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MATRIX APPROACH TO Q-BINOMIAL THEOREM

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Abstract. The Leverrier-Takeno technique allows to construct the characteristic polynomial of a matrix. Here we use this process to obtain the Taylor expansion of $(-z;q)_n$, with special interest in the $q-$binomial theorem and the Jacobi triple product identity.

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1. Introduction

Here we study the $q-$Pochhammer symbol:

$$f_n(z) := (-z;q)_n = (1+z)(1+zq)(1+zq^2)\dots(1+zq^{n-1}), \tag{1}$$

which is a polynomial in z of degree n, and its Taylor expansion is given by:

$$f_n(z) = \sum_{k=0}^{n} \Big[\frac{d^k f_n}{dz^k}\Big]_{z=0} \frac{z^k}{k!}, \tag{2}$$

and it will imply an interesting deduction of the important $q-$binomial theorem [1-4]. Besides, we also consider the expansion $(-zq;q^2)_\infty$ because it participates in the fundamental Jacobi triple product identity [2-6].

In Sec. 3 we find that the $d^k f_n/dz^k$ are related with the coefficients of the characteristic polynomial $p(\lambda)$ [7-9] of the following diagonal matrix:

$$A_{n\times n} = Diag\bigg(\frac{1}{1+z}, \frac{q}{1+zq}, \frac{q^2}{1+zq^2}, \dots, \frac{q^{n-1}}{1+zq^{n-1}}\bigg), \tag{3}$$

and the traces of its powers are given by:

$$s_m = tr\ A^m = \sum_{j=1}^{n} \frac{q^{m(j-1)}}{(1+zq^{j-1})^m}, \quad m \geq 1, \tag{4}$$

which are employed by the Leverrier-Takeno method [10-15] to construct $p(\lambda)$ for A:

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n, \tag{5}$$

where the quantities a_r are in terms of the traces s_j; then in Sec. 2 we indicate the principal expressions of this method.

In Sec. 3 we obtain the result:

$$\frac{d^k f_n}{dz^k} = (-1)^k k! f_n a_k, \quad 0 \leq k \leq n, \tag{6}$$

thus (2) implies the relation:

$$(-z;q)_n=\sum_{k=0}^{n}[a_k]_{z=0}(-z)^k, \tag{7}$$

because $[f_n]_{z=0}=1$, and we find that:

$$[a_k]_{z=0}=(-1)^k q^{\frac{k(k-1)}{2}}\prod_{j=1}^{k}\frac{1-q^{n-j+1}}{1-q^j}=(-1)^k q^{\frac{k(k-1)}{2}}\binom{n}{k}_q, \tag{8}$$

such that $\binom{n}{k}_q$ is a q-binomial coefficient (Gaussian polynomial) [1-4]. Hence (7) and (8) give the Cauchy's generalization of the binomial theorem [1-4]:

$$(-z;q)_n=\sum_{k=0}^{n}\binom{n}{k}_q q^{\frac{k(k-1)}{2}}z^k. \tag{9}$$

In Sec. 4 we obtain the generating functions for sums of the reciprocals of divisors of an integer. Sec. 5 is dedicated to motivate the Jacobi triple product identity.

2. Leverrier-Takeno technique

For any matrix $A_{n\times n}$, in the approach of Leverrier-Takeno [10-15] the a_i are determined with the Newton's recurrence relation:

$$ra_r+s_1a_{r-1}+s_2a_{r-2}+\cdots+s_{r-1}a_1+s_r=0,\quad r=1,2,\ldots,n,\quad a_0=1, \tag{10}$$

therefore:

$$a_1=-s_1,\quad 2!\,a_2=(s_1)^2-s_2,\quad 3!\,a_3=-(s_1)^3+3s_1s_2-2s_3,$$

$$4!\,a_4=(s_1)^4-6(s_1)^2s_2+8s_1s_3+3(s_2)^2-6s_4, \tag{11}$$

$$5!\,a_5=-24s_5-(s_1)^5+10(s_1)^3s_2-20(s_1)^2s_3-15(s_2)^2s_1+30s_1s_4+20s_2s_3,etc.$$

in particular, $\det A=(-1)^n a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that A and its transpose have the same determinant. In [15-17] we find the general expression:

$$a_k=\frac{(-1)^k}{k!}\begin{vmatrix} s_1 & k-1 & 0 & ... & 0\\ s_2 & s_1 & k-2 & ... & 0\\ ... & ... & ... & ... & ...\\ ... & ... & ... & ... & ...\\ s_{k-1} & s_{k-2} & ... & ... & 1\\ s_k & s_{k-1} & ... & ... & s_1 \end{vmatrix},\qquad k=1,...,n, \tag{12}$$

which allows reproduce the values (11).

Besides, we can exhibit a relation to determine the coefficients a_j via the complete Bell polynomials [9, 18-24], in fact, we know the following representation [9, 25, 26]:

$$k!\,a_k=B_k(-0!\,s_1,-1!\,s_2,-2!\,s_3,-3!\,s_4,\ldots,-(k-2)!\,s_{k-1},-(k-1)!\,s_k). \tag{13}$$

such that [9, 27]:

$$(14)\quad B_m(x_1,x_2,...,x_m)=\begin{vmatrix} \binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & ... & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\ -1 & \binom{m-2}{0}x_1 & ... & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\ 0 & -1 & ... & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & ... & -1 & \binom{0}{0}x_1 \end{vmatrix},$$

therefore:

$$B_0=1,\ B_1=x_1,\ B_2=x_1^2+x_2, B_3=x_1^3+3x_1x_2+x_3,\ B_4=x_1^4+6x_1^2x_2+4x_1x_3+3x_2^2+x_4,$$

$$(15)\qquad B_5=x_1^5+10\,x_1^3x_2+10\,x_1^2x_3+15\,x_1x_2^2+5\,x_1x_4+10\,x_2x_3+x_5,\ \ ...$$

We see that (13) and (15) imply (11) if we employ $x_1=-s_1$, $x_2=-s_2$, $x_3=-2s_3$, $x_4=-6s_4$, $x_5=-24s_5$, ... ; thus the coefficients of the characteristic polynomial (5) are generated by the complete Bell polynomials.

3. Calculation of $\frac{d^k}{dz^k}(-z;q)_n$

Here we calculate the derivatives of the product (1). From (4) it is useful to observe that:

$$(16)\qquad \frac{d}{dz}s_m=-mgs_{m+1};$$

then:

$$\frac{df_n}{dz}=f_n\sum_{j=1}^{n}\frac{q^{j-1}}{1+zq^{j-1}}=f_ns_1=-a_1f_n,\quad \frac{d^2f_n}{dz^2}=f_n(s_1^2-s_2)=2!\,a_2f_n,$$

$$(17)\qquad \frac{d^3f_n}{dz^3}=f_n(s_1^3-3s_1s_2+2s_3)=-3!\,a_3f_n,$$

$$\frac{d^4f_n}{dz^4}=f_n(s_1^4-6s_1^2s_2+3s_2^2+8s_1s_3-6s_4)=4!\,a_4f_n,\dots,$$

which is the result (6), that is, the derivatives of (1) are determined by the coefficients of the characteristic polynomial of the matrix (3). From (4):

$$(18)\qquad [s_m]_{z=0}=\frac{1-q^{mn}}{1-q^m}$$

whose application in (11) gives the expressions:

$$[a_1]_{z=0}=-\frac{1-q^n}{1-q}=-\binom{n}{1}_q,\quad [a_2]_{z=0}=q\frac{(1-q^n)(1-q^{n-1})}{(1-q)(1-q^2)}=q\binom{n}{2}_q,$$

$$(19)\qquad [a_3]_{z=0}=-q^3\frac{(1-q^n)(1-q^{n-1})(1-q^{n-2})}{(1-q)(1-q^2)(1-q^3)}=-q^3\binom{n}{3}_q,$$

$$[a_4]_{z=0}=q^6\frac{(1-q^n)((1-q^{n-1})(1-q^{n-2})(1-q^{n-3})}{(1-q)(1-q^2)(1-q^3)(1-q^4)}=q^6\binom{n}{4}_q,\dots$$

in harmony with the relation (8); then (9) is immediate from (7) and (8). Thus, our process was a matrix approach to $q-$binomial theorem. The relation (9) implies the Euler identity [6, 28]:

$$(-z;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k}\, z^k, \tag{20}$$

because [3] $lim_{n\to\infty}\binom{n}{k}_q = \frac{1}{(q;q)_k}$.

4. Generating functions

We can use (13) for $z=0$ and the results (8) and (18) to obtain:

$$\binom{n}{k}_q q^{k(k-1)/2} = \frac{(-1)^k}{k!}\, B_k\Big(-0!\,\frac{1-q^n}{1-q}, -1!\,\frac{1-q^{2n}}{1-q^2}, \ldots, -(k-1)!\,\frac{1-q^{kn}}{1-q^k}\Big), \tag{21}$$

whose application in the $q-$binomial theorem (9) gives the interesting expression:

$$(-z;q)_n = \exp\Big(-\sum_{j=1}^{\infty} \frac{(-1)^j(1-q^{jn})}{j(1-q^j)}\, z^j\Big), \tag{22}$$

where we employ the generating function for the complete Bell polynomials:

$$\sum_{k=0}^{\infty}(-1)^k B_k(x_1,\ldots,x_k)\frac{z^k}{k!} = \exp\Big(\sum_{j=1}^{\infty}(-1)^j x_j\,\frac{z^j}{j!}\Big). \tag{23}$$

From (22):

$$(-z;q)_\infty = \exp\Big(-\sum_{j=1}^{\infty} \frac{(-1)^j}{j(1-q^j)}\, z^j\Big), \tag{24}$$

which for $z=-q$ implies:

$$(q;q)_\infty = \exp\Big(-\sum_{j=1}^{\infty}\frac{q^j}{j(1-q^j)}\Big) = \exp\Big(-\sum_{n=1}^{\infty}\Big(\sum_{d|n}\frac{1}{d}\Big)q^n\Big),$$

that is, the generating function for the sum of the inverses of divisors of an integer:

$$\sum_{n=1}^{\infty} D(n)q^n = -\log(q;q)_\infty\ , \quad D(n)=\sum_{d|n}\frac{1}{d} \tag{25}$$

Similarly, (24) for $z=q$ gives the generating function for the alternating sum of the reciprocals of divisors of an integer:

$$\sum_{n=1}^{\infty} E(n)q^n = -\log(-q;q)_\infty = \log(q;q^2)_\infty, \quad E(n)=\sum_{d|n}\frac{(-1)^d}{d}\ . \tag{26}$$

In (24) we realize the changes $q\to q^2$ and $z\to zq$ to deduce the relation:

$$(-zq;q^2)_\infty = \exp\Big(-\sum_{j=1}^{\infty}\frac{(-1)^j q^j}{j(1-q^{2j})}\, z^j\Big) \tag{27}$$

which for $z=-q$ gives the generating function for the sum of the inverses of even divisors of an integer:

$$\sum_{n=1}^{\infty} D_e(n)q^n = -\frac{1}{2}\log(q^2;q^2)_\infty, \quad D_e(n) = \sum_{even\ d|n} \frac{1}{d}; \tag{28}$$

besides:

$$D_o(n) = \sum_{odd\ d|n} \frac{1}{d} = D(n) - D_e(n) \quad \therefore \quad \sum_{n=1} D_o(n)q^n = \frac{1}{2}\log\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} = \frac{1}{2}\log\frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{29}$$

The sums $D_o(n)$ participate in an important recurrence relation [29] for the number of representations of an integer as the sum of squares [30, 31].

5. Jacobi triple product identity

From (27) for $z=-q$ & $z\to\frac{1}{z}$:

$$(-\frac{q}{z};q^2)_\infty = \exp\Big(-\sum_{j=1}^{\infty}\frac{(-1)^j q^j}{j(1-q^{2j})}\ \frac{1}{z^j}\Big), \qquad (q^2;q^2)_\infty = \exp\Big(-\sum_{j=1}^{\infty}\frac{q^{2j}}{j(1-q^{2j})}\Big). \tag{30}$$

then we construct the following product:

$$Q := (q^2;q^2)_\infty\,(-zq;q^2)_\infty\,(-\frac{q}{z};q^2)_\infty = \exp\Big[-\sum_{j=1}^{\infty}\frac{(-q)^j}{j(1-q^{2j})}\Big(z^j+\frac{1}{z^j}+(-q)^j\Big)\Big], \tag{31}$$

and we calculate its Taylor expansion with respect to $q=0$:

$$Q|_{q=0}=1, \quad \frac{dQ}{dq}\Big|_{q=0} = z+\frac{1}{z}, \quad \frac{d^2Q}{dq^2}\Big|_{q=0} = \frac{d^3Q}{dq^3}\Big|_{q=0} = 0, \quad \frac{d^4Q}{dq^4}\Big|_{q=0} = 24\Big(z^2+\frac{1}{z^2}\Big),\ldots \tag{32}$$

which is a strong motivation for the Jacobi triple product identity [2-6]:

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n = (q^2;q^2)_\infty\,(-zq;q^2)_\infty\,(-\frac{q}{z};q^2)_\infty\,, \tag{33}$$

this relation is one of the fundamental identities in $q-$series.

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