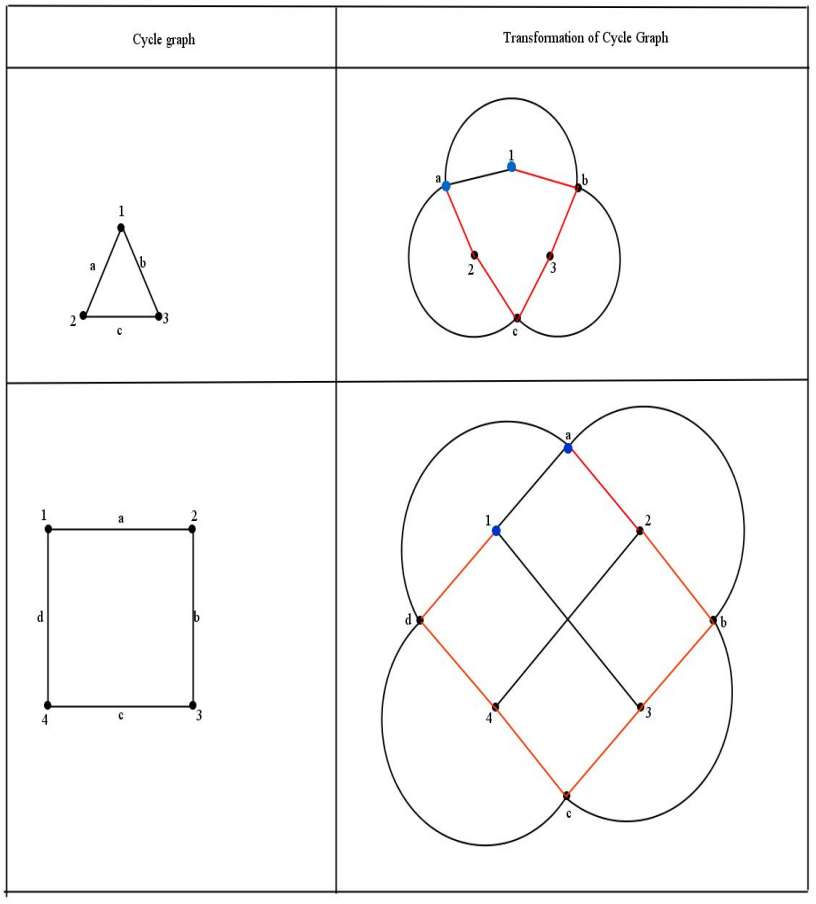


Figure 3. Total transformation graphs of C_n.

Proposition 21. *Let $\mathcal{Z} = W_n; n \geq 4$, then then W_n^{-++} is random Hamiltonian laceable for $1 \leq t \leq diam(W_n^{-++})$.*

Proof. Let $\mathcal{Z} = W_n$ be a cycle with $n \geq 4$. Since $W_n = C_{n-1} + K_1$ therefore $L(C_{n-1})$ and K_{n-1} are subgraphs of W_n^{-++}. By Theorem 14, W_n^{-++} is connected and by Theorem 15,

$diam(W_n^{-++}) \leq 3$. Also from Theorem 16, W_n^{-++} Hamiltonian. Since $\overline{\mathcal{Z}}$ and $L(\mathcal{Z})$ is a subgraphs of $\mathcal{Z}^{-++}$. Therefore $diam(W_n^{-++}) \leq 2$ due to the fact that if $diam(\mathcal{Z}) \geq 3$ then $diam(\overline{\mathcal{Z}}) \leq 2$ and $L(C_n) \cong C_n$. Hence we need to show that for every pair of adjacent vertices in W_n^{-++} there exists a Hamiltonian path. To prove this, consider the following cases:

Case 1: Suppose $v_i, v_j \in V(W_n) \setminus V(W_n^{-++})$. Let v_i and v_j are any two non-adjacent vertices in W_n. Then we need to show that there exists a Hamiltonian path between these two adjacent in W_n^{-++}. Then the following Hamiltonian path exists between v_i and v_j:
$P_H(v_i, v_j) : v_i \sim e_i \sim e_{i+1} \sim v_{i+1} \sim \cup_{2 \leq k \leq j} e_{i+k} v_{i+k} \sim e_j \sim v_j$.

Case 2: Suppose $v_i \in V(W_n) \setminus V(W_n^{-++})$ and $e_i \in E(W_n) \setminus V(W_n^{-++})$. Let $v_i \in V(W_n)$ and $e_i \in E(W_n)$ be any two adjacent vertices in W_n^{-++}. Then the following Hamiltonian path exists between v_i and e_i:
$P_H(v_i, v_i) : v_i \sim e_{i+1} \sim v_{i+1} \sim e_{i+2} \sim \cup_{3 \leq i \leq n-1} v_{i+k} e_{i+k} \sim v_{n-i+k} \sim e_{m-i+k} \sim e_i$.

Case 3: Suppose $e_i, e_j \in E(W_n) \setminus V(W_n^{-++})$. Let e_i and e_j be any two adjacent edges in W_n^{-++}. Then the following Hamiltonian path exists between e_i and e_j:
$P_H(e_i, e_j) : e_i \sim v_i \sim e_{i+2} \sim v_{i+1} \sim \cup_{3 \leq i \leq k} e_{i+m-k} v_{i+n-k-1} \sim e_{m-k-i} \sim e_j$.

Hence, W_n^{-++} contains Hamiltonian path between every pair of adjacent vertices. Therefore, W_n^{-++} random Hamiltonian laceable. □

Proposition 22. *Let $\mathcal{Z} = K_{r,s}; 2 \leq r \leq s$, then $K_{r,s}^{-++}$ is Hamiltonian connected.*

Proof. Let $\mathcal{Z} = K_{r,s}; 2 \leq r \leq s$ with $V(K_{r,s}) = \{v_1, v_2, v_3, \cdots, v_r, u_1, u_2, \cdots, u_s\}$, $E(K_{r,s}) = \{e_1, e_2, e_3, \cdots, e_{rs}\}$ and $H = K_{r,s}^{-++}$. Let $P_H(u, v)$ be a Hamiltonian path between vertices u and v in H. Let $V(H) = V_r \cup V_s \cup E_{rs}$, where $V_r = \{v_1, v_2, v_3, \cdots, v_r\}$, $V_s = \{u_1, u_2, \cdots, u_s\}$ and $E_{rs} = \{e_1, e_2, e_3, \cdots, e_{rs}\}$.

Case 1: Let $u = v_i$ and $v = v_j$, $1 \leq i \leq r$, $1 \leq j \leq r$ where $v_i, v_j \in V_r$. Then

$$\begin{aligned} P_H(u, v) &= v_i \sim \{e_i : 1 \leq i \leq s\} \sim v_{i+1} \sim \{e_{i+1} : 1 \leq i+1 \leq s\} \sim v_{i+2} \\ &\sim \{e_{i+2} : 1 \leq i+2 \leq s \sim v_{j-1} \sim \{e_{j-1} : 1 \leq j-1 \leq s\} \sim \{e_j : 1 \leq j \leq s\} \sim v_j \end{aligned}$$

Case 2: Let $u = v_i$ and $v = v_j$, $1 \leq i \leq r$, $1 \leq j \leq s$ where $v_i \in V_r$ and $v_j \in V_s$. Then

$$P_H(u, v) = v_i \sim \{v_j : 2 \leq j \leq r\} \sim e_r \sim \{e_i : r \leq i \leq rs\} \sim u_s \sim \{u_{s-i} : s+1 \leq i \leq 1\} = v_j$$

Case 3: Let $u = v_i$ and $v = e_j$, $1 \leq i \leq r$, $1 \leq j \leq rs$ where $v_i \in V_r$ and $e_j \in E_{rs}$.

Subcase 1: $u = v_i$, $1 \leq i \leq r$ and $v = e_i$, $1 \leq i \leq rs$

$$\begin{aligned} P_H(u, v) &= v_i \sim \{v_j : 2 \leq j \leq r\} \sim e_r \sim \{e_i : r \leq i \leq rs - 1\} \\ &\sim u_s \sim \{u_{s-i} : s+1 \leq i \leq 1\} \sim e_i \end{aligned}$$

Subcase 2: Let $u = v_i$ and $v = e_j$, $2 \leq j \leq r$, $1 \leq j \leq rs$ where $v_i \in V_r$ and $e_j \in E_{rs}$.

$$\begin{aligned} P_H(u, v) &= v_i \sim \{v_j : 2 \leq j \leq r\} \sim e_r \sim \{e_i : r \leq i \leq rs - 1\} \\ &\sim u_s \sim \{u_{s-i} : s+1 \leq i \leq 1\} \sim e_j \end{aligned}$$

Case 4: Let $u = e_i$ and $v = e_j$, $1 \leq i \leq rs$, $1 \leq j \leq rs$ where $e_i, e_j \in E_{rs}$. Then

$$P_H(u, v) = e_i \sim v_i \sim \{v_j : 2 \leq j \leq r\} \sim e_r \sim \{u_{r-1} : r-1 \leq i \leq s\} \sim \{e_{s+1} : s+1 \leq i\} \sim e_j$$

Case 5: Let $u = e_i$ and $v = u_j$, $1 \leq i \leq rs$, $1 \leq j \leq s$ where $e_i \in E_{rs}$ and $u_j \in V_s$. Then

$$\begin{aligned} P_H(u, v) &= e_i \sim v_i \sim \{v_j : 2 \leq j \leq r\} \sim e_{r+2} \sim \{e_j : r+2 \leq j \leq rs - 1\} \sim u_{s-1} \\ &\sim \{u_j : s-1 \leq j \leq s\} = u_j \end{aligned}$$

Case 6: Let $u = u_i$ and $v = u_j$, $1 \leq i \leq s$, $1 \leq j \leq s$ where $u_i, u_j \in V_s$. Then

$$\begin{aligned} P_H(u, v) &= u_i \sim e_1 \sim v_1 \sim \{v_i : 2 \leq i \leq r\} \sim e_{r+2} \sim \{e_j : r+2 \leq j \leq rs - 1\} \sim u_{s-1} \\ &\sim \{u_j : s-1 \leq j \leq s\} = u_j \end{aligned}$$

Thus, $K_{r,s}^{-++}$ is Hamiltonian connected. □

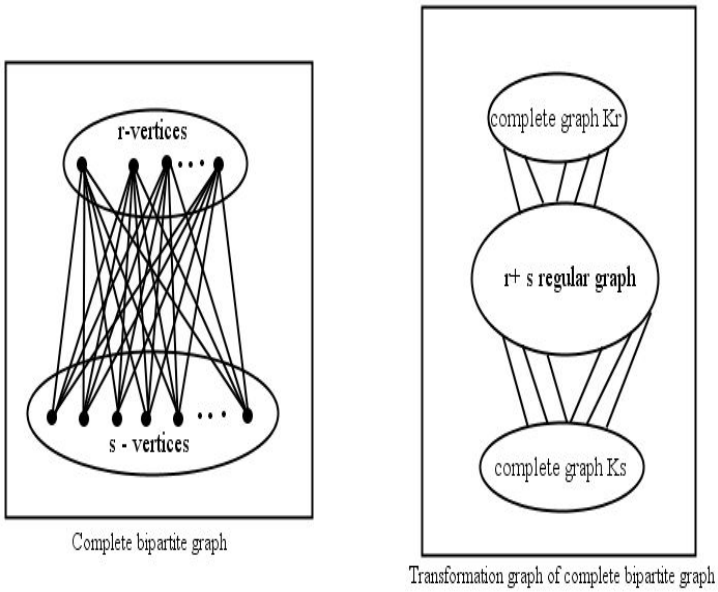


Figure 4. Total transformation graphs of $K_{r,s}$

Proposition 23. *Let $\mathcal{Z} = K_{1,n-1}; n \geq 4$, then $K_{1,n-1}^{-++}$ is Hamiltonian connected.*

Proof. Let $\mathcal{Z} = K_{1,n-1}; n \geq 4$ with $V(K_{1,n-1}) = \{v_1, v_2, v_3, \cdots, v_n\}$ where v_1 is the central vertex of a star with $deg(v_1) = n-1$, $E(K_{1,n-1}) = \{e_1, e_2, e_3, \cdots, e_m\}$ and $H = K_{1,n-1}^{-++}$. Consider the following cases.

Case 1: Let $u = v_1$ and $v = v_j$, $2 \leq j \leq n$. Then

$$P_H(u,v) \quad = \quad v_1 \sim \{e_i : 1 \leq i \leq m\} \sim \{v_j : 1 \leq j \leq n-1\}$$

Case 2: Let $u = v_i$ and $v = v_j$, $2 \leq i \leq n$, $2 \leq j \leq n$. Then

$$P_H(u,v) \quad = \quad v_i \sim e_i \sim v_1 \sim \{e_i : 2 \leq i \leq m\} \sim \{v_j : 2 \leq j \leq n-1\}$$

Case 3: Let $u = v_i$ and $v = e_j$, $2 \leq i \leq n$, $1 \leq j \leq m$. Then

$$P_H(u,v) \quad = \quad \{v_i : 2 \leq i \leq n\} \sim e_m \sim v_1 \sim \{e_j : 2 \leq i \leq m\}$$

Case 4: Let $u = e_i$ and $v = e_j$, $1 \leq i \leq m$, $1 \leq j \leq m$. Then

$$P_H(u,v) \quad = \quad e_i \sim \{v_i : 2 \leq i \leq n\} \sim e_m \sim v_1 \sim \{e_j : 2 \leq i \leq m\}$$

Thus, $K_{1,n-1}^{-++}$ is Hamiltonian connected. □

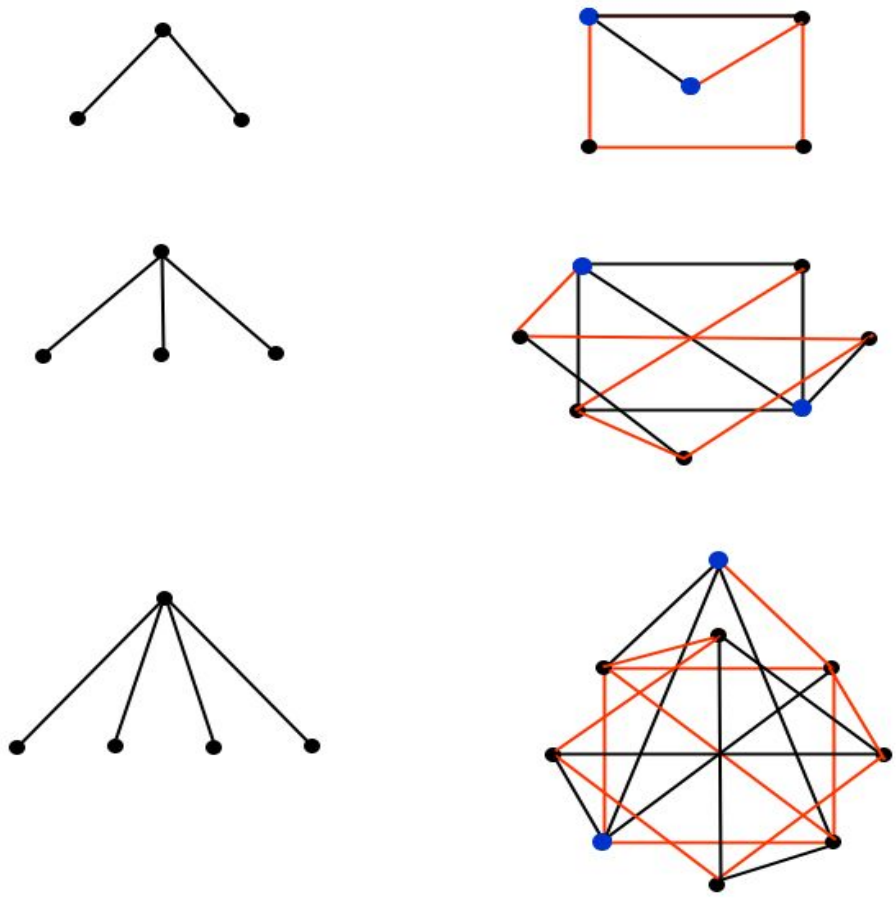


Figure 5. Total transformation graphs of $K_{1,n-1}$.

Conclusion

In this work we have proved the Hamiltonian connectedness of total transformation graphs $\mathcal{Z}^{+++}$, $\mathcal{Z}^{---}$, $\mathcal{Z}^{++-}$, $\mathcal{Z}^{+--}$ and $\mathcal{Z}^{-+-}$ and also shown that the graph $\mathcal{Z}^{-++}$ is random-Hamiltonian laceable for some standard class of graphs.

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