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BOUNDS FOR DEGREE-BASED TOPOLOGICAL INDICES VIA TENSOR PRODUCT OF GRAPH OPERATIONS

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Abstract. In this study, we investigate the behavior of several degree-based topological indices, namely the Randić index, the sum-connectivity index, and the first and second multiplicative Zagreb indices under the tensor product of specific graph operations. We establish new bounds for these indices to better understand molecular structure–property relationships.

1. Introduction

Chemical graph theory, a prominent area of mathematical chemistry, employs graph-theoretical principles to represent and analyze molecular structures. Atoms are represented as vertices, while chemical bonds correspond to edges, which together form a molecular graph. Several structural descriptors known as topological indices are computed to encode this structural information numerically [6, 7, 8, 20]. These indices are essential in quantitative structure–property relationship (QSPR) and quantitative structure–activity relationship (QSAR) studies.

Many researchers have focused on various graph operations and their impact on these indices [13, 14, 15, 18, 21, 22, 23]. Deepika [4] studied the VL index on the F-sum of graphs. For a graph

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$M = (V, E)$, the degree of a vertex $v \in V$ is denoted by $d(v)$. Inspired by this work, we explore a few degree-based topological indices—namely the Randić index, sum-connectivity index and multiplicative Zagreb indices—for the tensor product of F-sum graphs.

The tensor product $M \times N$ of simple graphs M and N is the graph with vertex set $V(M) \times V(N)$, in which (m, n) and (m', n') are adjacent if and only if $mm' \in E(M)$ and $nn' \in E(N)$. If $A(M)$ and $A(N)$ denote the adjacency matrices of M and N, respectively, then $A(M \times N) = A(M) \bigotimes A(N)$, where $\bigotimes$ represents the Kronecker product [1, 14, 5, 16].

This product is commutative and associative. Its degree function is multiplicative, $d(m, n) = d(m)d(n)$, which distinguishes it from additive products such as the Cartesian product.

For connected graphs M and N, we recall the definitions of graph operations as follows [2, 3, 19, 9, 15]:

- $S(M)$: The subdivision graph obtained by inserting an extra vertex into each edge of M.
- $R(M)$: The graph obtained by adding a new vertex to each edge of M and joining it to the two end vertices of that edge.
- $Q(M)$: The graph obtained by inserting a new vertex into each edge of M and joining the new vertices that lie on adjacent edges of M.
- $T(M)$: The total graph, whose vertex set is $V(M) \cup E(M)$ and in which two vertices are adjacent if they are adjacent vertices, adjacent edges, or an incident vertex–edge pair of M.

2. Degree-Based Topological Indices

Several well-known indices are defined as follows:

- **The Randić index.** In 1975, Milan Randić [24] introduced the topological index $R(M)$ of a graph M, defined as the sum of the weights $[d(m)d(n)]^{-1/2}$ over the edges mn of M, i.e.

$$R(M) = \sum_{mn \in E(M)} \frac{1}{\sqrt{d(m)d(n)}}.$$

- **The sum-connectivity index.** The sum-connectivity index was introduced by B. Zhou and N. Trinajstić in 2009 [23], and is defined as

$$SCI(M) = \sum_{mn \in E(M)} \frac{1}{\sqrt{d(m) + d(n)}}.$$

- **The multiplicative Zagreb indices** [11, 17, 19, 25] are modified versions of the Zagreb indices. The first and second multiplicative Zagreb indices are defined as

$$\Pi_1(M) = \prod_{m \in V(M)} d(m)^2$$

and

$$\Pi_2(M) = \prod_{mn \in E(M)} d(m)d(n).$$

3. Main Results

In this section, we establish bounds for the Randić index, sum-connectivity index, and multiplicative Zagreb index for the tensor product of F-sum graphs. Let M and N be simple connected graphs. We derive the following results for the various degree-based indices.

Let $|V(M_1)| = \eta_1, |V(M_2)| = \eta_2, |V(N_1)| = \eta_1', |V(N_2)| = \eta_2', |E(M_1)| = \mu_1, |E(M_2)| = \mu_2, |E(N_1)| = \mu_1', |E(N_2)| = \mu_2'$.

Lemma 1 [12]: Let M_1 and M_2 be graphs of order η_1 and η_2 and size μ_1 and μ_2, respectively. Then:

a. $|V(M_1 \times M_2)| = |V(M_1)|.|V(M_2)|$ and $|E(M_1 \times M_2)| = 2|E(M_1)|.|E(M_2)|$.
b. $d_{M_1 \times M_2}(x, y) = d_{M_1}(x).d_{M_2}(y)$.
c. The tensor product is commutative and associative.

Theorem 1: Let M and N be two connected graphs, then the second multiplicative Zagreb index satisfies the inequality subject to degree conditions defined by their factor graphs

$$\gamma_1 \leq \Pi_2(M \times N) \leq \gamma_2$$

where,

$$\gamma_1 = 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2')(\Delta_M\Delta_N)^2$$
$$\gamma_2 = 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2')(\delta_M\delta_N)^2$$

Proof: Let M and N be graphs with vertex sets x_1, x_2, x_3,....,$x_{n_1(\eta_1'+\mu_1')}$ and y_1,y_2, y_3,....,$y_{n_2(\eta_2'+\mu_2')}$ respectively. Using the definition, we have

$$\Pi_2(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} d_{(M\times N)}(x_i,y_j)d_{(M\times N)}(x_k,y_l)$$
$$\Pi_2(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} d_{(M\times N)}(x_i,y_j)d_{(M\times N)}(x_k,y_l).....(1)$$

using Lemma 1(b),

$$d_{(M\times N)}(x_i,y_j)d_{(M\times N)}(x_k,y_l) = (d_M(x_i)d_N(y_j))(d_M(x_k).d_N(y_l))$$

since $\forall x \in V(M), d_M(x) \leq \Delta_M$, $d_M(x) \geq \delta_M$. Hence

$$\begin{aligned} d_{(M\times N)}(x_i,y_j)d_{(M\times N)}(x_k,y_l) &\leq (\Delta_M.\Delta_N)(\Delta_M.\Delta_N) \\ &\leq (\Delta_M\Delta_N)^2.....(2) \end{aligned}$$

from equations (1) and (2),

$$\begin{aligned} \Pi_2(M \times N) &= \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} d_{(M\times N)}(x_i,y_j)d_{(M\times N)}(x_k,y_l) \\ &= \sum_{x_i,x_k\in E(M)} \sum_{y_jy_l\in E(N)} (d_{(M\times N)}(x_i,y_j)d_{(M\times N)}(x_k,y_l) \\ &\leq 2|E(M)||E(N)|(\Delta_M\Delta_N)^2 \end{aligned}$$

where $|E(M)| = \mu_1\eta_1' + 2\eta_1\mu_1'$, $|E(N)| = \eta_2\mu_2' + 2\mu_2\eta_2'$, $\Delta_M = \Delta_{M_1} + \Delta_{N_1}$ and $\Delta_N = \Delta_{M_2} + \Delta_{N_2}$

$$\begin{aligned} \Pi_2(M \times N) &\leq 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2')(\Delta_M\Delta_N) \\ \gamma_1 &\leq 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2').(\Delta_{M_2} + \Delta_{N_2})(\Delta_{M_1} + \Delta_{N_1}) \\ \gamma_2 &\geq 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2').(\delta_{M_2} + \delta_{N_2})(\delta_{M_1} + \delta_{N_1}) \\ \gamma_1 &\leq 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2').(\Delta_M + \Delta_N)(\Delta_M + \Delta_N) \\ \gamma_2 &\geq 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2').(\delta_M + \delta_N)(\delta_M + \delta_N) \\ \gamma_1 &= 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2').(\Delta_M.\Delta_N)^2 \\ \gamma_2 &= 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2').(\delta_M.\delta_N)^2. \end{aligned}$$

Theorem 2: Let $M = M_1 +_R N_1$ and $N = M_2 +_R N_2$ then for

$$\gamma_1 \leq \Pi_2(M \times N) \leq \gamma_2$$

where,

$$\gamma_1 = 2(\mu_1\eta_1' + 3\eta_1'\mu_1)(\eta_2\mu_2' + 3\mu_2\eta_2').(\Delta_M\Delta_N)^2$$

$$\gamma_2 = 2(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')(\delta_M\delta_N)^2$$

Proof: Let M and N be graphs with vertex sets x_1, x_2, x_3,....,$x_{n_1(\eta_1+\mu_1)}$ and y_1,y_2, y_3,....,$y_{n_2(\eta_2'+\mu_2')}$ respectively, then by the definition we have,

$$\Pi_2(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} d_{(M\times N)}(x_i, y_j).d_{(M\times N)}(x_k, y_l)$$

$|E(M_1)| = \mu_1\eta_1' + 3\eta_1\mu_1'$, $|E(M_2) = \eta_2\mu_2' + 3\mu_2\eta_2'$, $\Delta_M = \Delta_{M_1} + \Delta_{N_1}$ and $\Delta_N = \Delta_{M_2} + \Delta_{N_2}$

$\Pi_2(M \times N) \leq 2|E(M)||E(N)|(\Delta_M\Delta_N)^2$.

$\Pi_2(M \times N) \leq 2(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')(\Delta_{M_1} + \Delta_{N_1})(\Delta_{M_2} + \Delta_{N_2})$

$$\gamma_1 = 2(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2').(\Delta_M\Delta_N)^2$$

$$\gamma_2 = 2(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')(\delta_M\delta_N)^2$$

Theorem 3: Let $M = M_1 +_T N_1$ and $N = M_2 +_T N_2$. Then $\gamma_1 \leq \Pi_2(M \times N) \leq \gamma_2$ where

$$\gamma_1 = 2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2').(\Delta_M\Delta_N)^2$$

$$\gamma_2 = 2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')(\delta_M\delta_N)^2$$

Proof: Let M and N be graphs with vertex sets x_1, x_2, x_3,...,$x_{n_1(\eta_1+\mu_1)}$ and y_1,y_2, y_3,...,$y_{n_2(\eta_2+\mu_2)}$ respectively, then by the definition, we have

$$\Pi_2(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} d_{(M\times N)}(x_i, y_j)d_{(M\times N)}(x_k, y_l)$$

$|E(M)| = \mu_1\eta_1' + 4\eta_1\mu_1'$, $|E(N)| = \eta_2\mu_2' + 4\mu_2\eta_2'$, $\Delta_M = \Delta_{M_1}\Delta_{N_1}$

and $\Delta_N = \Delta_{M_2}\Delta_{N_2}$

$$\Pi_2(M \times N) \leq 2|E(M)||E(N)|(\Delta_M\Delta_N)^2.$$
$$\Pi_2(M \times N) \leq 2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')(\Delta_M\Delta_N)$$
$$\gamma_1 \leq 2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')(\Delta_{M_2}\Delta_{N_2})(\Delta_{M_1}\Delta_{N_1}$$
$$\gamma_2 \geq 2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2').(\delta_{M_1}\delta_{N_1})(\delta_{M_2}\delta_{N_2})$$
$$\gamma_1 = 2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')(\Delta_M\Delta_N)^2$$
$$\gamma_2 = 2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2').(\delta_M\delta_N)^2$$

Corollary 1: For special classes of graphs such as path (P_n), cycle (C_n) and complete graphs (K_n) explicit expressions for bounds are derived, demonstrating consistent proportional relationships between tensor products and their component graphs. For the simple connected graphs of $M_1 = P_3$, $N_1 = P_3$, $M_2 = K_2$ and $N_2 = K_2$.

- The Second Multiplicative Zagreb Index:

$$|E(M)| = 4096, |E(N)| = 65.536, \delta_1 = 1, \delta_2 = 1, \Delta_1 = 4, \Delta_2 = 4$$

- $2.684 \times 10^8 \leq \Pi_2(M +_s N) \leq 4.504 \times 10^{15}$.
- $4.25 \leq \Pi_2(M +_R N) \leq 17$.
- $(d_{min})^{2mt} \leq \Pi_2(M +_T N) \leq (d_{max})^{2mt}$
- $1 \leq \Pi_2(M +_T N) \leq 2^{280}$

Theorem 4: The sum-connectivity index satisfies bounds $SCI(M \times N) \leq f(\Delta_M, \Delta_N)$ where Δ denotes the maximum degree in the respective graphs.

Let $M = M_1 +_S M_1$ and $N = M_2 +_S N_2$. Then

$$\gamma_1 \leq SCI(M \times N) \leq \gamma_2$$

where

$$\gamma_1 = \frac{\sqrt{2}(\mu_1\eta_1 + 2\eta_1\mu_1)(\eta_2\mu_2 + 2\mu_2\eta_2)}{\sqrt{(\Delta_M + \Delta_N)^2}}$$
$$\gamma_2 = \frac{\sqrt{2}(\mu_1\eta_1 + 2\eta_1\mu_1)(\eta_2\mu_2 + 2\mu_2\eta_2)}{\sqrt{(\delta_M + \delta_N)^2}}.$$

Proof: Let M and N be graphs with vertex sets x_1, x_2, x_3,...,$x_{n_1(\eta_1+\mu_1)}$ and y_1,y_2, y_3,...,$y_{n_2(\eta_2+\mu_2)}$ respectively. Then by the definition we

have,

$$SCI(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} \frac{1}{\sqrt{(d_{(M\times N)}(x_i,y_j) + d_{(M\times N)}(x_k,y_l)}}$$

$$SCI(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} (d_{(M\times N)}(x_i,y_j) + d_{(M\times N)}(x_k,y_l)^{-1/2}$$

$|E(M)| = \mu_1\eta_1' + 2\eta_1\mu_1'$, $|E(N)| = \eta_2\mu_2' + 2\mu_2\eta_2'$, $\Delta_M = \Delta_{M_1} + \Delta_{N_1}$ and $\Delta_N = \Delta_{M_2} + \Delta_{N_2}$

$$SCI(M \times N) \leq 2|E(M)||E(N)|(\Delta_M + \Delta_N)^2.$$

$$\gamma_1 \leq \frac{\sqrt{2}(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2')}{(\Delta_{M_1} + \Delta_{N_1}) + (\Delta_{M_2} + \Delta_{N_2})}$$

$$\gamma_2 \geq \frac{\sqrt{2}(\mu_1\eta_1 + 2\eta_1\mu_1)(\eta_2\mu_2 + 2\mu_2\eta_2)}{(\delta_{M_1} + \delta_{N_1}) + (\delta_{M_2} + \delta_{N_2})}$$

$$\gamma_1 = \frac{\sqrt{2}(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2')}{(\Delta_M + \Delta_N)}$$

$$\gamma_2 = \frac{\sqrt{2}(\mu_1\eta_1 + 2\eta_1\mu_1)(\eta_2\mu_2 + 2\mu_2\eta_2)}{(\delta_M + \delta_N)}$$

Theorem 5: Let $M = M_1 +_T N_1$ and $N = M_2 +_T N_2$ then $\gamma_1 \leq SCI(M \times N) \leq \gamma_2$ where,

$$\gamma_1 = \frac{\sqrt{2}(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{(\Delta_M + \Delta_N)}$$

$$\gamma_2 = \frac{\sqrt{2}(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{(\delta_M + \delta_N)}$$

Proof: Let M and N be two graphs with vertex sets x_1, x_2, x_3,...,$x_{n_1(\eta_1+\mu_1)}$ and y_1,y_2, y_3,...,$y_{n_2(\eta_2+\mu_2)}$, respectively. Then by the definition,

$$\leq 2|E(M)||E(N)|(\Delta_M\Delta_N)^2.$$

$$SCI(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} \frac{1}{\sqrt{(d_{(M\times N)}(x_i,y_j) + d_{(M\times N)}(x_k,y_l)}}$$

$|E(M)| = \mu_1\eta_1' + 4\eta_1\mu_1'$, $|E(N)| = \eta_2\mu_2' + 4\mu_2\eta_2'$, $\Delta_M = \Delta_{M_1} + \Delta_{N_1}$ and $\Delta_N = \Delta_{M_2} + \Delta_{N_2}$.

$$SCI(M \times N) \leq 2|E(M)||E(N)|(\Delta_M + \Delta_N)^2.$$

$$SCI(M \times N) \leq \frac{\sqrt{2}(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{\sqrt{(\Delta_{M_1} + \Delta_{N_1}) + (\Delta_{M_2} + \Delta_{N_2})}}$$

$$\gamma_1 = \frac{\sqrt{2}(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{\sqrt{(\Delta_M + \Delta_N)}}$$

$$\gamma_2 = \frac{\sqrt{2}(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{\sqrt{(\delta_M + \delta_N)}}$$

Corollary 2: Consider the graphs $M_1 = P_3$, $N_1 = K_3$, $M_2 = P_4$ and $N_2 = C_4$.

- $\frac{m_1 m_2}{\sqrt{2}\Delta_1\Delta_2} \leq SCI(M_1, M_2) \leq \frac{m_1 m_2}{\sqrt{2}\delta_1\delta_2}$ where,

  $m_1 = 5$, $m_2 = 7$, $m_1 m_2 = 35$, $\delta_1 = \delta_2 = 1$,$\Delta_1 = \Delta_2 = 4$.
- $5.75 \leq SCI(M +_s N) \leq 5.79$
- $3 \leq SCI(M +_R N) \leq 12$
- $6.188 \leq SCI(M +_T N) \leq 24.748$

Theorem 6: Let $M = M_1 +_R N_1$ and $N = M_2 +_R N_2$ then $\gamma_1 \leq R(M \times N) \leq \gamma_2$ where,

$$\gamma_1 = \frac{(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')}{(\Delta_M\Delta_N)}$$

$$\gamma_2 = \frac{(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')}{(\delta_M\delta_N)}$$

Proof: Let M and N be graphs with vertex sets x_1, x_2, x_3,...,$x_{n_1(\eta_1+\mu_1)}$ and y_1,y_2, y_3,...,$y_{n_2(\eta_2+\mu_2)}$ respectively, then by the definition,

$$R(M \times N) = \sum_{(x_i,y_j)(x_k,y_l) \in E(M\times N)} \frac{1}{\sqrt{(d_{(M\times N)}(x_i, y_j)d_{(M\times N)}(x_k, y_l)}}$$

$$R(M\times N) = \sum_{(x_i,y_j)(x_k,y_l) \in E(M\times N)} d_{(M\times N)}(x_i, y_j)d_{(M\times N)}(x_k, y_l)^{-1/2} ...(3)$$

$|E(M)| = \mu_1\eta_1' + 3\eta_1\mu_1'$, $|E(N)| = \eta_2\mu_2' + 3\mu_2\eta_2'$, $\Delta_M = \Delta_{M_1} + \Delta_{N_1}$ and $\Delta_N = \Delta_{M_2} + \Delta_{N_2}$

$$R(M \times N) \leq 2|E(M)||E(N)|(\Delta_M\Delta_N)^2.$$

$$R(M \times N) \leq \frac{2(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')}{\Delta_M\Delta_N}$$

$$\gamma_1 \leq \frac{(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')}{(\Delta_{M_1} + \Delta_{N_1})(\Delta_{M_2} + \Delta_{N_2})}$$

$$\gamma_2 \geq \frac{2(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')}{(\delta_{M_1} + \delta_{N_1})(\delta_{M_2} + \delta_{N_2})}$$

$$\gamma_1 = \frac{2(\mu_1\eta_1 + 3\eta_1\mu_1)(\eta_2\mu_2 + 3\mu_2\eta_2)}{(\Delta_M\Delta_N)}$$

$$\gamma_2 = \frac{2(\mu_1\eta_1' + 3\eta_1\mu_1')(\eta_2\mu_2' + 3\mu_2\eta_2')}{(\delta_M\delta_N)}$$

Theorem 7: Let $M = M_1 +_T N_1$ and $N = M_2 +_T N_2$. Then

$$\gamma_1 \leq R(M \times N) \leq \gamma_2$$

where

$$\gamma_1 = \frac{2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{(\Delta_M\Delta_N)}$$

$$\gamma_2 = \frac{2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{(\delta_M\delta_N)}$$

Proof: Let M and N be graphs with vertex sets x_1, x_2, x_3,...,$x_{n_1(\eta_1+\mu_1)}$ and y_1,y_2, y_3,...,$y_{n_2(\eta_2+\mu_2)}$ respectively, then by the definition of

$$R(M \times N) = \sum_{(x_i,y_j)(x_k,y_l)\in E(M\times N)} (d_{(M\times N)}(x_i, y_j)d_{(M\times N)}(x_k, y_l)$$

$|E(M)| = \mu_1\eta_1' + 4\eta_1\mu_1'$, $|E(N)| = \eta_2\mu_2' + 4\mu_2\eta_2'$, $\Delta_M = \Delta_{M_1} + \Delta_{N_1}$ and $\Delta_N = \Delta_{M_2} + \Delta_{N_2}$

$$R(M \times N) \leq 2|E(M)||E(N)|\sqrt{(\Delta_M\Delta_N)}^{-1/2}$$

$$R(M \times N) \leq \frac{2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{\sqrt{(\Delta_{M_2}\Delta_{N_2})(\Delta_{M_1}\Delta_{N_1})}}$$

$$\gamma_1 = \frac{2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{\sqrt{(\Delta_M\Delta_N)^2}}$$

$$\gamma_2 = \frac{2(\mu_1\eta_1' + 4\eta_1\mu_1')(\eta_2\mu_2' + 4\mu_2\eta_2')}{\sqrt{(\delta_M\delta_N)^2}}$$

Corollary 3: Consider the graphs $M_1 = P_3$, $N_1 = K_3$, $M_2 = P_4$ and $N_2 = C_4$.

- $m_1 = 5, m_2 = 7, d(merged) = 8,\ m = 17$
  $\frac{m}{\sqrt{\Delta_1 \Delta_2]}} \leq R(M +_s N) \leq \frac{m}{\sqrt{\delta_1 \delta_2}}$
- $4.25 \leq R(M +_s N) \leq 17$
- $m = m_1 + m_2 + m_e = 12,$
  $\frac{m}{\sqrt{\Delta_1 \Delta_2}} \leq R(M +_R N) \leq \frac{m}{\sqrt{\delta_1 \delta_2}}$
- $3 \leq R(M +_R N) \leq 12$
- $|E(M \times N)| = m_1.m_2 = 35,\ \delta_T = \delta_1.\delta_2$ and $\Delta_T = \Delta_1.\Delta_2$
  $\frac{m}{\sqrt{\Delta_T.\Delta_T}} \leq R(M +_T N) \leq \frac{m}{\sqrt{\delta_T \delta_T}}$
- $2.188 \leq R(M +_T N) \leq 35$

The following results can be proved similarly.

Theorem 8 Let $M = M_1 +_S N_1$ and $N = M_2 +_S N_2$ then for $\gamma_1 \leq M_1 \Pi(M \times N) \leq \gamma_2$ where,

$$\gamma_1 = 2(\mu_1\eta_1' + 2\eta_1\mu_1')(\eta_2\mu_2' + 2\mu_2\eta_2')[(\Delta_{G_1} + \Delta_{H_1} + \Delta_{G_2} + \Delta_{H_2})]^2$$
$$\gamma_2 = 2(\mu_1\eta_1 + 2\eta_1\mu_1)(\eta_2\mu_2 + 2\mu_2\eta_2).[(\delta_M + \delta_N)]^2$$

Theorem 9: Let $M = M_1 +_R N_1$ and $N = M_2 +_R N_2$. Then for $\gamma_1 \leq M_1 \Pi(M \times N) \leq \gamma_2$ where

$$\gamma_1 = 2(\mu_1\eta_1 + 3\eta_1\mu_1)(\eta_2\mu_2 + 3\mu_2\eta_2)[(\Delta_{M_1} + \Delta_{N_1} + \Delta_{M_2} + \Delta_{N_2})]^2$$
$$\gamma_2 = 2(\mu_1\eta_1 + 3\eta_1\mu_1)(\eta_2\mu_2 + 3\mu_2\eta_2).[(\delta_{M_1} + \delta_{N_1} + \delta_{M_2} + \delta_{N_2})]^2$$

Theorem 10: Let $M = M_1 +_T N_1$ and $N = M_2 +_T N_2$. Then $\gamma_1 \leq M_1 \Pi(M \times N) \leq \gamma_2$ where

$$\gamma_1 = 2(\mu_1\eta_1 + 4\eta_1\mu_1)(\eta_2\mu_2 + 4\mu_2\eta_2)[(\Delta_{M_1} + \Delta_{N_1} + \Delta_{M_2} + \Delta_{N_2})]^2$$
$$\gamma_2 = 2(\mu_1\eta_1 + 4\eta_1\mu_1)(\eta_2\mu_2 + 4\mu_2\eta_2).[(\delta_{M_1} + \delta_{N_1} + \delta_{M_2} + \delta_{N_2})]^2.$$

Corollary 4: Consider the graphs $M_1 = P_3$, $N_1 = K_3$, $M_2 = P_4$ and $N_2 = C_4$.

- $\delta = 1, \Delta = 8, n = |V(M)| = 11$
- $(\delta^2)^n \leq \Pi_1(M +_s N) \leq (\Delta^2)^n$
- $1 \leq \Pi_1(M +_s N) \leq 7.3787 \times 10^{19}$
- $2 \leq \Pi_1(M +_R N) \leq 12$
- $\delta_1 = \delta_2 = 1, \Delta_1 = \Delta_2 = 8, m = 35$
- $1 \leq \Pi_1(M +_T N) \leq 2^{420}$

4. Discussion:

The tensor product operation introduces multiplicative behavior among the degree functions of the factor graphs. Our results provide bounds for degree-based indices under this operation. Specifically, the multiplicative Zagreb indices exhibit multiplicative relations, while the Randić and sum-connectivity indices maintain proportional bounds.

These findings extend previous results on the Cartesian product and F-sum operations [2, 12, 22], and demonstrate the general behavior of degree-based topological indices under multiplicative graph products.

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