Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 4, pp. 631 – 640

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.4.002

FIRST GENERAL ZAGREB INDEX OF POWER FOUR OF GRAPHS

Pushpalatha Mahalank[1],

Ismail Naci Cangul[2,*]

[1]Dept of Maths, SDM College of Eng and Tech, Dharwad-580002, Karnataka, INDIA
mahalankpushpalatha@gmail.com
[2]Dept of Maths, Bursa Uludag University, 16059 Bursa, Turkey
cangul@uludag.edu.tr
* Corresponding author
The authors have no competing interests.

Abstract

Edge and vertex operations on graphs are very useful methods in graph theory. They enable us to find the required properties of a graph by means of similar graphs. In this paper, as a sequel to recent papers on edge deletion and addition, we study the change in the first general Zagreb index of a simple graph G for power four when an arbitrary edge is added. This can be used to calculate the first general Zagreb index of larger graphs in terms of the first general Zagreb indices of smaller graphs. As some examples, some inequalities for the change of Zagreb indices for path, cycle, star, complete, complete bipartite and tadpole graphs are given.

Keywords: first general Zagreb index, graph, vertex degree, topological index, edge addition.
Mathematics Subject Classification: 05C07, 05C30, 05C38.

1 Introduction

We take $G = (V, E)$ as a simple graph with order n and size m. If $v \in V(G)$, the degree of v is denoted by $d_G(v)$. A vertex with degree one is called a pendant vertex. As usual, we denote the path, cycle, star, complete, complete bipartite and tadpole graphs by $P_n, C_n, S_n, K_n, K_{r,s}$ and $T_{r,s}$, respectively.

Topological graph indices are defined and used in many areas to study several properties of different objects such as atoms and molecules. Several topological graph indices have been defined and studied by many scientists as graphs are generated from molecules by replacing atoms with vertices and chemical bonds with edges. Two of the most important topological graph indices are called the

first and second Zagreb indices denoted by $M_1(G)$ and $M_2(G)$, respectively:

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v). \tag{1.1}$$

They were first defined in 1972 by Gutman and Trinajstic, [3], and are referred to due to their uses in QSAR and QSPR studies. In [1], some results on the first Zagreb index together with some other indices are given. In [2], the multiplicative versions of these indices are studied. For graph operations, these indices are calculated in [4]. Some relations between Zagreb indices and some other indices such as ABC, GA and Randic indices are obtained in [5]. Zagreb indices of subdivision graphs were studied in [7] and these were calculated for the line graphs of the subdivision graphs in [6]. A more generalized version of subdivision graphs is called r-subdivision graphs and Zagreb indices of r-subdivision graphs are calculated in [8]. These indices are calculated for several important graph classes in [9].

In [10, 11, 12], the same authors obtained some formulae and inequalities for the change of the first and second Zagreb indices when one or more edges are deleted from a given graph. In this paper, we calculate the change of the first general Zagreb index of power four when a new edge e is added to the given graph. We specially require that both the graphs G and $G + \{e\}$ to be simple. One can use this result to obtain the first general Zagreb index of graphs with $n+1$ vertices by means of the first general Zagreb index of graphs with n vertices where these n vertices are in the set of $n + 1$ vertices, or vice versa.

Finally we give examples to the change of first general Zagreb index of power four for some well-known graphs when a new edge is added.

2 Zagreb indices

Similarly to first and second Zagreb indices, third Zagreb index is defined by

$$M_3(G) = F(G) = \sum_{v \in V(G)} dv^3 = \sum_{uv \in E(G)} (du^2 + dv^2).$$

It is also called as the forgotten index and therefore, denoted by F in some papers.

Several other variations of Zagreb indices have been defined and studied. For example, the first general Zagreb index was defined by

$$M_1^\alpha(G) = \sum_{v \in V(G)} dv^\alpha = \sum_{uv \in E(G)} (du^{\alpha-1} + dv^{\alpha-1}).$$

Note that in particular,

$$\text{for } \alpha = 1,\ M_1^1(G) = \sum_{v \in V(G)} dv = 2m; (m = \mid V(G) \mid)$$

$$\text{for } \alpha = 2,\ M_1^2(G) = \sum_{v \in V(G)} dv^2 = M_1(G) \text{ and}$$

$$\text{for } \alpha = 3,\ M_1^3(G) = M_3(G).$$

For $\alpha = 4$, the first general Zagreb index of power four is defined by

$$M_1^4(G) = \sum_{v \in V(G)} dv^4 = \sum_{uv \in E(G)} (du^3 + dv^3).$$

3 Adding a new edge to a graph

In this section, we will determine the amount of change in the first general Zagreb index of power four when a new edge is added to any simple graph. We have

Theorem 3.1. *Let G be a simple graph. Let us add an edge e to form a larger graph $G + \{e\}$.*

i) *If the added edge e is a pendant edge connecting the vertex v_i of degree d_i in G with a new pendant vertex v_{n+1} of degree $d_{n+1} = 1$, then*

$$M_1^4(G + \{e\}) = M_1^4(G) + 4d_i^3 + 6d_i^2 + 4d_i + 2. \tag{3.1}$$

ii) *If the added edge is not a pendant edge, in other words, new edge e connects any two vertices v_i, v_j of the graph G with degrees d_i, d_j respectively, then*

$$M_1^4(G + \{e\}) = M_1^4(G) + 4(d_i^3 + d_j^3) + 6(d_i^2 + d_j^2) + 4(d_i + d_j) + 2. \tag{3.2}$$

Proof. ***i)*** Let the graph G have vertices $v_1, v_2, \cdots, v_n$ with degrees $d_1, d_2, \cdots, d_n$, respectively. Let us add a pendant edge e connecting the vertex $v_i \in G$ of degree d_i in G with a new vertex v_{n+1} of degree $d_{n+1} = 1$ which is not a vertex of G.

Recall that $M_1^4(G) = \sum\limits_{v \in V(G)} dv^4 = \sum\limits_{uv \in E(G)} (du^3 + dv^3)$

$$\begin{aligned} M_1^4(G + e) &= \sum_{\substack{k=1 \\ k \neq i}}^{n} d_k^4 - d_i^4 + (d_i + 1)^4 + 1^4 \\ &= \sum_{k=1}^{n} d_k^4 - d_i^4 + (d_i + 1)^4 + 1^4 \\ &= M_1^4(G) + 4d_i^3 + 6d_i^2 + 4d_i + 2. \end{aligned}$$

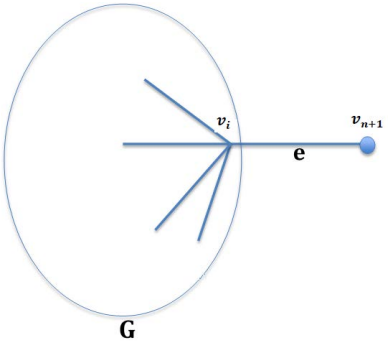


Figure 1: Adding a pendant edge e to the graph G

which gives the result.

ii) Secondly, consider adding a new edge $e = v_iv_j$ where $v_i,\ v_j \in V(G)$. That is, the new edge joins two existing vertices. Adding e increases the degrees of v_i and v_j by 1.

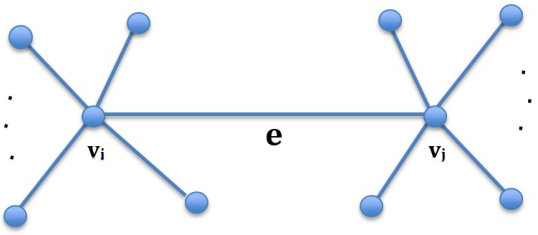


Figure 2: Adding a non-pendant edge e to the graph G

$$\begin{aligned} M_1^4(G+e) &= M_1^4(G) - d_i^4 - d_j^4 + (d_i+1)^4 + (d_j+1)^4 \\ &= M_1^4(G) + 4(d_i^3 + d_j^3) + 6(d_i^2 + d_j^2) + 4(d_i + d_j) + 2 \end{aligned}$$

Hence the theorem is proved. □

4 Effect of edge addition on M_1^4 for some classes of graphs

In this section, using the above two results in **(i)** for adding a pendant edge, **(ii)** for adding a non-pendant edge, we get easy results for graph classes $P_n,\ C_n,\ S_n,\ K_n,\ K_{r,s}$

and $T_{r,s}$.

4.1 Path graph P_n

First note that $M_1^4(P_n) = 16n - 30$. Therefore by Theorem 3.1, we can obtain

$$M_1^4(P_n{+}e) = \begin{cases} 16n - 30 + 4d_i^3 + 6d_i^2 + 4d_i + 2 & \text{if a pendant edge } e \text{ is added at } v_i \\ 16n - 30 + 4(d_i^3 + d_j^3) & \\ \quad +6(d_i^2 + d_j^2) + 4(d_i + d_j) + 2 & \text{if } e \text{ connects } v_i,\, v_j \end{cases}$$

So we have the following possibilities for P_n:

a) If a pendant edge e is added to one of the end points of degree 1, then

$$M_1^4(P_n + e) = 16n - 30 + 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 2 = 16n - 14;$$

b) If a pendant edge e is added to one of the midponts of P_n of degree 2, then

$$M_1^4(P_n + e) = 16n - 30 + 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 2 = 16n - 36;$$

c) If a non-pendant edge e is added at vertices v_i, v_j so that $d_i = d_j = 1$, then

$$M_1^4(P_n + e) = 16n - 30 + 4(1^3 + 1^3) + 6(1^2 + 1^2) + 4(1 + 1) + 2 = 16n;$$

d) If a non-pendant edge e is added at vertices v_i, v_j so that $d_i = 1, d_j = 2$ or vice versa, then

$$M_1^4(P_n + e) = 16n - 30 + 4(1^3 + 2^3) + 6(1^2 + 2^2) + 4(1 + 2) + 2 = 16n + 50;$$

and finally
e) If a non-pendant edge e is added at vertices v_i, v_j so that $d_i = d_j = 2$, then

$$M_1^4(P_n + e) = 16n - 30 + 4(2^3 + 2^3) + 6(2^2 + 2^2) + 4(2 + 2) + 2 = 16n + 100;$$

So we proved.

Corollary 4.1. *When a new edge is added to a path graph P_n, its first general Zagreb index of power four increases by either 16, 30, 66, 80 or 130.*

4.2 Cycle graph C_n

First note that $M_1^4(C_n) = 16n$. Hence by Theorem 3.1, we can obtain

$$M_1^4(C_n{+}\{e\}) = \begin{cases} 16n + 4d_i^3 + 6d_i^2 + 4d_i + 2 & \text{if a pendant edge } e \text{ is added at } v_i \\ 16n + 4(d_i^3 + d_j^3) & \\ \quad +6(d_i^2 + d_j^2) + 4(d_i + d_j) + 2 & \text{if } e \text{ connects } v_i,\, v_j. \end{cases}$$

So, as all vertices are of degree 2, we have the following possibilities for C_n:

a) If a pendant edge e is added to any one of the vertices v_i of C_n of degree 2, then

$$M_1^4(C_n + e) = 16n + 4 \cdot 2^3 + 6 \cdot 2^3 + 4 \cdot 2 + 2 = 16n + 66;$$

b) If a non-pendant edge e is added at vertices v_i, v_j, so that, $d_i = d_j = 2$, then

$$M_1^4(C_n + e) = 16n + 4(2^3 + 2^3) + 6(2^2 + 2^2) + 4(2 + 2) + 2 = 16n + 130.$$

So we proved.

Corollary 4.2. *When a new edge is added to a cycle graph C_n, its first general Zagreb index of power four increases by either 66 or 130.*

4.3 Star graph S_n

It is easy to see that for $n \geq 3$, $M_1^4(S_n) = (n-1)^4 + (n-1)$. Hence we have

$$M_1^4(S_n+e) = \begin{cases} M_1^4(S_n) + 4d_i^3 + 6d_i^2 + 4d_i + 2 & \text{if a pendant edge } e \text{ is added at } v_i \\ M_1^4(S_n) + 4(d_i^3 + d_j^3) & \\ \quad +6(d_i^2 + d_j^2) + 4(d_i + d_j) + 2 & \text{if } e \text{ connects } v_i,\ v_j \end{cases}$$

As a vertex of S_n is either pendant or of degree $n-1$, we have the following possibilities for S_n:

a) If a pendant edge e is added to a vertex v_i of S_n of degree 1, then

$$M_1^4(S_n+e) = (n-1)^4 + (n-1) + 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 2 = (n-1)^4 + (n-1) + 16;$$

b) If a pendant edge e is added to the vertex v_i of S_n of degree $n-1$, then

$$M_1^4(S_n+e) = (n-1)^4 + (n-1) + 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 2 = n^4 + n;$$

c) If a non-pendant edge e is added at the vertices v_i, v_j of degree both 1, then

$$M_1^4(S_n+e) = (n-1)^4+(n-1)+4(1^3+1^3)+6(1^2+1^2)+4(1+1)+2 = (n-1)^4+(n-1)+30;$$

d) If a non-pendant edge e is added at vertices v_i, v_j of degrees 1 and $n-1$, then

$$M_1^4(S_n+e) = (n-1)^4+(n-1)+4(1^3+(n-1)^3)+6(1^2+(n-1)^2)+4(1+(n-1))+2 = n^4+n+14;$$

So we proved.

Corollary 4.3. *When a new edge is added to a star graph S_n, its first general Zagreb index of power four increases by either 16, 30, $4n^3 - 6n^2 + 4n$ or $4n^3 - 6n^2 + 4n + 14$.*

4.4 Complete graph K_n

We can calculate $M_1^4(K_n) = n \cdot (n-1)^4$ and by Theorem 3.1, we can write

$$M_1^4(K_n{+}e) = \begin{cases} n(n-1)^4 + 4d_i^3 + 6d_i^2 + 4d_i + 2 & \text{if a pendant edge } e \text{ is added at } v_i \\ n(n-1)^4 + 4(d_i^3 + d_j^3) & \\ \quad +6(d_i^2 + d_j^2) + 4(d_i + d_j) + 2 & \text{if } e \text{ connects } v_i,\, v_j \end{cases}$$

As all vertices of K_n are of degree $n-1$, we have the following possibilities for K_n:

a) If a pendant edge e is added to a vertex v_i of K_n of degree $n-1$, then

$$M_1^4(K_n + e) = n^5 - 4n^4 + 10n^3 - 10n^2 + 5n;$$

b) If a non-pendant edge e is added at the vertices v_i, v_j of degree both $n-1$, then

$$M_1^4(K_n + e) = n^5 - 4n^4 + 14n^3 - 16n^2 + 9n - 2;$$

So we proved.

Corollary 4.4. *When a new edge is added to a complete graph K_n, its first general Zagreb index of power four increases by either $4n^3 - 6n^2 + 4n$ or $8n^3 - 12n^2 + 8n - 2$.*

4.5 Complete bipartite graph $K_{r,s}$

It can be easily seen that $M_1^4(K_{r,s}) = rs(r^3 + s^3)$. By Theorem 3.1, we can write

$$M_1^4(K_{r,s}{+}e) = \begin{cases} rs(s^3 + r^3) + 4d_i^3 + 6d_i^2 + 4d_i + 2 & \text{if a pendant edge } e \text{ is added at } v_i \\ rs(s^3 + r^3) + 4(d_i^3 + d_j^3) & \\ \quad +6(d_i^2 + d_j^2) + 4(d_i + d_j) + 2 & \text{if } e \text{ connects } v_i,\, v_j \end{cases}$$

As the degree sequence of $K_{r,s}$ is $\{r^s, s^r\}$, we have the following possibilities for $K_{r,s}$:

a) If we add a pendant edge e to a vertex of degree s, then

$$M_1^4(K_{r,s} + e) = rs(r^3 + s^3) + 4s^3 + 6s^2 + 4s + 2;$$

b) Similarly if we add a pendant edge e to a vertex of degree r, then

$$M_1^4(K_{r,s} + e) = rs(r^3 + s^3) + 4r^3 + 6r^2 + 4r + 2;$$

c) If we add a non-pendant edge e to two vertices v_i, v_j both of degree r, then

$$M_1^4(K_{r,s} + e) = rs(r^3 + s^3) + 8r^3 + 12r^2 + 8r + 2;$$

d) If we add a non-pendant edge e to two vertices v_i, v_j both of degree s, then

$$M_1^4(K_{r,s}+e) = rs(r^3+s^3)+8s^3+12s^2+8s+2;$$

e) If we add a non-pendant edge e to two vertices v_i of degree r and v_j of degree s, then

$$M_1^4(K_{r,s}+e) = rs(r^3+s^3)+2(2r^3+2s^3+3r^2+3s^2+2r+2s+1)$$

which are all compatible with Theorem 3.1. So we proved.

Corollary 4.5. *When a new edge is added to a complete bipartite graph $K_{r,s}$, its first general Zagreb index of power four increases by either $4s^3+6s^2+4s+2$, $4r^3+6r^2+4r+2$, $8r^3+12r^2+8r+2$, $8s^3+12s^2+8s+2$ or $2(2r^3+2s^3+3r^2+3s^2+2r+2s+1)$.*

4.6 Tadpole graph $T_{r,s}$

As $M_1^4(T_{r,s}) = 16(r+s)+50$, by Theorem 3.1, we have

$$M_1^4(T_{r,s}+e) = \begin{cases} 16(r+s)+50+4d_i^3+6d_i^2+4d_i+2 & \text{if a pendant edge } e \text{ is added at } v_i \\ 16(r+s)+50+4(d_i^3+d_j^3) & \\ \quad +6(d_i^2+d_j^2)+4(d_i+d_j)+2 & \text{if } e \text{ connects } v_i,\, v_j \end{cases}$$

The degree sequence of $T_{r,s}$ is $\{1^{(1)}, 2^{(r+s-2)}, 3^{(1)}\}$, hence we have the following possibilities for $T_{r,s}$:

a) If we add a pendant edge e to a vertex of degree 1, then

$$M_1^4(T_{r,s}+e) = 16(r+s)+50+4\cdot 1^3+6\cdot 1^2+4\cdot 1+2 = 16(r+s)+66;$$

b) If we add a pendant edge e to a vertex of degree 2, then

$$M_1^4(T_{r,s}+e) = 16(r+s)+50+4\cdot 2^3+6\cdot 2^2+4\cdot 2+2 = 16(r+s)+116;$$

c) If we add a pendant edge e to a vertex of degree 3, then

$$M_1^4(T_{r,s}+e) = 16(r+s)+50+4\cdot 3^3+6\cdot 3^2+4\cdot 3+2 = 16(r+s)+226;$$

d) If we add a non-pendant edge e to two vertices v_i, v_j, so that $d_i = 1$ and $d_j = 2$, then

$$M_1^4(T_{r,s}+e) = 16(r+s)+50+4(1^3+2^3)+6(1^2+2^2)+4(1+2)+2 = 16(r+s)+130;$$

e) If we add a non-pendant edge e to two vertices v_i, v_j, so that $d_i = 1, d_j = 3$, then

$$M_1^4(T_{r,s}+e) = 16(r+s)+50+4(1^3+3^3)+6(1^2+3^2)+4(1+3)+2 = 16(r+s)+240;$$

f) If we add a non-pendant edge e to two vertices v_i, v_j, so that $d_i = 2, d_j = 3$, then

$$M_1^4(T_{r,s}+e) = 16(r+s)+50+4(2^3+3^3)+6(2^2+3^2)+4(2+3)+2 = 16(r+s)+290;$$

which are all compatible with Theorem 3.1. So we proved.

Corollary 4.6. *When a new edge is added to a tadpole graph $T_{r,s}$, its first general Zagreb index of power four increases by either 16, 66, 80, 176, 190 or 240.*

5 Summary and conclusions

In this paper, the effect of edge addition to any connected simple graph on its first general Zagreb index of power four is studied. This effect mainly differs for adding a pendant edge or non-pendant edge. Both effects are formulized and all possible changes are calculated for six most frequently used graph classes. The same problem is solved for the first, second and third Zagreb indices and there will be a continuation of this paper for other indices.

6 Declarations

6.1 Competing interests

The authors declare that they have no competing interests.

6.2 Funding

The authors declare that they have no financial or non-financial support.

6.3 Authors' contributions

INC proposed the problem and recalled the existing results, PM has done all the calculations, and constructed the statements and checked calculations.

References

[1] Das, KC, Akgunes, N, Togan, M, Yurttas, A, Cangul, IN, Cevik, AS: On the first Zagreb index and multiplicative Zagreb coindices of graphs. Analele Stiintifice ale Universitatii Ovidius Constanta. **24** (1) (2016), 153-176.

[2] Das, KC, Yurttas, A, Togan, M, Cangul, IN, Cevik, AS: The multiplicative Zagreb indices of graph operations. Journal of Inequalities and Applications. **90** (2013), 1-14.

[3] Gutman, I, Trinajstic, N: Graph theory and molecular orbitals III, Total π-electron energy of alternant hydrocarbons. Chem. Phys. Lett. **17** (1972), 535-538.

[4] Khalifeh, MH, Yousefi-Azari, H, Ashrafi, AR: The First and second Zagreb indices of some graph operations. Discrete Appl. Math. **157** (2009), 804-811.

[5] Lokesha, V, Shetty, SB, Ranjini, PS, Cangul, IN: Computing ABC, GA, Randic and Zagreb Indices. Enlightments of Pure and Applied Mathematics. **1** (1) (2015), 17-28.

[6] Ranjini, PS, Lokesha, V, Cangul, IN: On the Zagreb indices of the line graphs of the subdivision graphs. Appl. Math. Comput. **218** (2011), 699-702.

[7] Ranjini, PS, Rajan, MA, Lokesha, V: On Zagreb Indices of the Sub-division Graphs. Int. J. of Math. Sci. & Eng. Appns. **4** (2010), 221-228.

[8] Togan, M, Yurttas, A, Cangul, IN: Some formulae and inequalities on several Zagreb indices of r-subdivision graphs. Enlightments of Pure and Applied Mathematics. **1** (1) (2015), 29-45.

[9] Togan, M, Yurttas, A, Cangul, IN: All versions of Zagreb indices and coindices of subdivision graphs of certain graph types. Advanced Studies in Contemporary Mathematics. **26** (1) (2016), 227-236.

[10] Togan, M, Yurttas, A, Cevik, AS, Cangul, IN: Effect of Edge Deletion and Addition on Zagreb Indices of Graphs. In: Tas, K, Baleanu, D, Machado, J (eds), Mathematical Methods in Engineering, Theoretical Aspects, Nonlinear Systems and Complexity, Springer. **23** (2019), 191-201.

[11] Mahalank, P, Cangul, IN: Third Zagreb Index of Graphs with Added Edges, Bulletin of Computational Applied Mathematics. **10** (1) (2022), 81-92.

[12] Yurttas, A, Togan, M, Cangul, IN: Zagreb Indices of Graphs with Added Edges. Proceedings of the Jangjeon Mathematical Society. **21** (3) (2018), 385-392.