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SOME IDENTITIES FOR DEGENERATE STIRLING NUMBERS OF THE SECOND KIND VIA BOSON OPERATORS

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Abstract. In this paper, we investigate the properties of degenerate Stirling numbers of the second kind. Employing the boson operator algebra and the normal ordering of the number operator, we derive a new relation for $\left\{ {n+m \atop l} \right\}_{\lambda}$. We also demonstrate that our result recovers the classical identity in the limit $\lambda \to 0$.

Keywords: Degenerate Stirling numbers; Boson operators; Normal ordering; Combinatorial identities.

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1. Introduction

In recent years, the study of degenerate versions of special polynomials and numbers has regained significant interest among mathematicians and physicists, leading to new insights into their underlying algebraic structures (see[7]). The foundation of this field was established by L. Carlitz, who introduced the degenerate Bernoulli and Eulerian numbers. These degenerate versions are obtained by replacing the power x^n with the degenerate falling factorial $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$. This shift from classical calculus to degenerate calculus provides a broader framework for understanding combinatorial identities(see[4]).

Among various special numbers, the Stirling numbers of the second kind, $\left\{ {n \atop k} \right\}$, play a fundamental role in combinatorics as they represent the number of ways to partition a set of n elements into k nonempty subsets.This combinatorial significance has inspired numerous extensions within various operational settings, including shift algebras and generalized ordering frameworks (see[10]). The degenerate Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}_{\lambda}$, serve as a natural

generalization. These numbers have been shown to satisfy various properties, including recurrence relations and generating functions, which mirror their classical counterparts(see[7]).

A particularly powerful method for investigating these numbers is the use of boson operator algebra (see[1, 2, 12, 13]). In quantum mechanics, the creation operator $a^\dagger$ and the annihilation operator a satisfy the commutation relation $[a, a^\dagger] = 1$. The normal ordering of these operators where all creation operators are placed to the left of annihilation operators provides an algebraic path to deriving complex combinatorial identities. T. Kim and his collaborators have recently demonstrated that the degenerate number operator $(a^\dagger a)_{n,\lambda}$ can be used to systematically derive new identities for degenerate Stirling numbers(see[8, 9]).

The aim of this paper is to extend these results by investigating the addition type relations for degenerate Stirling numbers of the second kind, building upon recent algebraic developments in the study of r-Stirling operator analogues (see[6]). Specifically, we focus on the expansion of $(a^\dagger a)_{n+m,\lambda}$ and utilize the degenerate binomial addition formula. Applying the identity of the interaction between the annihilation and creation operators, we derive a new consolidated formula for $\left\{ {n+m \atop l} \right\}_\lambda$. We also verify that our results are consistent with the classical case taking the limit $\lambda \to 0$.

2. Preliminaries

For any $\lambda \in \mathbb{R}$, the degenerate falling factorial sequence $(x)_{n,\lambda}$ is defined by:

$$(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda) \quad (n \geq 1), \tag{2.1}$$

with $(x)_{0,\lambda} = 1$. Note that as $\lambda \to 0$, $(x)_{n,\lambda}$ reduces to the classical power x^n.

The degenerate Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_\lambda$ are defined by the following expansion of the degenerate falling factorial (see[11]) :

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_\lambda (x)_k. \tag{2.2}$$

Let a and $a^\dagger$ be the annihilation and creation operators, respectively, satisfying the commutation relation (see [5]):

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1. \tag{2.3}$$

The number operator is defined as $\hat{n} = a^\dagger a$. A fundamental property of these operators is the normal ordering identity which establishes the

connection between boson operator powers and special combinatorial numbers (see[5, 11]):

$$(a^\dagger a)^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (a^\dagger)^k a^k, \tag{2.4}$$

where $\left\{ {n \atop k} \right\}$ denotes the classical Stirling numbers of the second kind.

The degenerate version of the normal ordering for the number operator is given by([9, 8]):

$$(\hat{n})_{n,\lambda} = (a^\dagger a)_{n,\lambda} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\lambda} (a^\dagger)^k a^k. \tag{2.5}$$

3. Main Results

In this section, we provide the derivation of our main identity. Let a and $a^\dagger$ be the boson annihilation and creation operators satisfying the commutation relation

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1. \tag{3.1}$$

We observe that for $r \geq 0$

$$\begin{aligned} a(a^\dagger)^r =& aa^\dagger(a^\dagger)^{r-1} = (1 + a^\dagger a)(a^\dagger)^{r-1} \\ =& (a^\dagger)^{r-1} + a^\dagger(a(a^\dagger)^{r-1}) \\ =& (a^\dagger)^{r-1} + a^\dagger[(a^\dagger)^{r-2} + a^\dagger(a(a^\dagger)^{r-2})] \\ =& 2(a^\dagger)^{r-1} + (a^\dagger)^2(a(a^\dagger)^{r-2}) \\ =& 2(a^\dagger)^{r-1} + (a^\dagger)^2[(a^\dagger)^{r-3} + a^\dagger(a(a^\dagger)^{r-3})] \\ =& 3(a^\dagger)^{r-1} + (a^\dagger)^3(a(a^\dagger)^{r-3}) \\ =& \cdots \\ =& r(a^\dagger)^{r-1} + (a^\dagger)^r a = \sum_{k=0}^{1} \binom{1}{k} \frac{r!}{(r-1+k)!} (a^\dagger)^{r-1+k} a^k. \end{aligned} \tag{3.2}$$

From (3.2), we note that:

$$\begin{aligned} a^2(a^\dagger)^r =& ra(a^\dagger)^{r-1} + a(a^\dagger)^r a \\ =& r((r-1)(a^\dagger)^{r-2} + (a^\dagger)^{r-1}a) + (r(a^\dagger)^{r-1} + (a^\dagger)^r a)a \\ =& r(r-1)(a^\dagger)^{r-2} + 2r(a^\dagger)^{r-1}a + (a^\dagger)^r a^2 \\ =& \sum_{k=0}^{2} \binom{2}{k} \frac{r!}{(r-2+k)!} (a^\dagger)^{r-2+k} a^k \end{aligned} \tag{3.3}$$

Continuing the same process, we get

$$a^n(a^\dagger)^r = \sum_{k=0}^{n} \binom{n}{k} \frac{r!}{(r-n+k)!} (a^\dagger)^{r-n+k} a^k. \tag{3.4}$$

Theorem 3.1. *For $n, r \geq 0$, we have the following identities*

$$\begin{aligned} a(a^\dagger)^r =& r(a^\dagger)^{r-1} + (a^\dagger)^r a, \\ a^n(a^\dagger)^r =& \sum_{k=0}^{n} \binom{n}{k} \frac{r!}{(r-n+k)!} (a^\dagger)^{r-n+k} a^k. \end{aligned}$$

From equation (2.4), we note that

$$(a^\dagger a)^{n+m} = \sum_{l=0}^{n+m} \left\{ {n+m \atop l} \right\} (a^\dagger)^l a^l. \tag{3.5}$$

On the other hand, using Equation (2.4) and Theorem 3.1, we have

$$\begin{aligned} (a^\dagger a)^{n+m} =& (a^\dagger a)^n (a^\dagger a)^m \\ =& \left(\sum_{i=0}^{n} \left\{ {n \atop i} \right\} (a^\dagger)^i a^i \right) \left(\sum_{j=0}^{m} \left\{ {m \atop j} \right\} (a^\dagger)^j a^j \right) \\ =& \sum_{i=0}^{n} \sum_{j=0}^{m} \left\{ {n \atop i} \right\} \left\{ {m \atop j} \right\} (a^\dagger)^i a^i (a^\dagger)^j a^j \\ =& \sum_{i=0}^{n} \sum_{j=0}^{m} \left\{ {n \atop i} \right\} \left\{ {m \atop j} \right\} (a^\dagger)^i \sum_{k=0}^{i} \binom{i}{k} \frac{j!}{(j-i+k)!} (a^\dagger)^{j-i+k} a^k a^j \\ =& \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{i} \left\{ {n \atop i} \right\} \left\{ {m \atop j} \right\} \binom{i}{k} \frac{j!}{(j-i+k)!} (a^\dagger)^{j+k} a^{j+k} \quad (i \geq k) \\ =& \sum_{k=0}^{n} \sum_{i=0}^{n} \sum_{j=0}^{m} \left\{ {n \atop i} \right\} \left\{ {m \atop j} \right\} \binom{i}{k} \frac{j!}{(j-i+k)!} (a^\dagger)^{j+k} a^{j+k} \\ =& \sum_{l=0}^{n+m} \sum_{j=0}^{l} \sum_{i=0}^{n} \left\{ {n \atop i} \right\} \left\{ {m \atop j} \right\} \binom{i}{l-j} \frac{j!}{(l-i)!} (a^\dagger)^l a^l. \end{aligned} \tag{3.6}$$

By comparing (3.5) and (3.6) we have the following addition formula, which builds upon the structural properties of algebraic ordering

relations for boson operators (see[3, 5]),

$$\left\{ {n+m \atop l} \right\} = \sum_{j=0}^{l} \sum_{i=0}^{n} \left\{ {n \atop i} \right\} \left\{ {m \atop j} \right\} \binom{i}{l-j} \frac{j!}{(l-i)!}. \tag{3.7}$$

Theorem 3.2. *For $m, n \geq 0$, we have*

$$\left\{ {n+m \atop l} \right\} = \sum_{j=0}^{l} \sum_{i=0}^{n} \left\{ {n \atop i} \right\} \left\{ {m \atop j} \right\} \binom{i}{l-j} \frac{j!}{(l-i)!}.$$

From Equation (2.5), we note that

$$(a^\dagger a)_{n+m,\lambda} = \sum_{l=0}^{n+m} \left\{ {n+m \atop l} \right\}_\lambda (a^\dagger)^l a^l. \tag{3.8}$$

On the other hand, using Equation (2.5) and Theorem 3.1, we get

$$\begin{aligned}
(a^\dagger a)_{n+m,\lambda} =& (a^\dagger a)(a^\dagger a - \lambda)(a^\dagger a - 2\lambda) \cdots (a^\dagger a - (n-1)\lambda) \\
& \times (a^\dagger a - n\lambda) \cdots (a^\dagger a - (n+m-1)\lambda) \\
=& (a^\dagger a)_{n,\lambda} (a^\dagger a - n\lambda)_{m,\lambda} \\
=& \left(\sum_{i=0}^{n} \left\{ {n \atop i} \right\}_\lambda (a^\dagger)^i a^i \right) \left(\sum_{j=0}^{m} \binom{m}{j} (a^\dagger a)_{j,\lambda} (-n\lambda)_{m-j,\lambda} \right) \\
=& \sum_{i=0}^{n} \sum_{j=0}^{m} \left\{ {n \atop i} \right\}_\lambda \binom{m}{j} (a^\dagger)^i a^i (a^\dagger a)_{j,\lambda} (-n\lambda)_{m-j,\lambda} \\
=& \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{j} \left\{ {n \atop i} \right\}_\lambda \binom{m}{j} \left\{ {j \atop k} \right\}_\lambda (-n\lambda)_{m-j,\lambda} (a^\dagger)^i a^i (a^\dagger)^k a^k \\
=& \sum_{i=0}^{n} \sum_{k=0}^{m} \sum_{j=k}^{m} \sum_{p=0}^{i} \left\{ {n \atop i} \right\}_\lambda \binom{m}{j} \left\{ {j \atop k} \right\}_\lambda \binom{i}{p} (-n\lambda)_{m-j,\lambda} \frac{k!}{(k-i+p)!} \\
& \times (a^\dagger)^i (a^\dagger)^{k-i+p} a^p a^k \\
=& \sum_{p=0}^{n} \sum_{i=0}^{n} \sum_{k=0}^{m} \sum_{j=0}^{m} \left\{ {n \atop i} \right\}_\lambda \binom{m}{j} \left\{ {j \atop k} \right\}_\lambda \binom{i}{p} (-n\lambda)_{m-j,\lambda} \frac{k!}{(k-i+p)!} \\
& \times (a^\dagger)^{k+p} a^{k+p} \\
=& \sum_{l=0}^{n+m} \sum_{k=0}^{l} \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \left\{ {n \atop i} \right\}_\lambda \left\{ {j \atop k} \right\}_\lambda \binom{i}{l-k} (-n\lambda)_{m-j,\lambda} \frac{k!}{(l-i)!} \\
& \times (a^\dagger)^l a^l.
\end{aligned} \tag{3.9}$$

Therefore, by (3.8) and (3.9), we obtain the following theorem.

Theorem 3.3. *For $n, m, l \geq 0$, we have*

$$\left\{ {n+m \atop l} \right\}_{\lambda} = \sum_{k=0}^{l} \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \left\{ {n \atop i} \right\}_{\lambda} \left\{ {j \atop k} \right\}_{\lambda} \binom{i}{l-k} (-n\lambda)_{m-j,\lambda} \frac{k!}{(l-i)!}.$$

We note that when $\lambda \to 0$, the degenerate combinatorial numbers reduce to the classical case. Specifically, since

$$\lim_{\lambda \to 0} \left\{ {n \atop k} \right\}_{\lambda} = \left\{ {n \atop k} \right\}, \tag{3.10}$$

and

$$\begin{aligned} \lim_{\lambda \to 0} (-n\lambda)_{m-j,\lambda} &= \lim_{\lambda \to 0} (-n\lambda)(-n\lambda - \lambda)(-n\lambda - 2\lambda) \cdots (-n\lambda - (m-j-1)\lambda) \\ &= \lim_{\lambda \to 0} (\lambda)^{m-j} (-n(-n-1)(-n-2) \cdots (-n-(m-j-1)) \\ &= \lim_{\lambda \to 0} (\lambda)^{m-j} (-n)_{m-j} \end{aligned}$$

$$\lim_{\lambda \to 0} (-n\lambda)_{m-j,\lambda} = \begin{cases} 0, & \text{if } m-j \neq 0, \\ 1, & \text{if } m-j = 0. \end{cases} \tag{3.11}$$

Therefore, from (3.10) and (3.11) we get the following remark:

Remark. *We note that when $\lambda \to 0$, Theorem 3.3 reduces to the classical case:*

$$\left\{ {n+m \atop l} \right\} = \sum_{k=0}^{l} \sum_{i=0}^{n} \left\{ {n \atop i} \right\} \left\{ {m \atop k} \right\} \binom{i}{l-k} \frac{k!}{(l-i)!}.$$

4. Conclusion

In this paper, we have successfully applied the algebraic framework of boson operator algebra to investigate the structural and combinatorial properties of degenerate Stirling numbers of the second kind. By employing the normal ordering techniques of creation and annihilation operators alongside the properties of the degenerate number operator, we established a novel addition-type identity for $\{ {n+m \atop l} \}_{\lambda}$ expressed as a consolidated multi-sum formula. Furthermore, we demonstrated that our main result satisfies a crucial consistency check: taking the limit $\lambda \to 0$ systematically recovers the standard non-degenerate addition formula for the classical Stirling numbers of the second kind.

The methods developed here underscore the elegance and utility of combining quantum mechanical operator algebra with degenerate calculus. In future research, we intend to extend this operational approach

to study more general families of special configurations, such as the degenerate r-Whitney numbers and generalized degenerate Dowling polynomials, establishing their underlying normal ordering relationships.

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