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On new subclasses of bi-univalent functions defined by Tremblay fractional derivative with bounded boundary rotation

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Abstract

In this article, we introduce three innovative subclasses of bi-univalent functions associated with bounded boundary rotation, employing Tremblay fractional derivatives in an open unit disk $\mathbb{E}$. For these new classes, we establish the initial bounds for the first two coefficients. Additionally, we derive the well-known Fekete-Szegö inequality for these newly defined subclasses. Moreover, we outline corollaries that provide improved outcomes relative to those already documented in the literature.

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1. Introduction

Let $\mathcal{A}$ represent the family of all functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{E}$. If $0 \leq \delta < 1$. In 1936, Robertson [19] introduced the $\mathcal{S}^*(\delta)$ containing all starlike functions of order δ. A function f is said to be in the class $\mathcal{S}^*(\delta)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta, \quad z \in \mathbb{E}.$$

Similarly, $\mathcal{K}(\delta)$ containing all convex functions of order δ. A function f is said to be in the class $\mathcal{K}(\delta)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta, \quad z \in \mathbb{E}.$$

It is also known that a function $f \in \mathcal{K}(\delta)$ if and only if $zf' \in \mathcal{S}^*(\delta)$. Let $\mathcal{R}(\delta)$ represent the family of all functions $f \in \mathcal{A}$ satisfying

$$\Re(f'(z)) > \delta, \quad z \in \mathbb{E}.$$

Functions in $\mathcal{R}(\delta)$ are called of bounded turning of order δ. For $\delta = 0$, $\mathcal{R}(\delta) \equiv \mathcal{R}(0) \equiv \mathcal{R}$, is the family of bounded turning functions or sometimes called functions of bounded rotation introduced by Goodman [8].

As per the Koebe one-quarter [7] theorem, the image of the unit disk $\mathbb{E}$ under any univalent function f from the class $\mathcal{A}$ is a disk of radius 1/4. This implies that every function f in the class $\mathcal{S}$ has an inverse f^{-1}, which satisfies the relationships

$$f'(f(z)) = z, \quad z \in \mathbb{E}$$

and

$$f(f'(w)) = w, \quad |w| < r_0(f);\ r_0(f) \geq \frac{1}{4}.$$

In the case where f^{-1} has an analytic continuation to the domain $\mathbb{E}$, the function f is referred to as bi-univalent in $\mathbb{E}$ if both f and f^{-1} are univalent in $\mathbb{E}$ and are represented by

$$f^{-1}(w) := \xi(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (a_4 - 5a_2a_3 + 5a_2^3)w^4 + \cdots . \tag{1.2}$$

Let Σ represent the category of bi-univalent functions within the unit disk $\mathbb{E}$ as defined by (1.1). The concept of the class Σ, which encompasses analytic bi-univalent functions, was initially presented by Lewin [11], who established that $|a_2| \leq 1.51$. Subsequently, Brannan and Clunie [4] enhanced Lewin's findings, demonstrating that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [13] determined that the maximum value of $|a_2|$ for functions in Σ is 4/3. The functions

$$\frac{z}{1-z} \quad \textit{and} \quad -\log(1-z)$$

belong to the class Σ. In contrast, the well-known Koebe function does not belong to Σ. In their studies, Brannan and Taha [5] and Taha [29] examined certain subclasses of Σ and identified non-sharp estimates for $|a_2|$ and $|a_3|$. Furthermore, Sivasubramanian et al. [21], Breaz et al. [6] substantiated Brannan and Clunie's conjecture, demonstrating that $|a_2| \leq \sqrt{2}$ for various subclasses of Σ.

Similar to the well-known subclasses $\mathcal{S}^*(\delta)$ and $\mathcal{K}(\delta)$ associated with starlike and convex functions of order δ, where $0 \leq \delta < 1$, Brannan and Taha [5] proposed specific subclasses of Σ, namely $\mathcal{S}^*_\Sigma(\delta)$ and $\mathcal{K}_\Sigma(\delta)$, which pertain to bi-starlike and bi-convex functions of the same order. For functions f belonging to $\mathcal{S}^*_\Sigma(\delta)$ and $\mathcal{K}_\Sigma(\delta)$, they derived non-sharp estimates for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. The study of bi-univalent functions gained concentration as well as thrust mainly due to the investigation of Srivastava et al [24].

Let $m \geq 2$ and $0 \leq \delta < 1$. Define the class of functions $\mathcal{P}_m(\delta)$ as those functions p that are analytic and normalized with $p(0) = 1$, which satisfy the following condition:

$$\int_0^{2\pi} \left| \frac{\Re(p(z)) - \delta}{1 - \delta} \right| d\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{E}$. This class was introduced by Padmanabhan and Parvatham [17]. The functions $p_1(z) = 1 + z$ and $p_2(z) = 1 + 3z$ are members of the class $\mathcal{P}_m(\delta)$. In the case where

$\delta = 0$, we denote $\mathcal{P}_m(0)$ as $\mathcal{P}_m$. The class $\mathcal{P}_m$, as introduced by Pinchuk [18], includes analytic functions p that are defined in the unit disk $\mathbb{E}$, with the property that $p(0) = 1$. A function p in the class $\mathcal{P}_m$ can be represented as

$$p(z) = \frac{1}{2}\int_0^{2\pi} \left|\frac{1 - ze^{it}}{1 + ze^{it}}\right| d\mu(t),$$

where μ is a real-valued function of bounded variation, fulfilling the conditions

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m, \quad m \geq 2. \tag{1.3}$$

Let $\mathcal{R}_m(\delta)$ denote the set of analytic functions $f(z)$ defined in the unit disk $\mathbb{E}$, where $f(0) = 0$, $f'(0) = 1$, and the condition

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}_m(\delta)$$

is satisfied. This class extends the starlike function class $\mathcal{S}^*(\delta)$ of order δ, as explored by Robertson [20]. When $\delta = 0$, we obtain the class $\mathcal{R}_m(0) \equiv \mathcal{R}_m$, which encompasses all functions with bounded radius of rotation. Consequently, functions $f \in \mathcal{R}_m$ can be expressed as

$$f(z) = z \exp\left\{\int_0^{2\pi} -\log(1 - ze^{it}) d\mu(t)\right\},$$

where, μ is a real-valued function characterized by bounded variation, as specified in (1.3). The function $f_2(z) = ze^z$ is a member of the class $\mathcal{R}_m(\delta)$.

Let $\mathcal{V}_m(\delta)$ represent the set of all functions $f(z)$ defined in $\mathbb{D}$, which are normalized such that $f(0) = 0$ and $f'(0) = 1$. These functions must satisfy the condition

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_m(\delta), \quad 0 \leq \delta < 1.$$

When $\delta = 0$, we obtain the class $\mathcal{V}_m(0) \equiv \mathcal{V}_m$, which encompasses all functions with bounded boundary rotation, as investigated by Paatero [16]. Consequently, functions $f \in \mathcal{V}_m$ can be expressed as

$$f'(z) = \exp\left\{\int_0^{2\pi} -\log\left(1 - ze^{it}\right) d\mu(t)\right\},$$

where, μ is a real-valued function characterized by bounded variation, as specified in (1.3). The function $f_2(z) = e^z - 1$ is a member of the class $\mathcal{V}_m(\delta)$.

Operators play a significant role in the study of geometric function theory and its related domains. Several researchers [1, 3, 10, 22] have presented and examined different subclasses of the class Σ by employing various operators.

Definition 1. [27] *For a function f, the fractional integral of order η is expressed as follows:*

$$\mathfrak{D}_z^{-\eta} f(z) = \frac{1}{\Gamma(\eta)} \int_0^z \frac{f(\chi)}{(z - \chi)^{1-\eta}} d\chi \ (\eta > 0).$$

Here, f is an analytic function within a simply-connected region of the complex z-plane that includes the origin and the multiplicity of $(z-\chi)^{1-\eta}$ is removed by requiring, $\log(z-\chi)$ to be real when $z-\chi>0$.

Definition 2. [27] *For a function f, the fractional derivative of order η is defined by the equation:*

$$\mathfrak{D}_z^{\eta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \int_0^z \frac{f(\chi)}{(z-\chi)^{\eta}} d\chi; \quad (0 \leq \eta < 0).$$

Here, f is subject to specific constraints, and the complexity of $(z-\chi)^{-\eta}$ is addressed as per Definition 1.

For further details on the fractional integral and fractional derivative, see [14, 15, 25].

Definition 3. [27] *Based on the conditions specified in Definition 2, the fractional derivative of order $(k+\eta)$ can be defined by the equation:*

$$\mathfrak{D}_z^{k+\eta} f(z) = \frac{d^k}{dz^k} \mathfrak{D}_z^{\eta} f(z), \quad (0 \leq \eta < 0, \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Based on the implications of Definitions 1, 2, and 3, we observe that

$$\mathfrak{D}_z^{-\eta} z^k = \frac{\Gamma(k+1)}{\Gamma(k+\eta+1)} z^{k+\eta} \quad (k \in \mathbb{N} \ \ \eta > 0)$$

and

$$\mathfrak{D}_z^{\eta} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\eta+1)} z^{k-\eta} \quad (0 \leq \eta < 0).$$

The Tremblay fractional derivative operator for a function $f \in \mathcal{A}$ is defined for all $z \in \mathbb{D}$ as follows:

$$\mathcal{T}_z^{\alpha,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} z^{1-\gamma} \mathfrak{D}_z^{\alpha-\gamma} z^{\alpha-1} f(z)$$

with the following conditions on the parameters:

$$(0 < \alpha \leq 1; \ 0 < \gamma \leq 1; \ \alpha > \gamma; \ 0 < \alpha - \gamma < 1).$$

Definition 4. [9] *The modified Tremblay fractional derivative operator for a function $f \in \mathcal{A}$ is represented as $\mathfrak{T}^{\alpha,\gamma} : \mathcal{A} \to \mathcal{A}$ and is defined by the equation:*

$$\mathfrak{T}_z^{\alpha,\gamma} f(z) = \frac{\gamma}{\alpha} \mathcal{T}_z^{\alpha,\gamma} f(z) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+1)} z^{1-\gamma} \mathfrak{D}_z^{\alpha-\gamma} z^{\alpha-1} f(z)$$

Consequently, we have:

$$\mathfrak{T}_z^{\alpha,\gamma} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\gamma+k)}{\Gamma(\alpha+k)} a_k z^k.$$

The parameters are constrained by:

$$(0 < \alpha \leq 1; \ 0 < \gamma \leq 1; \ \alpha > \gamma; \ 0 < \alpha - \gamma < 1).$$

Here, $\mathcal{T}_z^{\alpha,\gamma}$ refers to the Tremblay fractional derivative operator. For further details regarding the Tremblay Operator, see [26].

Lemma 1. [2] *Let us define the function* $\Psi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$ *for* z *in the open unit disk* $\mathbb{E}$, *under the condition that* $\Psi \in \mathcal{P}_m(\delta)$. *Consequently, we have the following inequality for* $k \geq 1$*:*

$$|B_k| \leq m(1-\delta). \tag{1.4}$$

Consider the functions $p, q \in \mathcal{P}_m(\delta)$, defined as follows:

$$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \tag{1.5}$$

and

$$q(z) = 1 + \sum_{k=1}^{\infty} c_k z^k. \tag{1.6}$$

According to Lemma 1, we can derive the following inequalities:

$$|b_k| \leq m(1-\delta), \quad \forall \quad k \geq 1 \tag{1.7}$$

and

$$|c_k| \leq m(1-\delta), \quad \forall \quad k \geq 1. \tag{1.8}$$

In this article, we introduce three innovative subclasses of bi-univalent functions associated with bounded boundary rotation, employing Tremblay fractional derivatives in an open unit disk $\mathbb{E}$. For these new classes, we establish the initial bounds for the first two coefficients. Additionally, we derive the well-known Fekete-Szegö inequality for these newly defined subclasses. Moreover, we outline corollaries that provide improved outcomes relative to those already documented in the literature.

2. Main definitions and Theorems

Definition 5. *For* $0 < \alpha \leq 1,\ 0 < \gamma \leq 1,\ \alpha > \gamma,\ 0 < \alpha - \gamma < 1,\ 0 \leq \delta < 1$ *and* $2 \leq m \leq 4$. *A function* $f \in \Sigma$ *as defined in* (1.1) *is considered to belong to the class* $\mathcal{H}_{\Sigma}^{m,\delta}(\alpha, \gamma)$ *if the following conditions hold:*

$$(\mathfrak{T}_z^{\alpha,\gamma} f)'(z) \in \mathcal{P}_m(\delta)$$

and

$$(\mathfrak{T}_w^{\alpha,\gamma} \xi)'(w) \in \mathcal{P}_m(\delta),$$

where ξ *represents the analytic continuation of* f^{-1} *to the domain* $\mathbb{D}$.

Remark 1. (i) *By opting for* $\alpha = \gamma$ *the class* $\mathcal{H}_{\Sigma}^{m,\delta}(\alpha, \gamma)$ *leads to the family* $\mathcal{H}_{\Sigma}(m, \delta)$, *which was established by Sharma* [23] *in 2023. This family consists of the functions* $f \in \mathcal{A}$ *that fulfill the requirements*

$$f'(z) \in \mathcal{P}_m(\delta)$$

and

$$\xi'(w) \in \mathcal{P}_m(\delta).$$

(ii) *By opting for* $\alpha = \gamma$ *and* $m = 2$ *the class* $\mathcal{H}_{\Sigma}^{m,\delta}(\alpha, \gamma)$ *leads to the family* $\mathcal{H}_{\Sigma}(\delta)$, *which was established by Srivastava* [24] *in 2010. This family consists of the functions* $f \in \mathcal{A}$ *that fulfill the requirements*

$$\Re(f'(z)) > \delta$$

and

$$\Re(\xi'(w)) > \delta.$$

(iii) *By opting for* $m = 2$ *the class* $\mathcal{H}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ *leads to the family* $\mathcal{H}_{\Sigma}^{\delta}(\alpha,\gamma)$, *which is known as the class of bi-bounded turning functions of order* δ *defined by Tremblay fractional derivative operator. This family consists of the functions* $f \in \mathcal{A}$ *that fulfill the requirements*

$$\Re((\mathfrak{T}_z^{\alpha,\gamma} f)'(z)) > \delta$$

and

$$\Re((\mathfrak{T}_w^{\alpha,\gamma} \xi)'(w)) > \delta.$$

In the next step, we determine the initial bounds for the coefficients and evaluate $|a_3 - \delta a_2^2|$ for the class $\mathcal{H}_{\Sigma}^{m,\delta}(\alpha,\gamma)$.

Theorem 1. *Assume that* $0 < \alpha \leq 1,\ 0 < \gamma \leq 1,\ \alpha > \gamma,\ 0 < \alpha - \gamma < 1,\ 0 \leq \delta < 1$ *and* $2 \leq m \leq 4$. *A function* $f \in \mathcal{H}_{\Sigma}^{m,\delta}(\alpha,\gamma)$, *then*

$$|a_2| \leq \sqrt{\frac{m(1-\delta)(\gamma+1)(\gamma+2)}{3(\alpha+1)(\alpha+2)}}, \tag{2.1}$$

$$|a_3| \leq \frac{m(1-\delta)(\gamma+1)(\gamma+2)}{3(\alpha+1)(\alpha+2)} \tag{2.2}$$

and

$$|a_3 - \delta a_2^2| \leq \begin{cases} \dfrac{m(1-\delta)(\gamma+1)(\gamma+2)(1-\aleph)}{3(\alpha+1)(\alpha+2)} & \text{for } \aleph < 0, \\ \dfrac{m(1-\delta)(\gamma+1)(\gamma+2)}{3(\alpha+1)(\alpha+2)} & \text{for } 0 \leq \aleph < 2, \\ \dfrac{m(1-\delta)(\gamma+1)(\gamma+2)(\aleph-1)}{3(\alpha+1)(\alpha+2)} & \text{for } \aleph \geq 2. \end{cases} \tag{2.3}$$

Proof. Assume a function f is an element of $\mathcal{H}_{\Sigma}^{m,\delta}(\alpha,\gamma)$. In this case, there exist polynomials $p(z)$ and $q(w)$ within $\mathcal{P}_m(\delta)$ such that:

$$(\mathfrak{T}_z^{\alpha,\gamma} f)'(z) = p(z) \tag{2.4}$$

and

$$(\mathfrak{T}_w^{\alpha,\gamma} \xi)'(w) = q(w). \tag{2.5}$$

The functions $p(z)$ and $q(w)$ are defined as follows:

$$p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \tag{2.6}$$

and

$$q(w) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + \cdots. \tag{2.7}$$

Therefore, based on equations (2.4), (2.5), (2.6), and (2.7), we can conclude that

$$\frac{2(\alpha+1)}{(\gamma+1)} a_2 = b_1, \tag{2.8}$$

$$\frac{3(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3 = b_2, \tag{2.9}$$

$$-\frac{2(\alpha+1)}{(\gamma+1)}a_2 = c_1 \tag{2.10}$$

and

$$\frac{3(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}(2a_2^2 - a_3) = c_2. \tag{2.11}$$

From the analysis of equations (2.8) and (2.10), we determine that $b_1 + c_1 = 0$. In conjunction, equations (2.9) and (2.11) yield

$$\frac{6(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_2^2 = b_2 + c_2. \tag{2.12}$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.12), we get

$$\frac{3(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_2^2 \leq m(1-\delta). \tag{2.13}$$

From equation (2.13), we derive (2.1). By taking (2.11) away from (2.9) and utilizing (2.12), we obtain

$$\frac{3(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3 = b_2. \tag{2.14}$$

Utilizing the triangle inequality along with (1.7), we get

$$\frac{3(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3 \leq m(1-\delta). \tag{2.15}$$

From equation (2.15), we derive (2.2). Furthermore, equation (2.12) can be expressed as

$$a_2^2 = \frac{(\gamma+1)(\gamma+2)}{6(\alpha+1)(\alpha+2)}b_2 + \frac{(\gamma+1)(\gamma+2)}{6(\alpha+1)(\alpha+2)}c_2. \tag{2.16}$$

For any $\aleph \in \mathbb{R}$, we have

$$a_3 - \aleph a_2^2 = \frac{b_2(\gamma+1)(\gamma+2)}{6(\alpha+1)(\alpha+2)}[2-\aleph] - \frac{c_2(\gamma+1)(\gamma+2)}{6(\alpha+1)(\alpha+2)}\aleph. \tag{2.17}$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.17), we get

$$|a_3 - \aleph a_2^2| \leq \frac{m(1-\delta)(\gamma+1)(\gamma+2)}{6(\alpha+1)(\alpha+2)}\left[|2-\aleph| + |\aleph|\right]. \tag{2.18}$$

From equation (2.18), we derive (2.3). This completes the proof of Theorem 1. □

Definition 6. *For* $0 < \alpha \leq 1,\ 0 < \gamma \leq 1,\ \alpha > \gamma,\ 0 < \alpha - \gamma < 1,\ 0 \leq \delta < 1$ *and* $2 \leq m \leq 4$. *A function* $f \in \Sigma$ *as defined in* (1.1) *is considered to belong to the class* $\mathcal{R}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ *if the following conditions hold:*

$$\frac{z(\mathfrak{T}_z^{\alpha,\gamma}f)'(z)}{(\mathfrak{T}_z^{\alpha,\gamma}f)(z)} \in \mathcal{P}_m(\delta)$$

and

$$\frac{w(\mathfrak{T}_w^{\alpha,\gamma}\xi)'(w)}{(\mathfrak{T}_w^{\alpha,\gamma}\xi)(w)} \in \mathcal{P}_m(\delta),$$

where ξ represents the analytic continuation of f^{-1} to the domain $\mathbb{D}$.

Remark 2. (i) *By opting for $\alpha = \gamma$ the class $\mathcal{R}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ leads to the family $\mathcal{R}_{\Sigma}(m,\delta)$, which was established by Li [12] in 2020. This family consists of the functions $f \in \mathcal{A}$ that fulfill the requirements*

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}_m(\delta)$$

and

$$\frac{w\xi'(w)}{\xi(w)} \in \mathcal{P}_m(\delta).$$

(ii) *By opting for $\alpha = \gamma$ and $m = 2$ the class $\mathcal{R}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ leads to the family $\mathcal{S}_{\Sigma}^{*}(\delta)$, which was established by Brannan and Taha [5] in 1985. This family consists of the functions $f \in \mathcal{A}$ that fulfill the requirements*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta$$

and

$$\Re\left(\frac{w\xi'(w)}{\xi(w)}\right) > \delta.$$

(iii) *By opting for $m = 2$ the class $\mathcal{R}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ leads to the family $\mathcal{S}_{\Sigma}^{*,\delta}(\alpha,\gamma)$, which is known as the class of bi-starlike functions of order δ defined by Tremblay fractional derivative operator. This family consists of the functions $f \in \mathcal{A}$ that fulfill the requirements*

$$\Re\left(\frac{z(\mathfrak{T}_z^{\alpha,\gamma}f)'(z)}{(\mathfrak{T}_z^{\alpha,\gamma}f)(z)}\right) > \delta$$

and

$$\Re\left(\frac{w(\mathfrak{T}_w^{\alpha,\gamma}\xi)'(w)}{(\mathfrak{T}_w^{\alpha,\gamma}\xi)(w)}\right) > \delta.$$

In the next step, we determine the initial bounds for the coefficients and evaluate $|a_3 - \aleph a_2^2|$ for the class $\mathcal{R}_{\Sigma}^{m,\delta}(\alpha,\gamma)$.

Theorem 2. *Assume that $0 < \alpha \leq 1$, $0 < \gamma \leq 1$, $\alpha > \gamma$, $0 < \alpha - \gamma < 1$, $0 \leq \delta < 1$ and $2 \leq m \leq 4$. A function $f \in \mathcal{R}_{\Sigma}^{m,\delta}(\alpha,\gamma)$, then*

$$|a_2| \leq \sqrt{\frac{\rho m(1-\delta)(1+\gamma)}{(1+\alpha)}}, \tag{2.19}$$

$$|a_3| \leq \frac{\rho m(1-\delta)(1+\gamma)}{(1+\alpha)} \tag{2.20}$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} \dfrac{\rho m(1-\delta)(\gamma+1)(1-\aleph)}{(\alpha+1)} & \text{for } \aleph < \lambda, \\ \rho m(1-\delta)\left[\dfrac{(\gamma+1)}{(\alpha+1)} - \dfrac{(\gamma+2)}{2(\alpha+2)}\right] & \text{for } \lambda \leq \aleph < 2-\lambda, \\ \dfrac{\rho m(1-\delta)(\gamma+1)(\aleph-1)}{(\alpha+1)} & \text{for } \aleph \geq 2-\lambda, \end{cases} \tag{2.21}$$

where

$$\rho = \frac{(\alpha+1)(\gamma+1)}{\gamma(\alpha+3)+2} \quad \textit{and} \quad \lambda = \frac{(\alpha+1)(\gamma+2)}{2(\alpha+1)(\gamma+2)}.$$

Proof. Assume a function f is an element of $\mathcal{R}_{\Sigma}^{m,\delta}(\alpha,\gamma)$. In this case, there exist polynomials $p(z)$ and $q(w)$ within $\mathcal{P}_m(\delta)$ such that:

$$\frac{z(\mathfrak{T}_z^{\alpha,\gamma}f)'(z)}{(\mathfrak{T}_z^{\alpha,\gamma}f)(z)} = p(z) \tag{2.22}$$

and

$$\frac{w(\mathfrak{T}_w^{\alpha,\gamma}\xi)'(w)}{(\mathfrak{T}_w^{\alpha,\gamma}\xi)(w)} = q(w), \tag{2.23}$$

where the functions $p(z)$ and $q(w)$ are defined in (2.6) and (2.7). Now, we have

$$\frac{z(\mathfrak{T}_z^{\alpha,\gamma}f)'(z)}{(\mathfrak{T}_z^{\alpha,\gamma}f)(z)} = 1 + \frac{(\alpha+1)}{(\gamma+1)}a_2 z + \left[\frac{2(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3 - \frac{(\alpha+1)^2}{(\gamma+1)^2}a_2^2\right]z^2 + \cdots \tag{2.24}$$

and

$$\frac{w(\mathfrak{T}_w^{\alpha,\gamma}g)'(w)}{(\mathfrak{T}_w^{\alpha,\gamma}g)(w)} = 1-1+\frac{(\alpha+1)}{(\gamma+1)}a_2 z+\left\{\left[\frac{4(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)} - \frac{(\alpha+1)^2}{(\gamma+1)^2}\right]a_2^2 - \frac{2(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3\right\}w^2+\cdots. \tag{2.25}$$

Therefore, based on equations (2.6), (2.7), (2.24) and (2.25), we can conclude that

$$\frac{(\alpha+1)}{(\gamma+1)}a_2 = b_1, \tag{2.26}$$

$$\frac{2(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3 - \frac{(\alpha+1)^2}{(\gamma+1)^2}a_2^2 = b_2, \tag{2.27}$$

$$-\frac{(\alpha+1)}{(\gamma+1)}a_2 = c_1 \tag{2.28}$$

and

$$\left[\frac{4(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)} - \frac{(\alpha+1)^2}{(\gamma+1)^2}\right]a_2^2 - \frac{2(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3 = c_2. \tag{2.29}$$

From the analysis of equations (2.26) and (2.28), we determine that $b_1+c_1=0$. In conjunction, equations (2.27) and (2.29), yield

$$\frac{2(\alpha+1)}{(\gamma+1)}\left[\frac{2(\alpha+2)}{(\gamma+2)} - \frac{(\alpha+1)}{(\gamma+1)}\right]a_2^2 = b_2 + c_2. \tag{2.30}$$

Equation (2.30) can we written as

$$\frac{2(\alpha+1)}{(\gamma+1)\rho}a_2^2 = b_2 + c_2, \tag{2.31}$$

where

$$\rho = \frac{(\alpha+1)(\gamma+1)}{\gamma(\alpha+3)+2}.$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.31), we get

$$|a_2|^2 \leq \frac{m\rho(1-\delta)(1+\gamma)}{(\alpha+1)}. \tag{2.32}$$

From equation (2.32), we derive (2.19). By taking (2.29) away from (2.27) and utilizing (2.31), we obtain

$$\frac{4(\alpha+1)(\alpha+2)}{\rho(\gamma+1)(\gamma+2)}a_3 = \left[\frac{4(\alpha+2)}{(\gamma+2)} - \frac{(\alpha+1)}{(\gamma+1)}\right] b_2 + \frac{(\alpha+1)}{(\gamma+1)}c_2. \tag{2.33}$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.33), we get

$$\frac{(\alpha+1)(\alpha+2)}{\rho(\gamma+1)(\gamma+2)}|a_3| \leq \frac{m(1-\delta)(\alpha+2)}{(\gamma+2)}. \tag{2.34}$$

From equation (2.34), we derive (2.20). Equation (2.34) gives (2.20). For any $\aleph \in \mathbb{R}$, and based on equations (2.31) and (2.33), we get

$$a_3 - \aleph a_2^2 = \rho b_2 \left[\frac{(\gamma+1)}{(\alpha+1)} - \frac{(\gamma+2)}{4(\alpha+2)} - \frac{\aleph(\gamma+1)}{2(\alpha+1)}\right] + \rho c_2 \left[\frac{(\gamma+2)}{(4(\alpha+2)} - \frac{\aleph(\gamma+1)}{4(\alpha+2)}\right]. \tag{2.35}$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.35), we get

$$|a_3 - \aleph a_2^2| \leq \rho m(1-\delta)\left[\left|\frac{(\gamma+1)}{(\alpha+1)} - \frac{(\gamma+2)}{4(\alpha+2)} - \frac{\aleph(\gamma+1)}{2(\alpha+1)}\right| + \left|\frac{(\gamma+2)}{(4(\alpha+2)} - \frac{\aleph(\gamma+1)}{4(\alpha+2)}\right|\right]. \tag{2.36}$$

From equation (2.36), we derive (2.21). This completes the proof of Theorem 2. □

Definition 7. *For $0 < \alpha \leq 1,\ 0 < \gamma \leq 1,\ \alpha > \gamma,\ 0 < \alpha - \gamma < 1,\ 0 \leq \delta < 1$ and $2 \leq m \leq 4$. A function $f \in \Sigma$ as defined in (1.1) is considered to belong to the class $\mathcal{V}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ if the following conditions hold:*

$$1 + \frac{z(\mathfrak{T}_z^{\alpha,\gamma} f)''(z)}{(\mathfrak{T}_z^{\alpha,\gamma} f)'(z)} \in \mathcal{P}_m(\delta)$$

and

$$1 + \frac{w(\mathfrak{T}_w^{\alpha,\gamma} \xi)''(w)}{(\mathfrak{T}_w^{\alpha,\gamma} \xi)'(w)} \in \mathcal{P}_m(\delta),$$

where ξ represents the analytic continuation of f^{-1} to the domain $\mathbb{D}$.

Remark 3. (i) *By opting for $\alpha = \gamma$ the class $\mathcal{V}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ leads to the family $\mathcal{V}_{\Sigma}(m,\delta)$, which was established by Li [12] in 2020. This family consists of the functions $f \in \mathcal{A}$ that fulfill the requirements*

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_m(\delta)$$

and

$$1 + \frac{w\xi''(w)}{\xi'(w)} \in \mathcal{P}_m(\delta).$$

(ii) *By opting for $\alpha = \gamma$ and $m = 2$ the class $\mathcal{V}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ leads to the family $\mathcal{C}_{\Sigma}(\delta)$, which was*

established by Brannan and Taha [5] in 1985. This family consists of the functions $f \in \mathcal{A}$ that fulfill the requirements

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \delta$$

and

$$\Re\left(1+\frac{w\xi''(w)}{\xi'(w)}\right) > \delta.$$

(iii) *By opting for $m = 2$ the class $\mathcal{V}_{\Sigma}^{m,\delta}(\alpha,\gamma)$ leads to the family $\mathcal{C}_{\Sigma}^{\delta}(\alpha,\gamma)$, which is known as the class of bi-convex functions of order δ defined by Tremblay fractional derivative operator. This family consists of the functions $f \in \mathcal{A}$ that fulfill the requirements*

$$\Re\left(1+\frac{z(\mathfrak{T}_z^{\alpha,\gamma}f)''(z)}{(\mathfrak{T}_z^{\alpha,\gamma}f)'(z)}\right) > \delta$$

and

$$\Re\left(1+\frac{w(\mathfrak{T}_w^{\alpha,\gamma}\xi)''(w)}{(\mathfrak{T}_w^{\alpha,\gamma}\xi)'(w)}\right) > \delta.$$

In the next step, we determine the initial bounds for the coefficients and evaluate $|a_3 - \aleph a_2^2|$ for the class $\mathcal{V}_{\Sigma}^{m,\delta}(\alpha,\gamma)$.

Theorem 3. *Assume that $0 < \alpha \leq 1$, $0 < \gamma \leq 1$, $\alpha > \gamma$, $0 < \alpha - \gamma < 1$, $0 \leq \delta < 1$ and $2 \leq m \leq 4$. A function $f \in \mathcal{V}_{\Sigma}^{m,\delta}(\alpha,\gamma)$, then*

$$|a_2| \leq \sqrt{\frac{\zeta m(1-\delta)(1+\gamma)}{2(1+\alpha)}}, \tag{2.37}$$

$$|a_3| \leq \frac{\zeta m(1-\delta)(1+\gamma)}{2(1+\alpha)} \tag{2.38}$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} \dfrac{\zeta m(1-\delta)(\gamma+1)(1-\aleph)}{2(\alpha+1)} & \text{for } \aleph < \Delta, \\ \zeta m(1-\delta)\left[\dfrac{(\gamma+1)}{2(\alpha+1)} - \dfrac{(\gamma+2)}{3(\alpha+2)}\right] & \text{for } \Delta \leq \aleph < 2-\Delta, \\ \dfrac{\zeta m(1-\delta)(\gamma+1)(\aleph-1)}{2(\alpha+1)} & \text{for } \aleph \geq 2-\Delta, \end{cases} \tag{2.39}$$

where

$$\zeta = \frac{(\alpha+1)(\gamma+1)}{\gamma(\alpha+3)+2} \quad \text{and} \quad \Delta = \frac{2(\alpha+1)(\gamma+2)}{3(\gamma+1)(\alpha+2)}.$$

Proof. Assume a function f is an element of $\mathcal{V}_{\Sigma}^{m,\delta}(\alpha,\gamma)$. In this case, there exist polynomials $p(z)$ and $q(w)$ within $\mathcal{P}_m(\delta)$ such that:

$$1+\frac{z(\mathfrak{T}_z^{\alpha,\gamma}f)''(z)}{(\mathfrak{T}_z^{\alpha,\gamma}f)'(z)} = p(z) \tag{2.40}$$

and

$$1+\frac{w(\mathfrak{T}_w^{\alpha,\gamma}g)''(w)}{(\mathfrak{T}_w^{\alpha,\gamma}g)'(w)}=q(w), \tag{2.41}$$

where the functions $p(z)$ and $q(w)$ are defined in (2.6) and (2.7). Now, we have

$$1+\frac{z(\mathfrak{T}_z^{\alpha,\gamma}f)''(z)}{(\mathfrak{T}_z^{\alpha,\gamma}f)'(z)}=1+\frac{2(\alpha+1)}{(\gamma+1)}a_2z+\left[\frac{6(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3-\frac{4(\alpha+1)^2}{(\gamma+1)^2}a_2^2\right]z^2+\cdots \tag{2.42}$$

and

$$\begin{aligned}1+\frac{w(\mathfrak{T}_w^{\alpha,\gamma}g)''(w)}{(\mathfrak{T}_w^{\alpha,\gamma}g)'(w)}=&1-\frac{2(\alpha+1)}{(\gamma+1)}a_2z\\&+\left\{\left[\frac{12(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}-\frac{4(\alpha+1)^2}{(\gamma+1)^2}\right]a_2^2-\frac{6(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3\right\}w^2+\cdots.\end{aligned} \tag{2.43}$$

Therefore, based on equations (2.6), (2.7), (2.42) and (2.43), we can conclude that

$$\frac{2(\alpha+1)}{(\gamma+1)}a_2=b_1, \tag{2.44}$$

$$\frac{6(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3-\frac{4(\alpha+1)^2}{(\gamma+1)^2}a_2^2=b_2, \tag{2.45}$$

$$-\frac{2(\alpha+1)}{(\gamma+1)}a_2=c_1 \tag{2.46}$$

and

$$\left[\frac{12(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}-\frac{4(\alpha+1)^2}{(\gamma+1)^2}\right]a_2^2-\frac{6(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}a_3=c_2. \tag{2.47}$$

From the analysis of equations (2.44) and (2.46), we determine that $b_1+c_1=0$. In conjunction, equations (2.45) and (2.47), yield

$$\frac{4(\alpha+1)}{\zeta(\gamma+1)}a_2^2=b_2+c_2, \tag{2.48}$$

where

$$\zeta=\frac{\gamma(\alpha+4)+(2-\alpha)}{(\gamma+1)(\gamma+2)}.$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.48), we get

$$|a_2|^2\le\frac{\zeta m(1-\delta)(\gamma+1)}{2(\alpha+1)}. \tag{2.49}$$

From equation (2.49), we derive (2.37). By taking (2.47) away from (2.45) and utilizing (2.48), we obtain

$$a_3=\zeta\left[\frac{(\gamma+1)}{2(\alpha+1)}-\frac{(\gamma+2)}{6(\alpha+2)}\right]b_2+\frac{\zeta(\gamma+2)}{6(\alpha+2)}c_2. \tag{2.50}$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.50), we derive equation (2.38). For any $\aleph \in \mathbb{R}$, utilizing equations (2.48) and (2.50), we have

$$a_3 - \aleph a_2^2 = \zeta b_2 \left[\frac{(\gamma+1)}{2(\alpha+1)} - \frac{(\gamma+2)}{6(\alpha+2)} - \frac{(\gamma+1)}{4(\alpha+1)}\aleph\right] + \zeta c_2 \left[\frac{(\gamma+2)}{6(\alpha+2)} - \frac{(\gamma+1)}{4(\alpha+1)}\aleph\right]. \quad (2.51)$$

By applying the triangle inequality along with equations (1.7) and (1.8) in equation (2.51), we have

$$|a_3 - \aleph a_2^2| \leq \zeta m(1-\delta)\left[\left|\frac{(\gamma+1)}{2(\alpha+1)} - \frac{(\gamma+2)}{6(\alpha+2)} - \frac{(\gamma+1)}{4(\alpha+1)}\aleph\right| + \left|\frac{(\gamma+2)}{6(\alpha+2)} - \frac{(\gamma+1)}{4(\alpha+1)}\aleph\right|\right]. \quad (2.52)$$

From equation (2.52), we derive (2.39). This completes the proof of Theorem 3. □

3. **Corollaries and Consequences**

When we set $\alpha = \gamma$ in Theorem 1, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{H}_\Sigma(m,\delta)$.

Corollary 1. *For the range of* $0 \leq \delta < 1$ *and* $2 \leq m \leq 4$, *a function* $f(z)$ *defined by* (1.1) *is classified within the class* $\mathcal{H}_\Sigma(m,\delta)$, *then*

$$|a_2| \leq \sqrt{\frac{m(1-\delta)}{3}},$$

$$|a_3| \leq \frac{m(1-\delta)}{3}$$

and

$$|a_3 - \aleph a_2^2| \leq \begin{cases} \dfrac{m(1-\delta)(1-\aleph)}{3} & \text{for } \aleph < 0, \\ \dfrac{m(1-\delta)}{3} & \text{for } 0 \leq \aleph < 2, \\ \dfrac{m(1-\delta)(\aleph-1)}{3} & \text{for } \aleph \geq 2. \end{cases}$$

When we set $\alpha = \gamma$ and $m = 2$ in Theorem 1, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{H}_\Sigma(\delta)$.

Corollary 2. *For the range of* $0 \leq \delta < 1$, *a function* $f(z)$ *defined by* (1.1) *is classified within the class* $\mathcal{H}_\Sigma(\delta)$, *then*

$$|a_2| \leq \sqrt{\frac{2(1-\delta)}{3}},$$

$$|a_3| \leq \frac{2(1-\delta)}{3}$$

and

$$|a_3 - \aleph a_2^2| \leq \begin{cases} \dfrac{2(1-\delta)(1-\aleph)}{3} & \text{for } \aleph < 0, \\ \dfrac{2(1-\delta)}{3} & \text{for } 0 \leq \aleph < 2, \\ \dfrac{2(1-\delta)(\aleph-1)}{3} & \text{for } \aleph \geq 2. \end{cases}$$

When we set $m = 2$ in Theorem 1, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{H}_{\Sigma}^{\delta}(\alpha, \gamma)$.

Corollary 3. *Assume that* $0 < \alpha \leq 1,\ 0 < \gamma \leq 1,\ \alpha > \gamma,\ 0 < \alpha - \gamma < 1$ *and* $0 \leq \delta < 1$, *a function* $f(z)$ *defined by* (1.1) *is classified within the class* $\mathcal{H}_{\Sigma}^{\delta}(\alpha, \gamma)$, *then*

$$|a_2| \leq \sqrt{\frac{2(1-\delta)(\gamma+1)(\gamma+2)}{3(\alpha+1)(\alpha+2)}},$$

$$|a_3| \leq \frac{2(1-\delta)(\gamma+1)(\gamma+2)}{3(\alpha+1)(\alpha+2)}$$

and

$$|a_3 - \aleph a_2^2| \leq \begin{cases} \dfrac{2(1-\delta)(\gamma+1)(\gamma+2)(1-\aleph)}{3(\alpha+1)(\alpha+2)} & \text{for } \aleph < 0, \\ \dfrac{2(1-\delta)(\gamma+1)(\gamma+2)}{3(\alpha+1)(\alpha+2)} & \text{for } 0 \leq \aleph < 2, \\ \dfrac{2(1-\delta)(\gamma+1)(\gamma+2)(\aleph-1)}{3(\alpha+1)(\alpha+2)} & \text{for } \aleph \geq 2. \end{cases}$$

When we set $\alpha = \gamma$ in Theorem 2, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{R}_{\Sigma}(m, \delta)$.

Corollary 4. *For the range of* $0 \leq \delta < 1$ *and* $2 \leq m \leq 4$, *a function* $f(z)$ *defined by* (1.1) *is classified within the class* $\mathcal{R}_{\Sigma}(m, \delta)$, *then*

$$|a_2| \leq \sqrt{m(1-\delta)},$$

$$|a_3| \leq m(1-\delta)$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} m(1-\delta)(1-\aleph) & \text{for } \aleph < \dfrac{1}{2}, \\ \dfrac{m(1-\delta)}{2} & \text{for } \dfrac{1}{2} \leq \aleph < \dfrac{3}{2}, \\ m(1-\delta)(\aleph-1) & \text{for } \aleph \geq \dfrac{3}{2}. \end{cases}$$

When we set $\alpha = \gamma$ and $m = 2$ in Theorem 2, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{S}_{\Sigma}^{*}(\delta)$.

Corollary 5. *For the range of* $0 \leq \delta < 1$, *a function* $f(z)$ *defined by* (1.1) *is classified within the class* $\mathcal{S}_{\Sigma}^{*}(\delta)$, *then*

$$|a_2| \leq \sqrt{2(1-\delta)},$$

$$|a_3| \leq 2(1-\delta)$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} 2(1-\delta)(1-\aleph) & \text{for } \aleph < \dfrac{1}{2}, \\ \dfrac{2(1-\delta)}{2} & \text{for } \dfrac{1}{2} \leq \aleph < \dfrac{3}{2}, \\ 2(1-\delta)(\aleph-1) & \text{for } \aleph \geq \dfrac{3}{2}. \end{cases}$$

When we set $m = 2$ in Theorem 2, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{R}_{\Sigma}^{\delta}(\alpha, \gamma)$.

Corollary 6. *Assume that* $0 < \alpha \leq 1$, $0 < \gamma \leq 1$, $\alpha > \gamma$, $0 < \alpha - \gamma < 1$ *and* $0 \leq \delta < 1$, *a function* $f(z)$ *defined by* (1.1) *is classified within the class* $\mathcal{R}_{\Sigma}^{\delta}(\alpha, \gamma)$, *then*

$$|a_2| \leq \sqrt{\frac{2\rho(1-\delta)(1+\gamma)}{(1+\alpha)}},$$

$$|a_3| \leq \frac{2\rho(1-\delta)(1+\gamma)}{(1+\alpha)}$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} \dfrac{2\rho(1-\delta)(\gamma+1)(1-\aleph)}{(\alpha+1)} & \text{for } \aleph < \lambda, \\ 2\rho(1-\delta)\left[\dfrac{(\gamma+1)}{(\alpha+1)} - \dfrac{(\gamma+2)}{2(\alpha+2)}\right] & \text{for } \lambda \leq \aleph < 2-\lambda, \\ \dfrac{2\rho(1-\delta)(\gamma+1)(\aleph-1)}{(\alpha+1)} & \text{for } \aleph \geq 2-\lambda, \end{cases}$$

where

$$\rho = \frac{(\alpha+1)(\gamma+1)}{\gamma(\alpha+3)+2} \quad \text{and} \quad \lambda = \frac{(\alpha+1)(\gamma+2)}{2(\alpha+1)(\gamma+2)}.$$

When we set $\alpha = \gamma$ in Theorem 3, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{V}_{\Sigma}(m, \delta)$.

Corollary 7. *For the range of* $0 \leq \delta < 1$ *and* $2 \leq m \leq 4$, *a function* $f(z)$ *defined by* (1.1) *is classified within the class* $\mathcal{V}_{\Sigma}(m, \delta)$, *then*

$$|a_2| \leq \sqrt{\frac{m(1-\delta)}{2}},$$

$$|a_3| \leq \frac{m(1-\delta)}{2}$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} \dfrac{m(1-\delta)(1-\aleph)}{2} & \text{for } \aleph < \dfrac{2}{3}, \\ \dfrac{m(1-\delta)}{6} & \text{for } \dfrac{2}{3} \leq \aleph < \dfrac{4}{3}, \\ \dfrac{m(1-\delta)(\aleph-1)}{2} & \text{for } \aleph \geq \dfrac{4}{3}. \end{cases}$$

When we set $\alpha = \gamma$ and $m = 2$ in Theorem 3, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{C}_\Sigma(\delta)$.

Corollary 8. *For the range of $0 \leq \delta < 1$, a function $f(z)$ defined by (1.1) is classified within the class $\mathcal{C}_\Sigma(\delta)$, then*

$$|a_2| \leq \sqrt{1-\delta},$$
$$|a_3| \leq 1 - \delta$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} (1-\delta)(1-\aleph) & \text{for } \aleph < \dfrac{2}{3}, \\ \dfrac{(1-\delta)}{3} & \text{for } \dfrac{2}{3} \leq \aleph < \dfrac{4}{3}, \\ (1-\delta)(\aleph-1) & \text{for } \aleph \geq \dfrac{4}{3}. \end{cases}$$

When we set $m = 2$ in Theorem 3, we arrive at the following conclusion regarding functions that fall within the class $\mathcal{V}_\Sigma^\delta(\alpha, \gamma)$.

Corollary 9. *Assume that $0 < \alpha \leq 1$, $0 < \gamma \leq 1$, $\alpha > \gamma$, $0 < \alpha - \gamma < 1$ and $0 \leq \delta < 1$, a function $f(z)$ defined by (1.1) is classified within the class $\mathcal{V}_\Sigma^\delta(\alpha, \gamma)$, then*

$$|a_2| \leq \sqrt{\frac{\zeta(1-\delta)(1+\gamma)}{(1+\alpha)}},$$

$$|a_3| \leq \frac{\zeta(1-\delta)(1+\gamma)}{(1+\alpha)}$$

and

$$|a_3 - \aleph a_2| \leq \begin{cases} \dfrac{\zeta(1-\delta)(\gamma+1)(1-\aleph)}{(\alpha+1)} & \text{for } \aleph < \Delta, \\ \zeta(1-\delta)\left[\dfrac{2(\gamma+1)}{(\alpha+1)} - \dfrac{2(\gamma+2)}{3(\alpha+2)}\right] & \text{for } \Delta \leq \aleph < 2-\Delta, \\ \dfrac{\zeta(1-\delta)(\gamma+1)(\aleph-1)}{(\alpha+1)} & \text{for } \aleph \geq 2-\Delta, \end{cases}$$

where

$$\zeta = \frac{(\alpha+1)(\gamma+1)}{\gamma(\alpha+3)+2} \quad \text{and} \quad \Delta = \frac{2(\alpha+1)(\gamma+2)}{3(\gamma+1)(\alpha+2)}.$$

Remark 4. (i) *Corollary 1, verifies the results obtained by Sharma et al.* [23].

(ii) *Corollary 2, verifies the bound of* $|a_2|$ *and improves the bound of* $|a_3|$ *obtained by Srivastava et al.* [24].

(iii) *Corollary 4 and Corollary 7, verifies the bound of* $|a_2|$ *and improves the bound of* $|a_3|$ *obtained by Li et al.* [12].

(iv) *Corollary 4 and Corollary 7, verifies the bound of* $|a_2|$ *and* $|a_3|$ *obtained by Sharma et al.* [23].

(v) *Corollary 5 and Corollary 8, verifies the bound of* $|a_2|$ *and* $|a_3|$ *obtained by Brannan and Taha* [5].

Concluding remarks and observations

In this research, three new subclasses of bi-univalent functions with bounded boundary rotation are introduced, based on Tremblay fractional derivatives. The initial two non-sharp bounds for the Taylor–Maclaurin coefficients are derived for these new function classes. Furthermore, the well-known Fekete-Szegö inequality is derived for these function classes. In comparison to earlier published findings in the literature, several improved results are noted. Moreover, there are more corollaries that can be expressed regarding the choice of parameters in the context of Tremblay fractional derivatives.

With the presence of numerous differential operators, it is possible to utilize alternative operators for the classes considered here or their corresponding subclasses in $\mathcal{S}$. Additionally, similar outcomes can be derived for various other notable special functions found in the literature.

References

[1] I. Aldawish, P. Sharma, S.M. El-Deeb, M.R. Almutiri, and S. Sivasubramanian, Initial coefficient bounds for certain new subclasses of bi-univalent functions involving Mittag–Leffler function with bounded boundary rotation, Symmetry, **16**, (2024), 971.

[2] B. S. T. Alkahtani, P. Goswami and T. Bulboacă, Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions, Miskolc Math. Notes **17** (2016), no. 2, 739–748.

[3] T. Al-Hawary, Sufficient conditions for analytic functions defined by Frasin differential operator, Stud. Univ. Babeş-Bolyai Math. **66** (2021), no. 2, 353–359.

[4] D.A. Brannan and J. G. Clunie, Aspects of contemporary complex analysis, Academic Press, Inc., London, 1980.

[5] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in Mathematical analysis and its applications (Kuwait, 1985), 53–60, KFAS Proc. Ser., 3, Pergamon, Oxford.

[6] D. Breaz, P. Sharma, S. Sivasubramanian, and S.M. El-Deeb, On a new class of bi-close-to-convex functions with bounded boundary botation. Mathematics, **11(20)** (2023), 4376.

[7] P. L. Duren, Univalent functions, Grundlehren der mathematischen Wissenschaften, 259, Springer, New York, 1983

[8] A. W. GOODMAN, Univalent Functions, Mariner, Tampa, Florida, 1983.

[9] R. W. Ibrahim and J. M. Jahangiri, Boundary fractional differential equation in a complex domain, Bound. Value Probl. **2014**, 2014:66, 11 pp.

[10] J. Jothibasu, Certain subclasses of bi-univalent functions defined by Sălăgean operator, Electron. J. Math. Anal. Appl. **3** (2015), no. 1, 150–157.

[11] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. **18** (1967), 63–68.

[12] Y.Li, K. Vijaya, G. Murugusundaramoorthy and Huo Tang, On new subclasses of bi-starlike functions with bounded boundary rotation, AIMS Math. **5**(4) (2020), 3346–3356.

[13] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal. **32** (1969), 100–112.
[14] S. Owa, On the distortion theorems. I, Kyungpook Math. J. **18** (1978), no. 1, 53–59.
[15] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math. **39** (1987), no. 5, 1057–1077.
[16] V. Paatero, Über Gebiete von beschrankter Randdrehung, Ann. Acad. Sci. Fenn. Ser. A., 37 (1933), p. 9.
[17] K. S. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math. **31** (1975/76), no. 3, 311–323.
[18] B. Pinchuk, Functions of bounded boundary rotation, Israel J. Math. **10** (1971), 6–16.
[19] M. S. Robertson, Certain classes of starlike functions, Michigan Mathematical Journal, **32** (1985), no. 2, 135–140.
[20] M. S. Robertson, On the theory of univalent functions, Ann. of Math. (2) **37** (1936), no. 2, 374–408.
[21] S. Sivasubramanian, R. Sivakumar, S. Kanas and S. Kim, Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions, Ann. Polon. Math. **113** (2015), no. 3, 295–304.
[22] T.G. Shaba, On some new subclass of bi-univalent functions associated with Opoola differential operator. Open J. Math. Anal. **4** (2020), 74–79.
[23] P. Sharma, S. Sivasubramanian and N. E. Cho, Initial coefficient bounds for certain new subclasses of bi-univalent functions with bounded boundary rotation, AIMS Math. **8** (2023), no. 12, 29535–29554.
[24] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. **23** (2010), no. 10, 1188–1192.
[25] H. M. Srivastava and S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, Nagoya Math. J. **106** (1987), 1–28.
[26] H. M. Srivastava, S.S. Eker, S.G. Hamidi and J.M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Bull. Iranian Math. Soc. **44** (2018), no. 1, 149–157.
[27] H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in it Univalent functions, fractional calculus, and their applications (Koriyama, 1988), 329–354, Ellis Horwood Ser. Math. Appl, Horwood, Chichester.
[28] S. Sümer Eker and B. Şeker, On subclasses of bi-starlike functions defined by Tremblay fractional derivative operator, Konuralp J. Math. **6** (2018), no. 2, 226–232.
[29] T.S. Taha, Topics in Univalent Function Theory, Ph.D. Thesis, University of London, 1981.