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SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING GENERALIZED POLYLOGARITHM FUNCTIONS

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Abstract

In this work, we introduce a new class of meromorphically starlike functions defined by the generalized polylogarithm and investigate coefficient bounds, distortion properties, the radius of convexity, convex linear combinations, and integral transforms. Furthermore, we also obtain partial sums and neighborhood results.

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1 Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$E^* := \{z \in \mathbb{C} : 0 < |z| < 1\} = E \setminus \{0\}. \tag{1.2}$$

A function $f \in \Sigma$ is said to be *meromorphically starlike functions of order* δ if and only if

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \delta, \qquad (z \in E^*), \tag{1.3}$$

for some δ $(0 \leq \delta < 1)$. We denote by $\Sigma(\delta)$ the class of all meromorphically starlike functions of order δ.

Furthermore, a function $f \in \Sigma$ is said to be *meromorphically convex functions of order* δ if and only if

$$\Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \delta, \qquad (z \in E^*), \tag{1.4}$$

for some δ $(0 \leq \delta < 1)$. We denote by $\Sigma_k(\delta)$ the class of all meromorphically convex functions of order δ.

For functions $f \in \Sigma$ given by (1.1) and $g \in \Sigma$ of the form

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \tag{1.5}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.6}$$

Let Σ_p be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \geq 0, \tag{1.7}$$

which are analytic and univalent in E^*.

For $c \in \mathbb{N}$, the set of natural numbers with $c \geq 2$, the absolutely convergent series

$$Li_c(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^c} \tag{1.8}$$

is known as the polylogarithm function. This class of functions was introduced by Leibniz [2]. For further studies on polylogarithms and meromorphic functions, see [3, 7, 8].

We consider a linear operator $\Omega_c : \Sigma \to \Sigma$, defined by the Hadamard product (or convolution) as follows:

$$\begin{aligned}\Omega_c f(z) &= \phi_c(z) * f(z)\\ &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{a_n}{(n+2)^c} z^n, \end{aligned} \tag{1.9}$$

where

$$\phi_c(z) = z^{-2} Li_c(z).$$

Next, we define the linear operator $\sigma_c : \Sigma \to \Sigma$ as follows:

$$\begin{aligned}\sigma_c f(z) &= \Omega_c f(z) - \frac{a_0}{2^c}\\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{a_n}{(n+2)^c} z^n. \end{aligned} \tag{1.10}$$

For a function f in the class Σ_p, we define the linear operator $\mathfrak{D}^k_{c,\lambda} f(z)$ as follows:

$$\begin{aligned}\mathfrak{D}^0_{c,\lambda} f(z) &= \sigma_c f(z),\\ \mathfrak{D}^1_{c,\lambda} f(z) &= (1-\lambda)\sigma_c f(z) + \lambda \frac{\big(z^2 \sigma_c f(z)\big)'}{z}, \qquad \lambda \geq 0\\ &= (1+\lambda)\sigma_c f(z) + \lambda z \big(\sigma_c f(z)\big)'\\ &= \mathfrak{D}_{c,\lambda} f(z),\\ \mathfrak{D}^2_{c,\lambda} f(z) &= \mathfrak{D}_{c,\lambda}\big(\mathfrak{D}^1_{c,\lambda} f(z)\big),\\ &\vdots \end{aligned}$$

In general,

$$\begin{aligned}\mathfrak{D}^k_{c,\lambda} f(z) &= \mathfrak{D}_{c,\lambda}\big(\mathfrak{D}^{k-1}_{c,\lambda} f(z)\big)\\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\big[1+\lambda(n+1)\big]^k}{(n+2)^c} a_n z^n, \qquad k = 1, 2, \cdots. \end{aligned} \tag{1.11}$$

The object of the present paper is to introduce the class of meromorphically starlike functions which we denote by $\Sigma^*_p(\alpha, \beta, \lambda, c)$, for some α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$ and $\lambda \geq 0$. For the class $\Sigma^*_p(\alpha, \beta, \lambda, c)$ extend some of the results of Venkateswarlu et al. [8] and Thirupathi Reddy et al. [7] . we obtain coefficient bounds, distortion properties, radius of convexity, convex linear combinations, and integral transforms. Moreover, we derive partial sums and neighborhood results for this class.

Definition 1.1. Let the function $f(z)$ be defined by (1.7). Then $f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c)$ if and only if

$$\left| \frac{z\left(\mathfrak{D}_{c,\lambda}^k f(z)\right)'}{\mathfrak{D}_{c,\lambda}^k f(z)} + 1 \right| < \beta \left| \frac{z\left(\mathfrak{D}_{c,\lambda}^k f(z)\right)'}{\mathfrak{D}_{c,\lambda}^k f(z)} + 2\alpha - 1 \right|,$$

for some

$$\alpha \ (0 \leq \alpha < 1), \qquad \beta \ (0 < \beta \leq 1), \qquad \text{and} \qquad \lambda \geq 0,$$

for all $z \in E^*$.

2 Coefficient Estimates

In this section, we obtain necessary and sufficient conditions for a function to belong to the class $\Sigma_p^*(\alpha, \beta, \lambda, c)$.

Theorem 2.1. *Let*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \qquad (a_n \geq 0),$$

be a function in the class Σ_p. Then

$$f \in \Sigma_p^*(\alpha, \beta, \lambda, c)$$

if and only if

$$\sum_{n=1}^{\infty} [(1+\beta)n + (2\alpha - 1)\beta + 1] \left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c} \right] a_n \leq 2\beta(1-\alpha). \tag{2.1}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)(n+2)^c}{[(1+\beta)n + (2\alpha-1)\beta + 1]\,[1+\lambda(n+1)]^k} z^n. \tag{2.2}$$

Proof. First, assume that

$$f \in \Sigma_p^*(\alpha, \beta, \lambda, c).$$

Then, by the definition of the class $\Sigma_p^*(\alpha, \beta, \lambda, c)$, we have

$$\left| \frac{z\left(\mathfrak{D}_{c,\lambda}^k f(z)\right)'}{\mathfrak{D}_{c,\lambda}^k f(z)} + 1 \right| < \beta \left| \frac{z\left(\mathfrak{D}_{c,\lambda}^k f(z)\right)'}{\mathfrak{D}_{c,\lambda}^k f(z)} + 2\alpha - 1 \right|, \qquad (z \in E^*). \tag{2.3}$$

Using (1.11), we obtain

$$\mathfrak{D}_{c,\lambda}^k f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{[1+\lambda(n+1)]^k}{(n+2)^c} a_n z^n.$$

Differentiating, we get

$$\left(\mathfrak{D}^k_{c,\lambda}f(z)\right)' = -\frac{1}{z^2} + \sum_{n=1}^{\infty} n\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n z^{n-1}.$$

Substituting these expressions into (2.3), we have

$$\left|\frac{\sum_{n=1}^{\infty}(n+1)\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n z^{n+1}}{1-\sum_{n=1}^{\infty}\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n z^{n+1}}\right|$$
$$< \beta\left|\frac{2(1-\alpha)-\sum_{n=1}^{\infty}(n+2\alpha-1)\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n z^{n+1}}{1-\sum_{n=1}^{\infty}\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n z^{n+1}}\right|.$$

Choosing z to be real and letting $z \to 1^-$, we obtain

$$\sum_{n=1}^{\infty}(n+1)\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n \le \beta\left[2(1-\alpha)-\sum_{n=1}^{\infty}(n+2\alpha-1)\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n\right].$$

After simplification, we get

$$\sum_{n=1}^{\infty}[(1+\beta)n+(2\alpha-1)\beta+1]\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right]a_n \le 2\beta(1-\alpha),$$

which proves (2.1).

Conversely, suppose that (2.1) holds. Then

$$\left|z\left(\mathfrak{D}^k_{c,\lambda}f(z)\right)' + \mathfrak{D}^k_{c,\lambda}f(z)\right| - \beta\left|z\left(\mathfrak{D}^k_{c,\lambda}f(z)\right)' + (2\alpha-1)\mathfrak{D}^k_{c,\lambda}f(z)\right|$$
$$\le \sum_{n=1}^{\infty}(n+1)\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n - \beta\left(2(1-\alpha)-\sum_{n=1}^{\infty}(n+2\alpha-1)\frac{[1+\lambda(n+1)]^k}{(n+2)^c}a_n\right)$$
$$= \sum_{n=1}^{\infty}[(1+\beta)n+(2\alpha-1)\beta+1]\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right]a_n - 2\beta(1-\alpha)$$
$$\le 0.$$

Hence,

$$\left|\frac{z\left(\mathfrak{D}^k_{c,\lambda}f(z)\right)'}{\mathfrak{D}^k_{c,\lambda}f(z)}+1\right| < \beta\left|\frac{z\left(\mathfrak{D}^k_{c,\lambda}f(z)\right)'}{\mathfrak{D}^k_{c,\lambda}f(z)}+2\alpha-1\right|,$$

which shows that

$$f \in \Sigma_p^*(\alpha, \beta, \lambda, c).$$

Finally, equality in (2.1) is attained for the function given by (2.2). Hence the result is sharp. □

Corollary 2.2. *If*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \qquad a_n \geq 0,$$

belongs to $\Sigma_p^*(\alpha, \beta, \lambda, c)$, *then*

$$a_n \leq \frac{2\beta(1-\alpha)(n+2)^c}{\big[(1+\beta)n + (2\alpha-1)\beta + 1\big][1+\lambda(n+1)]^k}, \qquad n = 1, 2, \ldots \tag{2.4}$$

with equality attained for each n *by the function*

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)(n+2)^c}{\big[(1+\beta)n + (2\alpha-1)\beta + 1\big][1+\lambda(n+1)]^k} z^n, \qquad n = 1, 2, \cdots . \tag{2.5}$$

If $\beta = 1$ in Theorem 2.1, then we obtain the following corollary.

Corollary 2.3. *If*

$$f(z) \in \Sigma_p^*(\alpha, \lambda, c),$$

then

$$a_n \leq \frac{(1-\alpha)(n+2)^c}{(n+\alpha)[1+\lambda(n+1)]^k}, \qquad n = 1, 2, \cdots .$$

The result is sharp for the functions

$$f_n(z) = \frac{1}{z} + \frac{(1-\alpha)(n+2)^c}{(n+\alpha)[1+\lambda(n+1)]^k} z^n, \qquad n = 1, 2, \cdots .$$

If $\beta = 1$ and $\lambda = 0$ in Theorem 2.1, then we obtain the following corollary.

Corollary 2.4. *If*

$$f(z) \in \Sigma_p^*(\alpha, c),$$

then

$$a_n \leq \frac{(1-\alpha)(n+2)^c}{n+\alpha}, \qquad n = 1, 2, \cdots .$$

The result is sharp for the functions

$$f_n(z) = \frac{1}{z} + \frac{(1-\alpha)(n+2)^c}{n+\alpha} z^n, \qquad n = 1, 2, \cdots .$$

3 Distortion properties and radius of convexity estimates

In this section, we prove the distortion theorem and radius of convexity estimates for the class

$$\Sigma_p^*(\alpha, \beta, \lambda, c).$$

Theorem 3.1. *Let*

$$f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c).$$

Then, for $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{3^c\beta(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k} r \leq |f(z)| \leq \frac{1}{r} + \frac{3^c\beta(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k} r. \tag{3.1}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{3^c\beta(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k} z, \qquad \text{at } z = r \text{ and } z = ir. \tag{3.2}$$

Proof. Suppose that

$$f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c).$$

In view of Theorem 2.1, we have

$$\sum_{n=1}^{\infty} a_n \leq \frac{3^c\beta(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k}. \tag{3.3}$$

Thus, for $0 < |z| = r < 1$,

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \\ &\leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \\ &\leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n \\ &\leq \frac{1}{r} + \frac{3^c\beta(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k} r, \qquad \text{by (3.3).} \end{aligned}$$

This gives the right-hand side of (3.1). Also,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n\right| \\ &\geq \frac{1}{|z|}-\sum_{n=1}^{\infty}a_n|z|^n \\ &\geq \frac{1}{r}-r\sum_{n=1}^{\infty}a_n \\ &\geq \frac{1}{r}-\frac{3^c\beta(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k}r.\end{aligned}$$

This gives the left-hand side of (3.1). Hence the proof. □

Theorem 3.2. *Let the function*

$$f(z)\in\Sigma_p^*(\alpha,\beta,\lambda,c).$$

Then $f(z)$ is meromorphically convex of order δ $(0\leq\delta<1)$ in

$$|z|<r=r(\alpha,\beta,\lambda,c,\delta),$$

where

$$r(\alpha,\beta,\lambda,c,\delta)=\inf_{n\geq1}\left\{\frac{(1-\delta)\big[(1+\beta)n+(2\alpha-1)\beta+1\big]}{2\beta(1-\alpha)n(n+2-\delta)}\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right]\right\}^{\frac{1}{n+1}}, \tag{3.4}$$

for $n=1,2,\cdots$.

The bound for $|z|$ is sharp for each n, with the extremal function given by (2.5).

Proof. Let

$$f(z)\in\Sigma_p^*(\alpha,\beta,\lambda,c).$$

Then, by Theorem 2.1,

$$\sum_{n=1}^{\infty}\frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)}\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right]a_n\leq1. \tag{3.5}$$

In view of (2.1), it is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right|\leq1-\delta,\qquad |z|<r(\alpha,\beta,\lambda,c,\delta),$$

or equivalently,

$$\left|\frac{f'(z)+(zf'(z))'}{f'(z)}\right|\leq1-\delta,\qquad |z|<r(\alpha,\beta,\lambda,c,\delta). \tag{3.6}$$

Substituting the series expansions for $f'(z)$ and $(zf'(z))'$ into the left-hand side of (3.6), we obtain

$$\left|\frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2}+\sum_{n=1}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty} na_n|z|^{n+1}}.$$

This will be bounded above by $(1-\delta)$ if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n|z|^{n+1} \leq 1. \tag{3.7}$$

In view of (3.5), it follows that (3.7) holds if

$$\frac{n(n+2-\delta)}{1-\delta}|z|^{n+1} \leq \frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)}\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right].$$

That is,

$$|z| \leq \left\{\frac{(1-\delta)\big[(1+\beta)n+(2\alpha-1)\beta+1\big]}{2\beta(1-\alpha)n(n+2-\delta)}\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right]\right\}^{\frac{1}{n+1}}. \tag{3.8}$$

Setting

$$|z| = r(\alpha,\beta,\lambda,c,\delta)$$

in (3.8), the desired result follows.

The result is sharp, with the extremal function given by

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)}{(1+\beta)n+(2\alpha-1)\beta+1}\left[\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right] z^n, \qquad n = 1, 2, \cdots.$$

□

4 Convex linear combinations

In this section, we prove that the class

$$\Sigma_p^*(\alpha,\beta,\lambda,c)$$

is closed under convex linear combinations.

Theorem 4.1. *Let*

$$f_0(z) = \frac{1}{z}$$

and

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)}{(1+\beta)n + (2\alpha-1)\beta + 1}\left[\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right] z^n, \qquad n = 1, 2, \ldots.$$

Then

$$f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c)$$

if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \omega_n f_n(z),$$

where

$$\omega_n \geq 0 \qquad \textit{and} \qquad \sum_{n=0}^{\infty} \omega_n = 1.$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \omega_n f_n(z),$$

where

$$\omega_n \geq 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty} \omega_n = 1.$$

Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \omega_n f_n(z) \\ &= \omega_0 f_0(z) + \sum_{n=1}^{\infty} \omega_n f_n(z) \\ &= \left(1 - \sum_{n=1}^{\infty} \omega_n\right) f_0(z) + \sum_{n=1}^{\infty} \omega_n f_n(z) \\ &= \left(1 - \sum_{n=1}^{\infty} \omega_n\right) \frac{1}{z} \\ &\quad + \sum_{n=1}^{\infty} \omega_n \left[\frac{1}{z} + \frac{2\beta(1-\alpha)}{(1+\beta)n + (2\alpha-1)\beta + 1}\left(\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right) z^n\right] \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \omega_n \frac{2\beta(1-\alpha)}{(1+\beta)n + (2\alpha-1)\beta + 1}\left[\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right] z^n. \end{aligned}$$

Since

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)}\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right]\right\}\\
&\qquad\times\left\{\omega_n\frac{2\beta(1-\alpha)}{(1+\beta)n+(2\alpha-1)\beta+1}\left[\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right]\right\}\\
&=\sum_{n=1}^{\infty}\omega_n\\
&=1-\omega_0\le 1,
\end{aligned}$$

it follows from Theorem 2.1 that

$$f(z)\in\Sigma_p^*(\alpha,\beta,\lambda,c).$$

Conversely, suppose that

$$f(z)\in\Sigma_p^*(\alpha,\beta,\lambda,c).$$

Then

$$a_n\le\frac{2\beta(1-\alpha)}{(1+\beta)n+(2\alpha-1)\beta+1}\left[\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right],\qquad n=1,2,\ldots.$$

Define

$$\omega_n=\frac{(1+\beta)n+(2\alpha-1)\beta+1}{2\beta(1-\alpha)}\left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right]a_n,\qquad n=1,2,\ldots,$$

and

$$\omega_0=1-\sum_{n=1}^{\infty}\omega_n.$$

Then $\omega_n\ge 0$ for all n, and

$$\sum_{n=0}^{\infty}\omega_n=1.$$

Further,

$$\begin{aligned}
\sum_{n=0}^{\infty}\omega_n f_n(z)&=\omega_0\frac{1}{z}+\sum_{n=1}^{\infty}\omega_n\left[\frac{1}{z}+\frac{2\beta(1-\alpha)}{(1+\beta)n+(2\alpha-1)\beta+1}\left(\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right)z^n\right]\\
&=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n\\
&=f(z).
\end{aligned}$$

Hence,

$$f(z)=\sum_{n=0}^{\infty}\omega_n f_n(z).$$

This completes the proof. □

Theorem 4.2. *The class $\Sigma_p^*(\alpha, \beta, \lambda, c)$ is closed under convex linear combinations.*

Proof. Let the functions

$$F_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_{n,j} z^n, \qquad j = 1, 2, \ldots, m,$$

belong to the class $\Sigma_p^*(\alpha, \beta, \lambda, c)$.

It is sufficient to show that the function

$$H(z) = \omega F_1(z) + (1-\omega) F_2(z), \qquad (0 \leq \omega \leq 1),$$

also belongs to the class $\Sigma_p^*(\alpha, \beta, \lambda, c)$.

Since $0 \leq \omega \leq 1$, we have

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\omega f_{n,1} + (1-\omega) f_{n,2}\right] z^n.$$

Now,

$$\begin{aligned}
&\sum_{n=1}^{\infty} [(1+\beta)n + (2\alpha - 1)\beta + 1] \left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right] [\omega f_{n,1} + (1-\omega) f_{n,2}] \\
&= \omega \sum_{n=1}^{\infty} [(1+\beta)n + (2\alpha - 1)\beta + 1] \left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right] f_{n,1} \\
&\quad + (1-\omega) \sum_{n=1}^{\infty} [(1+\beta)n + (2\alpha - 1)\beta + 1] \left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right] f_{n,2} \\
&\leq 2\beta\omega(1-\alpha) + 2\beta(1-\omega)(1-\alpha) \\
&= 2\beta(1-\alpha).
\end{aligned}$$

Hence, by Theorem 2.1, we conclude that

$$H(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c).$$

□

5 Integral transforms

In this section, we consider integral transforms of functions in $\Sigma_p^*(\alpha, \beta, \lambda, c)$.

Theorem 5.1. *If $f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c)$, then the integral transform*

$$F_d(z) = d \int_0^1 u^d f(uz)\, du, \qquad 0 < d < \infty, \tag{5.1}$$

belongs to the class $\Sigma_p^*(\delta)$*, where*

$$\delta = \delta(\alpha, \beta, \lambda, c, d) = \frac{(1+\alpha\beta)(1+2\lambda)^k(d+2) - 3^c\beta d(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k(d+2) + 3^c\beta d(1-\alpha)}. \tag{5.2}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{3^c\beta(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k}\, z.$$

Proof. Suppose that

$$f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c).$$

Then

$$F_d(z) = d\int_0^1 u^d f(uz)\, du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{da_n}{n+d+1} z^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{n+\delta}{1-\delta} \cdot \frac{da_n}{n+d+1} \leq 1. \tag{5.3}$$

Since $f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c)$, we have

$$\sum_{n=1}^{\infty} \frac{(1+\beta)n + (2\alpha-1)\beta + 1}{2\beta(1-\alpha)} \left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right] a_n \leq 1. \tag{5.4}$$

Therefore, (5.3) will hold if

$$\frac{n+\delta}{1-\delta} \cdot \frac{d}{n+d+1} \leq \frac{(1+\beta)n + (2\alpha-1)\beta + 1}{2\beta(1-\alpha)} \left[\frac{[1+\lambda(n+1)]^k}{(n+2)^c}\right], \qquad n \geq 1.$$

This implies that

$$\delta \leq \frac{[(1+\beta)n + (2\alpha-1)\beta + 1]\,(n+d+1) - 2\beta(1-\alpha)nd\left[\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right]}{[(1+\beta)n + (2\alpha-1)\beta + 1]\,(n+d+1) + 2\beta(1-\alpha)nd\left[\frac{(n+2)^c}{[1+\lambda(n+1)]^k}\right]}. \tag{5.5}$$

Since the right-hand side of (5.5) is an increasing function of n, by taking $n = 1$ in (5.5), we obtain

$$\delta \leq \frac{(1+\alpha\beta)(1+2\lambda)^k(d+2) - 3^c\beta d(1-\alpha)}{(1+\alpha\beta)(1+2\lambda)^k(d+2) + 3^c\beta d(1-\alpha)}.$$

Hence the proof. □

6 Partial Sums

Let $f \in \Sigma_p$ be a function of the form (1.7). Motivated by Silverman [5] and Silvia [6], we define the partial sums f_m by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n, \qquad (m \in \mathbb{N}). \tag{6.1}$$

In this section, we consider partial sums of functions in the class $\Sigma_p^*(\alpha, \beta, \lambda, c)$ and obtain sharp lower bounds for the real parts of the ratios $\dfrac{f(z)}{f_m(z)}$ and $\dfrac{f_m(z)}{f(z)}$.

Theorem 6.1. *Let $f \in \Sigma_p^*(\alpha, \beta, \lambda, c)$ be given by (1.7), and define the partial sums $f_1(z)$ and $f_m(z)$ by*

$$f_1(z) = \frac{1}{z} \quad \text{and} \quad f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n, \qquad (m \in \mathbb{N} \setminus \{1\}). \tag{6.2}$$

Suppose also that

$$\sum_{n=1}^{\infty} d_n |a_n| \leq 1,$$

where

$$d_n \geq \begin{cases} 1, & \text{if } n = 1, 2, \ldots, m, \\ \dfrac{[(1+\beta)n + (2\alpha - 1)\beta + 1]\,[1 + \lambda(n+1)]^k}{2\beta(1-\alpha)(n+2)^c}, & \text{if } n = m+1, m+2, \ldots \end{cases}. \tag{6.3}$$

Then $f \in \Sigma_p^*(\alpha, \beta, \lambda, c)$. Furthermore,

$$\Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}}, \tag{6.4}$$

$$\Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{6.5}$$

Proof. For the coefficients d_n given by (6.3), it is easy to verify that

$$d_{m+1} > d_m > 1. \tag{6.6}$$

Therefore,

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} d_n |a_n| \leq 1, \tag{6.7}$$

by the hypothesis (6.3).

Define

$$g_1(z) = d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right)$$
$$= 1 + \frac{d_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{m} a_n z^{n+1}}.$$

It is sufficient to show that

$$\Re\big(g_1(z)\big) \geq 0, \qquad (z \in E^*),$$

or equivalently,

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq 1.$$

Using (6.7), we obtain

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq \frac{d_{m+1}\sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=1}^{m} |a_n| - d_{m+1}\sum_{n=m+1}^{\infty} |a_n|}$$
$$\leq 1, \qquad (z \in E^*).$$

This proves the assertion (6.4).

To show that the result is sharp, consider the function

$$f(z) = \frac{1}{z} + \frac{z^m}{d_{m+1}}. \tag{6.8}$$

For

$$z = re^{\frac{i\pi}{m+1}},$$

we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+1}}{d_{m+1}} \longrightarrow 1 - \frac{1}{d_{m+1}} \qquad \text{as } r \to 1^-.$$

Similarly, let

$$g_2(z) = (1 + d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1 + d_{m+1}}\right).$$

Again, by using (6.7), we obtain

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{m+1})\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{m}|a_n|-(1-d_{m+1})\sum_{n=m+1}^{\infty}|a_n|} \leq 1.$$

Hence, we obtain the assertion (6.5).

The bound in (6.5) is sharp for each $m \in \mathbb{N}$, with the extremal function given by (6.8). □

The proof of the following theorem is analogous to that of Theorem 6.1; therefore, we omit the details.

Theorem 6.2. *Let*

$$f(z) \in \Sigma_p^*(\alpha, \beta, \lambda, c)$$

be given by (1.7) *and satisfy the condition* (2.1). *Then*

$$\Re\left(\frac{f'(z)}{f'_m(z)}\right) > 1 - \frac{m+1}{d_{m+1}},$$
$$\Re\left(\frac{f'_m(z)}{f'(z)}\right) > \frac{d_{m+1}}{m+1+d_{m+1}},$$

where

$$d_n \geq \begin{cases} n, & \text{if } n = 2, 3, \ldots, m, \\ \dfrac{[(1+\beta)n + (2\alpha-1)\beta + 1]\,[1+\lambda(n+1)]^k}{2\beta(1-\alpha)(n+2)^c}, & \text{if } n = m+1, m+2, \ldots \end{cases}.$$

The bounds are sharp for the extremal function $f(z)$ given by (2.4).

7 Neighbourhood Results

In this section, we determine the neighbourhood for the class $\Sigma_p^{*\xi}(\alpha, \beta, \lambda, c)$, which we define as follows.

Definition 7.1. A function $f \in \Sigma_p$ is said to belong to the class $\Sigma_p^{*\xi}(\alpha, \beta, \lambda, c)$ if there exists a function

$$g \in \Sigma_p^*(\alpha, \beta, \lambda, c)$$

such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \xi, \qquad (z \in E^*,\ 0 \leq \xi < 1). \tag{7.1}$$

Following the earlier works on neighbourhoods of analytic functions by Goodman [1] and Ruscheweyh [4], we define the δ-neighbourhood of a function $f \in \Sigma_p$ by

$$N_\delta(f) = \left\{ g \in \Sigma_p : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}. \quad (7.2)$$

Theorem 7.2. *If*

$$g \in \Sigma_p^*(\alpha, \beta, \lambda, c)$$

and

$$\xi = 1 - \frac{\delta(1+\alpha\beta)(1+2\lambda)^k}{3^c \beta(1-\alpha)}, \quad (7.3)$$

then

$$N_\delta(g) \subset \Sigma_p^{*\xi}(\alpha, \beta, \lambda, c).$$

Proof. Let

$$f \in N_\delta(g).$$

Then, from (7.2), we have

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta. \quad (7.4)$$

This immediately implies that

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta. \quad (7.5)$$

Since

$$g \in \Sigma_p^*(\alpha, \beta, \lambda, c),$$

we obtain

$$\sum_{n=1}^{\infty} b_n \leq \frac{\beta(1-\alpha)3^c}{(1+\alpha\beta)(1+2\lambda)^k}.$$

Therefore,

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \\ &\leq \frac{\delta(1+\alpha\beta)(1+2\lambda)^k}{3^c \beta(1-\alpha)} \\ &= 1 - \xi, \end{aligned}$$

provided that ξ is given by (7.3).

Hence,

$$f \in \Sigma_p^{*\xi}(\alpha, \beta, \lambda, c).$$

This completes the proof. □

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