Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 4, pp. 701 – 718

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.4.007

ARITHMETIC PROPERTIES OF OPEND PARTITIONS

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Abstract. In 2023, Ballantine and Welch have considered various generalizations of POD and PED, wherein odd parts distinct (in case of POD) and even parts distinct (in case of PED), respectively. In the process, they were led to two new classes of POND and PEND, partitions which are integer partitions in which odd parts can not be distinct (in case of POND) and even parts can not be distinct (in case of PEND), respectively. In this paper, we obtain some infinite families of congruences for a new class of patrition function $OPEND_{j,k}(n)$, a positive integer partition of n wherein even parts not distinct (odd parts unrestricted) and first occurrence of a part congruent to j modulo k may be overlined.

2010 Mathematics Subject Classification. 11P83, 05A17.

Keywords and phrases. Partition identities, Theta–functions, Partition congruences, Regular partition.

1. Introduction

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$f_k := (q^k; q^k)_\infty.$$

Following Ramanujan, we define

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1 \tag{1}$$

and

$$f_1^3 = (q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}, \tag{2}$$

which are special cases of Ramanujan's general theta function [1, 3]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ \ |ab| < 1. \tag{3}$$

In Ramanujan's notation, Jacobi's famous triple product identity becomes,

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \tag{4}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $P(n)$ denotes the number of partitions of a positive integer n with $P(0) = 1$. The generating function for $P(n)$ is given by

$$\sum_{n=0}^{\infty} P(n)q^n = \frac{1}{(q,q)_\infty}.$$

In 2004, Corteel and Lovejoy [5] introduced the notion of overpartitions, which are partition of a positive integer n in which first occurrence of each part may be overlined. The generating function is given by

$$\sum_{n=0}^{\infty} \overline{P}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{5}$$

For example, there are 8 partitions for $\overline{P}(3)$, namely

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1.$$

In 2022, Mahadeva Naika, Harishkumar and Veeranayaka [9, 10] introduced notion of $[j,k]$-overpartition of a positive integer n is a non-increasing sequence of positive integers whose sum is n in which first occurrence of each part congruent to j modulo k may be overlined. Let $\overline{P}_{j,k}(n)$ denote the number of such partitions of a positive integer n with $\overline{P}_{j,k}(0) = 1$. The generating function is given by

$$\sum_{n=0}^{\infty} \overline{P}_{j,k}(n)q^n = \frac{(-q^j;q^k)_\infty}{(q,q)_\infty}. \tag{6}$$

For example, there are 5 partitions for $\overline{P}_{1,3}(3)$, namely

$$3, \qquad 2+1, \qquad 2+\overline{1}, \qquad 1+1+1, \quad \overline{1}+1+1.$$

In the following paper, Ballantine and Welch [2] considered various generalizations and refinements of POD and PED partitions. In the process they were led to two new classes of POND and PEND partitions which are integer partitions with odd parts can not be distinct (in case of POND) and even parts can not be distinct (in case of PEND) respectively.

In 2025, M. Prasad, Kaneez Raheela and M. S. Mahadeva Naika [11], have introduced and obtained some infinite families of congruences for a new class of partition function called $[j,k]-$ pond overpartitions or $OPOND$ partitions, which enumerate the number of partitions of a positive integer n in which the first occurrence of each part congruent to j modulo k may be overlined and odd parts are not distinct (even parts are unrestricted). Let $opond_{j,k}(n)$ denote the number of such partitions of a positive integer n and the generating function be given by

$$\sum_{n=0}^{\infty} opond_{j,k}(n)q^n = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}}(-q^j;q^k)_\infty. \tag{7}$$

For example. There are 10 partitions for $opond_{3,3}(6)$, namely

$$6, \quad \overline{6}, \quad 4+2, \quad 4+1+1, \quad 3+3, \quad \overline{3}+3, \quad 2+2+2, \quad 2+2+1+1, \quad 2+1+1+1+1,$$
$$1+1+1+1+1+1.$$

In the following papers [12, 13], Prasad, Kaneez Raheela and Mahadeva Naika, have introduced and obtained several infinite families of congruences for Singular POND and PEND overpartitions.

Motivated by the above works, in this paper we introduce and obtain some infinite families of congruences for a new class of partition function $OPEND(n)$ which enumerates the number of partition of a positive integer n in which the first occurrence of each part congruent to j modulo k may be overlined and even parts are not distinct (odd parts are unrestricted). Let $opend_{j,k}(n)$ denote the number of such partition of a positive integer n and the generating function be given by

$$\sum_{n=0}^{\infty} opend_{j,k}(n)q^n = \frac{f_2 f_{12}}{f_1 f_4 f_6}(-q^j;q^k)_\infty. \tag{8}$$

2. Preliminary results

Lemma 2.1. *The following* 2*-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{9}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{10}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{11}$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q\frac{f_4^{16}}{f_2^{24}} + 16q^2\frac{f_4^4 f_8^8}{f_2^{20}}, \tag{12}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{13}$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \tag{14}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^5 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{15}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{24}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{16}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{24}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}, \tag{17}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \tag{18}$$

and

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{19}$$

Proof. The equation (9) can be found in [3, Entry 25, p.40]. Replacing q by $-q$ in the equation (9), we obtain (10). The equation (11) can be found in [3, Entry 25, p.40]. From the equation (11), we obtain (12). The equation (13) is the same as (22.1.14) in [6]. Replacing q by $-q$ in the equation (13), we obtain (14). The equation (15) can be found in [6]. The equation (16) can be found in [14]. Replacing q by $-q$ in the equation (16), we obtain (17). The equation (18) can be found in [14]. The equation (19) can be found in [4]. □

Lemma 2.2. *The following* 3*-dissections hold:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^2} - 3q f_9^3, \tag{20}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}, \tag{21}$$

$$\frac{f_2}{f_1 f_4} = \frac{f_{18}^9}{f_3^2 f_9^3 f_{12}^2 f_{36}^3} + q \frac{f_6^2 f_{18}^3}{f_3^3 f_{12}^3} + q^2 \frac{f_6^4 f_9^3 f_{36}^3}{f_3^4 f_{12}^4 f_{18}^3}, \tag{22}$$

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \tag{23}$$

and

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q \frac{f_3 f_{18}^2}{f_6 f_9}. \tag{24}$$

Proof. The equation (20) is the same as (14.8.5) in [6]. Proofs of the equations (21)-(24) can be found in [8]. □

Lemma 2.3. [3, Ch. 19, Entry (v), p.270] *The following 5-dissection holds:*

$$f_1 = f_{25}\left(a - q - q^2/a\right), \tag{25}$$

where

$$a := a(q^5) = \frac{f(-q^{10}, -q^{15})}{f(-q^5, q^{20})}.$$

Lemma 2.4. [3, Ch.19, Entry 17 (v), p.303] *The following 7-dissection holds:*

$$f_1 = f_{49}\left(\frac{B}{C} - q\frac{A}{B} - q^2 + q^5\frac{C}{A}\right), \tag{26}$$

where $A := A(q^7) = f(-q^{21}, -q^{28})$, $\quad B := B(q^7) = f(-q^{14}, q^{35})$ *and* $\quad C := C(q^7) = f(-q^7, -q^{42})$.

3. Generating function for $OPEND_{j,k}(n)$

In this section, we provide proof for a generating function of a partition $OPEND_{j,k}(n)$.

Theorem 3.1. *For* $1 \leq j \leq k$*, we have*

$$\sum_{n=0}^{\infty} opend_{j,k}(n) q^n = \frac{f_2 f_{12}}{f_1 f_4 f_6} (-q^j; q^k)_\infty. \tag{27}$$

Proof. By the definition,

$$\begin{aligned}\sum_{n=0}^{\infty} opend_{j,k}(n)q^n &= \frac{(-q^j;q^k)_\infty}{(q;q^2)_\infty}\prod_{i=1}^{\infty}\left(\frac{1}{1-q^{2i}}-q^{2i}\right)\\ &= \frac{(-q^j;q^k)_\infty}{(q;q^2)_\infty}\prod_{i=1}^{\infty}\left(\frac{1-q^{2i}+(q^{2i})^2}{(1-q^{2i})}\right)\\ &= \frac{(-q^j;q^k)_\infty}{(q;q^2)_\infty}\prod_{i=1}^{\infty}\left(\frac{(1+q^{6i})}{(1-q^{2i})(1+q^{2i})}\right)\\ &= \frac{(-q^j;q^k)_\infty}{(q;q^2)_\infty}\frac{(-q^6;q^6)_\infty}{(q^4;q^4)_\infty}\\ &= \frac{f_2 f_{12}}{f_1 f_4 f_6}(-q^j;q^k)_\infty.\end{aligned}$$

□

4. Congruences for $OPEND_{3,3}(n)$

Theorem 4.1. *For $n \geq 0$ and $\alpha, \beta \geq 0$, we have for* (mod 8)

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 5^{2\alpha}n+\frac{17\cdot 5^{2\alpha}-1}{4}\right)q^n \equiv 4f_8 f_3^3, \tag{28}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 5^{2\alpha+1}n+\frac{13\cdot 5^{2\alpha+1}-1}{4}\right)q^n \equiv 4q^3 f_{40} f_{15}^3, \tag{29}$$

$$opend_{3,3}\left(6\cdot 5^{2\alpha+2}n+\frac{c_1\cdot 5^{2\alpha+1}-1}{4}\right) \equiv 0, \tag{30}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(18\cdot 5^{2\alpha}n+\frac{57\cdot 5^{2\alpha}-1}{4}\right)q^n \equiv 4f_1 f_6^3, \tag{31}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(18\cdot 5^{2\alpha+1}n+\frac{69\cdot 5^{2\alpha+1}-1}{4}\right)q^n \equiv 4q^3 f_5 f_{30}^3, \tag{32}$$

$$opend_{3,3}\left(18\cdot 5^{2\alpha+2}n+\frac{c_2\cdot 5^{2\alpha+1}-1}{4}\right) \equiv 0, \tag{33}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+1}n+\frac{9^{4\alpha+1}-1}{4}\right)q^n \equiv \frac{f_2^2 f_3^4}{f_1 f_6^2}+4q f_1^9 f_6^3, \tag{34}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+1}\cdot 5^{2\beta}n+\frac{17\cdot 9^{4\alpha+1}\cdot 5^{2\beta}-1}{4}\right)q^n \equiv 4f_8 f_3^3, \tag{35}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+1}\cdot 5^{2\beta+1}n+\frac{13\cdot 9^{4\alpha+1}\cdot 5^{2\beta+1}-1}{4}\right)q^n\equiv 4q^3f_{40}f_{15}^3, \tag{36}$$

$$opend_{3,3}\left(6\cdot 9^{4\alpha+1}\cdot 5^{2\beta+2}n+\frac{c_1\cdot 9^{4\alpha+1}\cdot 5^{2\beta+1}-1}{4}\right)\equiv 0, \tag{37}$$

$$opend_{3,3}\left(2\cdot 9^{4\alpha+2}n+\frac{57\cdot 9^{4\alpha+1}}{4}\right)\equiv 0, \tag{38}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+2}\cdot 5^{2\beta}n+\frac{17\cdot 9^{4\alpha+2}\cdot 5^{2\beta}-1}{4}\right)q^n\equiv 4f_8f_3^3, \tag{39}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+2}\cdot 5^{2\beta+1}n+\frac{13\cdot 9^{4\alpha+2}\cdot 5^{2\beta+1}-1}{4}\right)q^n\equiv 4q^3f_{40}f_{15}^3, \tag{40}$$

$$opend_{3,3}\left(6\cdot 9^{4\alpha+2}\cdot 5^{2\beta+2}n+\frac{c_3\cdot 9^{4\alpha+2}\cdot 5^{2\beta+1}-1}{4}\right)\equiv 0, \tag{41}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+3}\cdot 5^{2\beta}n+\frac{57\cdot 9^{4\alpha+2}\cdot 5^{2\beta}-1}{4}\right)q^n\equiv 4f_1f_6^3, \tag{42}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+3}\cdot 5^{2\beta+1}n+\frac{69\cdot 9^{4\alpha+2}\cdot 5^{2\beta+1}-1}{4}\right)q^n\equiv 4q^3f_5f_{30}^3, \tag{43}$$

$$opend_{3,3}\left(2\cdot 9^{4\alpha+3}\cdot 5^{2\beta+2}n+\frac{c_2\cdot 9^{4\alpha+2}\cdot 5^{2\beta+1}-1}{4}\right)\equiv 0, \tag{44}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+4}\cdot 5^{2\beta}n+\frac{17\cdot 9^{4\alpha+5}\cdot 5^{2\beta}-1}{4}\right)q^n\equiv 4f_8f_3^3, \tag{45}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+4}\cdot 5^{2\beta+1}n+\frac{13\cdot 9^{4\alpha+5}\cdot 5^{2\beta+1}-1}{4}\right)q^n\equiv 4q^3f_{40}f_{15}^3, \tag{46}$$

$$opend_{3,3}\left(6\cdot 9^{4\alpha+4}\cdot 5^{2\beta+2}n+\frac{c_1\cdot 9^{4\alpha+5}\cdot 5^{2\beta+1}-1}{4}\right)\equiv 0, \tag{47}$$

where $c_1\in\{13,37,61,109\}$, $c_2\in\{69,141,213,357\}$, $c_3\in\{117,333,549,981\}$.

Proof. Putting $j=3$ and $k=3$ in (27), we find that

$$\sum_{n=0}^{\infty} opend_{3,3}(n)q^n=\frac{f_2f_{12}}{f_1f_4f_3}. \tag{48}$$

Using (22) in (48), we find that

$$\sum_{n=0}^{\infty} opend_{3,3}(3n)q^n = \frac{f_6^9}{f_1^3 f_3^3 f_4 f_{12}}, \tag{49}$$

$$\sum_{n=0}^{\infty} opend_{3,3}(3n+1)q^n = \frac{f_2^2 f_6^3}{f_1^4 f_4^2} \tag{50}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}(3n+2)q^n = \frac{f_2^4 f_3^3 f_{12}^3}{f_1^5 f_4^3 f_6^3}. \tag{51}$$

Using (11) in (50), we get

$$\begin{aligned} \sum_{n=0}^{\infty} opend_{3,3}(6n+4)q^n &\equiv 4\frac{f_3^3 f_4^4}{f_1^8} \\ &\equiv 4f_8 f_3^3 \pmod 8, \end{aligned} \tag{52}$$

which implies that the result (28) is true for $\alpha = 0$. Suppose that the result (28) is true for some $\alpha \geq 0$. Employing (25) in (52) and then collecting the term involving q^{5n+2} on both side of the resulting equation, we arrive (29).
From the equation (29), we obtain (30) and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 5^{2\alpha+2}n + \frac{17\cdot 5^{2\alpha+2}-1}{4}\right) q^n \equiv 4f_8 f_3^3 \pmod 8, \tag{53}$$

which is $\alpha + 1$ case of (28). Hence, by mathematical induction the equation (28) is true for all $\alpha \geq 0$. Using (11) and (13) in (51), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} opend_{3,3}(6n+2)q^n &= \frac{f_2^{14} f_6^2}{f_1^{12} f_3 f_4^4} + 4q\frac{f_4^4 f_6^6}{f_1^6 f_2^2 f_3^3} \\ &\equiv \frac{f_2^2 f_6^2}{f_1^4 f_3} + 4q f_2^3 f_3^9 \pmod 8 \end{aligned} \tag{54}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} opend_{3,3}(6n+5)q^n &= \frac{f_2^{10} f_6^6}{f_1^{10} f_3^3 f_4^4} + 4\frac{f_2^2 f_4^4 f_6^2}{f_1^8 f_3} \\ &\equiv \frac{f_3^5 f_6^2}{f_1^2 f_2^2} + 4f_4^3 f_3^3 \pmod 8. \end{aligned} \tag{55}$$

Employing (20) and (21) in (54), we get

$$\sum_{n=0}^{\infty} opend_{3,3}(18n+2)q^n \equiv \frac{f_2^2 f_3^4}{f_1 f_6^2} + 4q f_1^9 f_6^3 \pmod 8, \tag{56}$$

$$\sum_{n=0}^{\infty} opend_{3,3}(18n+8)q^n \equiv 4f_2 f_3^3 + 4f_2 f_1^9 \pmod 8 \tag{57}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}(18n+14)q^n \equiv 4f_1f_6^3 \pmod 8, \tag{58}$$

which is $\alpha = 0$ case of (31). Suppose that the result (31) is true for some $\alpha \geq 0$. Employing (25) in (31) and then collecting the term involving q^{5n+4} on both side of the resulting equation, we arrive (32).
From the equation (32), we obtain (33) and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(18\cdot 5^{2\alpha+2}n + \frac{57\cdot 5^{2\alpha+2}-1}{4}\right)q^n \equiv 4f_1f_6^3 \pmod 8, \tag{59}$$

which is $\alpha+1$ case of (31). Hence, by induction the equation (31) is true for all $\alpha \geq 0$.
The equation (56) is $\alpha = 0$ case of (34). Suppose that the equation (34) is true for some $\alpha \geq 0$. Using (20) and (23) in (34) and then collecting the terms involving q^{3n+1} and q^{3n+2} on both sides of the equation, we find that

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+1}n + \frac{9^{4\alpha+2}-1}{4}\right)q^n \equiv \frac{f_1^4f_6^2}{f_2^2f_3} + 4f_1^9 + 4qf_2^3f_3^9 \pmod 8 \tag{60}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+1}n + \frac{17\cdot 9^{4\alpha+1}-1}{4}\right)q^n \equiv 4f_8f_3^3 \pmod 8, \tag{61}$$

which is $\beta = 0$ case of (35). Since, the rest of the proofs of (35)-(37) are similar to the proofs of (28)-(30), so we omit the details.
Using (20) and (24) in (60) and collecting the terms involving q^{3n} and q^{3n+2}, we get for (mod 8)

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+2}n + \frac{9^{4\alpha+2}-1}{4}\right)q^n \equiv \frac{f_2^2f_3^4}{f_1f_6^2} + 4f_1^3 + 4qf_3^9 + 4qf_1^9f_6^3 \tag{62}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+2}n + \frac{17\cdot 9^{4\alpha+2}-1}{4}\right)q^n \equiv 4f_1^3f_8. \tag{63}$$

Using (20) and (24) in (62), we get for (mod 8)

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+2}n + \frac{9^{4\alpha+3}-1}{4}\right)q^n \equiv \frac{f_1^4f_6^2}{f_2^2f_3} + 4f_3^3 + 4f_2^3f_3^9 \tag{64}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+2}n + \frac{17\cdot 9^{4\alpha+2}-1}{4}\right)q^n \equiv 4f_3^3f_8, \tag{65}$$

which is $\beta = 0$ case of (39). The rest of the proofs of (39)-(41) are similar to the proofs of (28)-(30), so we omit the details.

Using (20) and (24) in (64) and then collecting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we get for $\pmod 8$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+3}n+\frac{9^{4\alpha+3}-1}{4}\right)q^n \equiv \frac{f_2^2f_3^4}{f_1f_6^2}+4f_1^3+4qf_1^9f_6^3, \tag{66}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+3}n+\frac{33\cdot 9^{4\alpha+2}-1}{4}\right)q^n \equiv 4f_2f_3^3+4f_1^{11} \tag{67}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+3}n+\frac{57\cdot 9^{4\alpha+2}-1}{4}\right)q^n \equiv 4f_1f_6^3. \tag{68}$$

Using (20) and (23) in (66) and then collecting the terms involving q^{3n+1} and q^{3n+2}, we get for $\pmod 8$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+3}n+\frac{9^{4\alpha+4}-1}{4}\right)q^n \equiv \frac{f_1^4f_6^2}{f_2^2f_3}+4f_3^3+4f_1^9+4qf_2^3f_3^9 \tag{69}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+3}n+\frac{17\cdot 9^{4\alpha+3}-1}{4}\right)q^n \equiv 4f_3^3f_8. \tag{70}$$

Using (20) and (23) in (66) and then collecting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we get for $\pmod 8$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+4}n+\frac{9^{4\alpha+4}-1}{4}\right)q^n \equiv \frac{f_2^2f_3^4}{f_1f_6^2}+4qf_3^9+4qf_1^9f_6^3, \tag{71}$$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+4}n+\frac{33\cdot 9^{4\alpha+3}-1}{4}\right)q^n \equiv 4f_1^{11} \tag{72}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+4}n+\frac{57\cdot 9^{4\alpha+3}-1}{4}\right)q^n \equiv 4f_1f_6^3. \tag{73}$$

Employing (20) and (24) in (71) and then collecting the terms involving q^{3n+1} and q^{3n+2}, we arrive for $\pmod 8$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+4}n+\frac{9^{4\alpha+5}-1}{4}\right)q^n \equiv \frac{f_1^4f_6^2}{f_2^2f_3}+4qf_2^3f_3^9 \tag{74}$$

and

$$\sum_{n=0}^{\infty} opend_{3,3}\left(6\cdot 9^{4\alpha+4}n+\frac{17\cdot 9^{4\alpha+5}-1}{4}\right)q^n \equiv 4f_3^3f_8. \tag{75}$$

□

Employing (20) and (23) in (74) and then collecting the term involving q^{3n}, we arrive for $\pmod 8$

$$\sum_{n=0}^{\infty} opend_{3,3}\left(2\cdot 9^{4\alpha+5}n+\frac{9^{4\alpha+5}-1}{4}\right)q^n \equiv \frac{f_2^2f_3^4}{f_1f_6^2}+4qf_1^9f_6^3, \tag{76}$$

which is $\alpha+1$ case of (34). By mathematical induction, the equation (34) is true for all $\alpha \geq 0$.

5. Congruences for $OPEND_{3,6}(n)$

Theorem 5.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} opend_{3,6}(192n+88)q^n \equiv 512f_1^{11} \pmod{1024}, \tag{77}$$

$$opend_{3,6}(384n+352) \equiv 0 \pmod{1024}, \tag{78}$$

$$opend_{3,6}(1536n+1024) \equiv 0 \pmod{1024}, \tag{79}$$

$$opend_{3,6}(1536n+1408) \equiv 0 \pmod{1024}, \tag{80}$$

$$opend_{3,6}(12288n+3072i+1024) \equiv 0 \pmod{512}, \tag{81}$$

$$opend_{3,6}(3072n+1792) \equiv 0 \pmod{256}, \tag{82}$$

$$opend_{3,6}(6144n+256) \equiv 0 \pmod{64}, \tag{83}$$

$$opend_{3,6}(6144n+3328) \equiv 0 \pmod{128}, \tag{84}$$

$$opend_{3,6}(768n+448) \equiv 0 \pmod{64}, \tag{85}$$

$$opend_{3,6}(1536n+832) \equiv 0 \pmod{64}, \tag{86}$$

where $i = 1, 2, 3$.

Proof. Putting $j = 3$ and $k = 6$ in (27), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(n)q^n = \frac{f_2f_6}{f_1f_4f_3}. \tag{87}$$

Using (22) in (87), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(3n)q^n = \frac{f_2f_6^9}{f_1^3f_3^3f_4^2f_{12}^3}, \tag{88}$$

$$\sum_{n=0}^{\infty} opend_{3,6}(3n+1)q^n = \frac{f_2^3f_6^3}{f_1^4f_4^3} \tag{89}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(3n+2)q^n = \frac{f_2^5f_3^3f_{12}^3}{f_1^5f_4^4f_6^3}. \tag{90}$$

Employing (11) in (89), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(6n+1)q^n = \frac{f_2^{11}f_3^3}{f_1^{11}f_4^4} \tag{91}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(6n+4)q^n = 4\frac{f_3^3f_4^4}{f_1^7f_2}. \tag{92}$$

Using (9), (11) and (13) in (92), we deduce that

$$\sum_{n=0}^{\infty} opend_{3,6}(12n+4)q^n = 4\frac{f_2^{21}f_3^2f_4}{f_1^{22}f_6f_8^2} + 8q\frac{f_2^{19}f_6^3f_8^2}{f_1^{20}f_4^5} + 16q\frac{f_2^5f_4^9f_6^3}{f_1^{16}f_8^2} + 32q\frac{f_2^{11}f_3^2f_4^3f_8^2}{f_1^{18}f_6} \tag{93}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(12n+10)q^n = 4\frac{f_2^{17}f_4f_6^3}{f_1^{20}f_8^2} + 8\frac{f_2^{23}f_3^2f_8^2}{f_1^{22}f_4^5f_6} + 16\frac{f_2^9f_3^2f_4^9}{f_1^{18}f_6f_8^2} + 32q\frac{f_2^7f_4^3f_6^3f_8^2}{f_1^{16}}. \tag{94}$$

Using (11), (12) and (18) in (94), we get

$$\sum_{n=0}^{\infty} opend_{3,6}(24n+10)q^n \equiv 4\frac{f_2f_3^3}{f_1} \pmod 8 \tag{95}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(24n+22)q^n \equiv 48\frac{f_2^{18}f_3f_{12}}{f_1^{19}f_4f_6} + 112\frac{f_2^{11}f_3^3f_4^2}{f_1^{17}} + 416\frac{f_2f_4^5f_6^2}{f_{12}} \pmod{512}. \tag{96}$$

Employing (12), (13) and (15) in (96), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} opend_{3,6}(48n+22)q^n &\equiv 160\frac{f_2^{157}f_3^2}{f_1^{111}f_4^{48}f_6} + 416\frac{f_2^{61}f_3^2}{f_1^{47}f_4^{16}f_6}\\ &\equiv 160\frac{f_2^{13}f_3^2}{f_1^{15}f_6} + 416\frac{f_2^{13}f_3^2}{f_1^{15}f_6}\\ &\equiv 64\frac{f_2^{13}f_3^2}{f_1^{15}f_6} \pmod{512}\\ &\equiv 64f_1f_4f_2^3 \pmod{128}. \end{aligned} \tag{97}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} opend_{3,6}(48n+46)q^n &\equiv 256\frac{f_2^3f_6^3}{f_1} + 512f_1^{23} + 512f_1^{23}\\ &\equiv 256\frac{f_2^3f_6^2}{f_1} \pmod{1024}\\ &\equiv 256f_1f_4f_6^3 \pmod{512}. \end{aligned} \tag{98}$$

Using (11), (12) and (18) in (93), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(24n+4)q^n \equiv 4\frac{f_2^{75}f_6^2}{f_1^{54}f_4^{23}f_{12}} + 128q\frac{f_1^2f_2^{11}f_4f_6^2}{f_{12}} + 224q\frac{f_2^{12}f_3f_4^3f_{12}}{f_1^{17}f_6} + 416q\frac{f_2^5f_3^3f_4^6}{f_1^{15}} \pmod{1024} \tag{99}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(24n+16)q^n \equiv 8\frac{f_2^{72}f_3f_{12}}{f_1^{53}f_4^{21}f_6} + 24\frac{f_2^{65}f_3^3}{f_1^{51}f_4^{18}} + 112\frac{f_4^{17}f_6^2}{f_1^{50}f_2f_{12}} + 256q\frac{f_2^{18}f_3f_{12}}{f_1f_4f_6} + 512qf_8^5 + 768q\frac{f_2^{17}f_3^3}{f_1^3} \pmod{1024}. \tag{100}$$

Using (9), (12), (13) and (15) in (100), we get

$$\sum_{n=0}^{\infty} opend_{3,6}(48n+16)q^n \equiv 32\frac{f_2^9f_3^2f_4^5}{f_1^{14}f_6f_8^2} + 96q\frac{f_2^7f_6^3f_8^2}{f_1^{12}f_4} + 512qf_8^4 + 112\frac{f_2^{57}f_3^2}{f_1^{46}f_4^{11}f_6f_8^2} + 256q\frac{f_4^2f_6f_8f_{24}}{f_2f_{12}} \pmod{1024} \tag{101}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(48n+40)q^n \equiv 48\frac{f_2^{53}f_6^3}{f_1^{44}f_4^{11}f_8^2} + 256\frac{f_4^3f_8f_{12}^2}{f_{24}} + 288\frac{f_2^{11}f_3^2f_8^2}{f_1^{14}f_4f_6} + 512q\frac{f_4^7f_6^3}{f_2} \pmod{1024}. \tag{102}$$

Using (11), (12) and (19) in (102), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(96n+40)q^n \equiv 16\frac{f_2f_3^3}{f_1} \pmod{32} \tag{103}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} opend_{3,6}(96n+88)q^n\\
&\equiv 64\frac{f_2^3f_3^3f_4^2}{f_1} + 384\frac{f_2^9f_4f_6^2}{f_{12}} + 512\frac{f_2^7f_3^3}{f_1} + 576\frac{f_2^{10}f_3f_{12}}{f_1^3f_4f_6}\\
&\equiv \left(64f_2^3f_4^2 + 512f_2^7\right)\left(\frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}\right) + 384\frac{f_2^9f_4f_6^2}{f_2}\\
&+576\frac{f_2^{10}f_{12}}{f_4f_6}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2} + 3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right)\\
&\equiv 640\frac{f_2f_4^5f_6^2}{f_{12}} + 384\frac{f_2^9f_4f_6^2}{f_{12}} + 768qf_2^3f_4f_{12}^3 + 512f_2^{11} + 512qf_2^3f_4f_{12}^3\\
&\equiv 512f_2^{11} + 256qf_2^3f_4f_{12}^3 \pmod{1024},
\end{aligned} \tag{104}$$

which implies

$$\sum_{n=0}^{\infty} opend_{3,6}(192n+88)q^n \equiv 512f_1^{11} \pmod{1024} \tag{105}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(192n+184)q^n \equiv 256f_1f_4f_6^3 \pmod{512}. \tag{106}$$

Using (11), (12) and (19) in (101), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} opend_{3,6}(96n+16)q^n &\equiv 32\frac{f_2^3f_4f_6^2}{f_1^6f_{12}} + 112\frac{f_2^{51}f_6^2}{f_1^{38}f_4^{15}f_{12}} + 128q\frac{f_2f_3^3f_4^6}{f_1^7}\\ &+ 256q\frac{f_2f_4^7f_6^2}{f_1^2f_{12}} + 384q\frac{f_2^{12}f_3f_{12}}{f_1f_4f_6} + 512qf_4^7 \pmod{1024}\end{aligned} \tag{107}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} opend_{3,6}(96n+64)q^n &\equiv 96\frac{f_2^{25}f_3^3}{f_1^3f_4^{10}} + 192\frac{f_2^7f_4f_6^2}{f_1^2f_{12}}\\ &+ 256\frac{f_2^2f_3f_4f_{12}}{f_1f_6} + 288\frac{f_3f_4^3f_{12}}{f_1^5f_6} + 512f_4^4 \pmod{1024}.\end{aligned} \tag{108}$$

Employing (9), (13) and (17) in (108), we get

$$\begin{aligned}&\sum_{n=0}^{\infty} opend_{3,6}(192n+64)q^n\\ &\equiv 96\frac{f_2^9f_3^2f_4^5}{f_1^{14}f_6f_8^2}128q\frac{f_2f_4^6f_6f_{24}}{f_1^4f_8f_{12}} + 192\frac{f_1^2f_3^2f_2f_4^5}{f_6f_8^2} + 192q\frac{f_1^4f_6^3f_8^2}{f_2f_4} + 256\frac{f_2^2f_8f_{12}^2}{f_4f_{24}}\\ &+288\frac{f_2^{18}f_8f_{12}^2}{f_1^{16}f_4^5f_{24}} + 512f_8 \pmod{1024}\end{aligned} \tag{109}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} opend_{3,6}(192n+160)q^n &\equiv 96\frac{f_2^5f_4^5f_6^3}{f_1^{12}f_8^2} + 128\frac{f_2^2f_4^3f_8f_{12}^2}{f_1^4f_{24}}\\ &+ 256\frac{f_2f_6f_4^2f_{24}}{f_8f_{12}} + 288\frac{f_2^{17}f_6f_{24}}{f_1^{16}f_4^2f_8f_{12}} + 576\frac{f_1^2f_3^2f_2^3f_8^2}{f_4f_6} \pmod{1024}.\end{aligned} \tag{110}$$

Using (11), (12), (18) and (19) in (110), we get

$$\begin{aligned}\sum_{n=0}^{\infty} opend_{3,6}(384n+160)q^n &\equiv 256\frac{f_1f_3f_2^2f_{12}}{f_4f_6} + 384\frac{f_1^5f_4^6}{f_2^5f_3^3}\\ &+ 384q\frac{f_1f_3f_4^2f_6^4}{f_2} + 768\frac{f_2^3f_4f_6^2}{f_{12}} + 960\frac{f_3^3f_4^2}{f_1^5f_2} \pmod{1024}\end{aligned} \tag{111}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} opend_{3,6}(384n+352)q^n\\
&\equiv 128\frac{f_2^3f_3^3f_4^2}{f_1}+896\frac{f_1^3f_2f_4^2f_6^2}{f_3}+512f_2^{11}+512\frac{f_2^7f_3^3}{f_1}\\
&\equiv 128f_2^3f_4^2\left(\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}\right)+896f_2f_4^2f_6^2\left(\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}\right)\\
&+512f_2^{11}+512f_2^7\left(\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}\right)\\
&\equiv 512f_2^{11}+512\frac{f_2^5f_4^3f_6^2}{f_{12}}+512q\frac{f_2^7f_{12}^3}{f_4}+512qf_2^3f_4f_{12}^3\\
&\equiv 512f_2^{11}+512f_2^{11}+512qf_2^5f_{12}^3+512qf_2^5f_{12}^3 \pmod{1024}\\
&\equiv 0 \pmod{1024}, \qquad (112)
\end{aligned}$$

which implies (78).
Using (11), (12) and (19) in (109), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} opend_{3,6}(384n+64)q^n\\
&\equiv 192\frac{f_1^3f_2f_4^2f_6^8}{f_3^3f_{12}^4}+192q\frac{f_2^{13}f_3f_{12}^4}{f_1f_4^6f_6^4}+256q\frac{f_3f_4^5f_{12}}{f_1f_6}+256q\frac{f_3^3f_4^6}{f_1^3f_2}+256\frac{f_2f_4f_6^2}{f_1^2f_{12}}+384\frac{f_1^2f_4f_6^2}{f_2f_{12}}\\
&+512q\frac{f_3^3f_4^5}{f_1}+512f_4+512qf_4^7 \pmod{1024} \qquad (113)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} opend_{3,6}(384n+256)q^n &\equiv 192\frac{f_2^9f_3^3}{f_1^3f_4^2}+320\frac{f_3f_4^3f_{12}}{f_1^5f_6}+640\frac{f_1f_4^2f_6^2}{f_2f_3}\\
&+ 640\frac{f_4^5f_6^2}{f_1^2f_2f_{12}} \pmod{1024}. \qquad (114)
\end{aligned}$$

Using (9), (13) and (17) in (114), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} opend_{3,6}(768n+256)q^n &\equiv 128\frac{f_1^2f_2f_3^2f_4^5}{f_6f_8^2}+256q\frac{f_2^3f_4^3f_6^3}{f_1^4}\\
&+ 640\frac{f_2^2f_8f_{12}^2}{f_4f_{24}} \pmod{1024} \qquad (115)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} opend_{3,6}(768n+640)q^n &\equiv 128\frac{f_2f_4^5f_6^3}{f_1^4f_8^2}+256\frac{f_2f_3^2f_4^5}{f_1^2f_6}\\
&+ 384\frac{f_2f_4^2f_6f_{24}}{f_8f_{12}} \pmod{1024}. \qquad (116)
\end{aligned}$$

From the equation (115), we obtain (79) and

$$\sum_{n=0}^{\infty} opend_{3,6}(1536n+256)q^n \equiv 128\frac{f_1^2 f_4 f_6^2}{f_2 f_{12}} \pmod{512}. \tag{117}$$

From the equation (116), we obtain (80) and

$$\sum_{n=0}^{\infty} opend_{3,6}(1536n+640)q^n \equiv 128\frac{f_2 f_3^3}{f_1} \pmod{256}. \tag{118}$$

Using (10) in (117), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(3072n+256)q^n \equiv 128\frac{f_3^2 f_4}{f_1^2 f_2 f_6} \pmod{512} \tag{119}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(3072n+1792)q^n \equiv 256 f_4^3 \pmod{512}. \tag{120}$$

From the equation (120), we obtain (81) and

$$\sum_{n=0}^{\infty} opend_{3,6}(12288n+1792)q^n \equiv 256 f_1^3 \pmod{512}. \tag{121}$$

Using (18) in (118), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(1536n+640)q^n \equiv 128 f_1^5 \pmod{256} \tag{122}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(1536n+640)q^n \equiv 128\frac{f_1 f_6^3}{f_2} \pmod{256}. \tag{123}$$

□

Theorem 5.2. *For $\alpha, n \geq 0$, we have for* (mod 32)

$$\sum_{n=0}^{\infty} opend_{3,6}\left(7^{2\alpha}(192n+40)\right) q^n \equiv 16 f_1^5, \tag{124}$$

$$\sum_{n=0}^{\infty} opend_{3,6}\left(7^{2\alpha+1}(192n+88)\right) q^n \equiv 16 q f_7^5 \tag{125}$$

and

$$opend_{3,6}\left(7^{2\alpha+1}(1344n+a_1)\right) \equiv 0, \tag{126}$$

where $a_1 \in \{88, 472, 664, 856, 1048, 1240\}$.

Proof. Using (13) in (103), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(192n+40)q^n \equiv 16f_1^5 \pmod{32} \tag{127}$$

and

$$\sum_{n=0}^{\infty} opend_{3,6}(192n+136)q^n \equiv 16\frac{f_1 f_6^3}{f_2} \pmod{32}. \tag{128}$$

The equation (127) is $\alpha = 0$ case of (124). Suppose that the equation (124) is true for some $\alpha \geq 0$. Using (26) in (127) and then collecting the term involving q^{7n+3}, we obtain (125).
From the equation (125), we obtain (126) and

$$\sum_{n=0}^{\infty} opend_{3,6}\left(7^{2\alpha+2}(192n+40)\right)q^n \equiv 16f_1^5, \tag{129}$$

which implies (124) is true for $\alpha+1$. So, by mathematical induction the equation (124) is true for all $\alpha \geq 0$. □

Theorem 5.3. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then, for all $\alpha, k, m \geq 0$, we have*

$$opend_{3,6}\left(7^{2\alpha}\cdot p^{2k+1}(192m+40p)\right) \equiv 0 \pmod{32} \tag{130}$$

and

$$opend_{3,6}\left(7^{2\alpha+1}\cdot p^{2k+1}(192m+88p)\right) \equiv 0 \pmod{32}. \tag{131}$$

Proof. Using (1) in (124), we find that for (mod 32)

$$\sum_{n=0}^{\infty} opend_{3,6}\left(7^{2\alpha}(192n+40)\right)q^n \equiv 16\sum_{s,t=-\infty}^{\infty} q^{\frac{s(3s-1)+4t(3t-1)}{2}}, \tag{132}$$

implies $n = \frac{s(3s-1)+4t(3t-1)}{2}$. It is equivalent to

$$24n+5 = (6s-1)^2 + (12t-2)^2 = x^2+y^2.$$

Thus, if $24n+5 \neq (6s-1)^2+(12t-2)^2 = x^2+y^2$, then

$$opend_{3,6}\left(7^{2\alpha}(192m+40)\right) \equiv 0 \pmod{32}. \tag{133}$$

Since $24n+5 = p^{2k+1}(192m+40) \neq x^2+y^2$, from which (130) follows.
Since the proof of (131) is similar to the proof of (130), we omit the details. □

Theorem 5.4. *If n is not a sum of a pentagonal number, four times pentagonal number and six times triangular number, then we have*

$$opend_{3,6}(48n+46) \equiv 0 \pmod{512}, \tag{134}$$

$$opend_{3,6}(192n+184) \equiv 0 \pmod{512}. \tag{135}$$

Proof. Using (1) and (2) in (98), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(48n+46)q^n \equiv 256 \sum_{r,s,=-\infty}^{\infty} \sum_{t=0}^{\infty} q^{\frac{r(3r-1)}{2}+\frac{4s(3s-1)}{2}+\frac{6t(t+1)}{2}} \pmod{512}, \tag{136}$$

which implies $n = \frac{r(3r-1)}{2} + \frac{4s(3s-1)}{2} + \frac{6t(t+1)}{2}$.
Thus, if $n \neq \frac{r(3r-1)}{2} + \frac{4s(3s-1)}{2} + \frac{6t(t+1)}{2}$, we obtain (134). □

Since the proof of (135) is similar to the proof of (134). So, we omit the details.

Theorem 5.5. *If n is not a sum of eight times pentagonal number and a triangular number, then we have*

$$opend_{3,6}(48n+22) \equiv 0 \pmod{128}. \tag{137}$$

Proof. Using (1) and (2) in (97), we find that

$$\sum_{n=0}^{\infty} opend_{3,6}(48n+22)q^n \equiv 64 \sum_{s=-\infty}^{\infty} \sum_{t=0}^{\infty} q^{\frac{8s(3s-1)}{2}+\frac{t(t+1)}{2}} \pmod{128}, \tag{138}$$

which implies $n = \frac{8s(3s-1)}{2} + \frac{t(t+1)}{2}$.
Thus, if $n \neq \frac{8s(3s-1)}{2} + \frac{t(t+1)}{2}$, we obtain (137). □

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