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MULTIPLE ZETA VALUES AS PERIODS: ALGEBRAIC STRUCTURE AND TRANSCENDENCE CONJECTURES

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Abstract. L. Euler (1707–1783) studied the values of

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

for integer s, and B. Riemann (1826–1866) extended ζ to a function of a complex variable. For s a positive even integer, $\zeta(s)/\pi^s$ is rational, while our knowledge of $\zeta(s)$ for odd $s \geq 3$ remains extremely limited. Recent progress concerns the larger family of *multiple zeta values*

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}, \qquad s_1, \ldots, s_k \in \mathbb{Z}_{>0},\ s_1 \geq 2.$$

Two Bourbaki seminars (P. Cartier, March 2001; P. Deligne, January 2012) have been devoted to this circle of ideas. Three $\mathbb{Q}$-algebras are intertwined here: the algebra of multiple zeta values (MZV) sitting inside $\mathbb{R}$; the algebra of *formal* MZV, built from the known combinatorial relations among them; and the algebra of *motivic* zeta values, arising from the pro-unipotent fundamental group and mixed Tate motives. The central conjecture asserts that the first two algebras are isomorphic, and outstanding progress toward this — largely due to Francis Brown — has come from the study of the third.

1. Historical background

1.1. **The Basel problem (1644).** In 1644 Pietro Mengoli asked for the exact value of

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = 1.644934\ldots,$$

a question resolved by Euler in 1734/1735. Related telescoping and alternating series were already known to Leibniz and to Johann Bernoulli:

$$\sum_{n\geq 1} \frac{1}{n(n+1)} = 1, \qquad \sum_{n\geq 0} \frac{1}{(2n+1)(2n+2)} = \log 2.$$

Example 1.1. From $\log(1+t) = \sum_{n\geq 1}(-1)^{n-1}t^n/n$ for $-1 < t \leq 1$, taking $t = 1$ gives

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

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1.2. **Euler's proof.** If $1 + a_1 z + \cdots + a_n z^n = (1 - \alpha_1 z) \cdots (1 - \alpha_n z)$, then $\alpha_1 + \cdots + \alpha_n = -a_1$: the sum of the reciprocals of the roots equals $-f'(0)$ when $f(0) = 1$. Applying this formally to

$$\frac{\sin \sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \cdots,$$

whose zeros are $\pi^2, 4\pi^2, 9\pi^2, \ldots$, together with the product formula

$$\frac{\sin x}{x} = \prod_{n \geq 1} \Big(1 - \frac{x^2}{n^2 \pi^2}\Big) = 1 - \frac{x^2}{6} + \cdots,$$

yields $\displaystyle\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

1.3. **Calabi's proof.** Using $\dfrac{1}{1 - x^2 y^2} = \sum_{n \geq 0} x^{2n} y^{2n}$ and $\int_0^1 x^{2n} dx = \dfrac{1}{2n+1}$,

$$\int_0^1 \int_0^1 \frac{dx\, dy}{1 - x^2 y^2} = \sum_{n \geq 0} \frac{1}{(2n+1)^2}.$$

The substitution $x = \sin u / \cos v,\ y = \sin v / \cos u$ transforms the left side into the area of the triangle $u, v \geq 0,\ u + v \leq \pi/2$, giving

$$\sum_{n \geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \qquad \text{hence} \qquad \zeta(2) = \frac{4}{3} \sum_{n \geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6}.$$

2. Special values of the Riemann zeta function

For $n \geq 1$, $\pi^{-2n} \zeta(2n) \in \mathbb{Q}$, via the Bernoulli numbers B_n defined by

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n \geq 1} (-1)^{n+1} B_n \frac{t^{2n}}{(2n)!}, \qquad B_1 = \tfrac{1}{6},\ B_2 = \tfrac{1}{30},\ B_3 = \tfrac{1}{42}, \ldots$$

$$\boxed{\zeta(2n) = 2^{2n-1} \frac{B_n}{(2n)!}\, \pi^{2n} \qquad (n \geq 1)}, \qquad \zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945},\ \zeta(8) = \frac{\pi^8}{9450}.$$

The function $\zeta(s)$ extends meromorphically to $\mathbb{C}$, with a single simple pole at $s = 1$ of residue 1, and

$$\lim_{s \to 1} \Big(\zeta(s) - \frac{1}{s-1}\Big) = \gamma,$$

the Euler–Mascheroni constant $\gamma = 0.5772156649\ldots$

Theorem 2.1 (Functional equation)**.**

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\, \Gamma(1-s)\, \zeta(1-s).$$

The trivial zeros of ζ are $-2, -4, -6, \ldots$; the *Riemann Hypothesis* asserts that all other zeros have real part $1/2$.

Theorem 2.2 (Lindemann [2])**.** *π is transcendental; consequently so is $\zeta(2k)$ for every $k \geq 1$.*

Theorem 2.3 (Hermite–Lindemann [1, 2])**.** *For α, β nonzero algebraic numbers with $\log\alpha \neq 0$, both $\log\alpha$ and e^β are transcendental.*

2.1. **Odd zeta values.** Determining the algebraic relations among $\zeta(2), \zeta(3), \zeta(5), \zeta(7), \ldots$ remains largely open.

Conjecture 2.4. *The numbers $\zeta(2), \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent; in particular each $\zeta(2n+1)$ and each $\zeta(2n+1)/\pi^{2n+1}$ is conjecturally transcendental.*

Theorem 2.5 (Apéry, 1978 [14])**.** *$\zeta(3)$ is irrational.*

Theorem 2.6 (Rivoal, 2000 [8])**.** *For every $\varepsilon > 0$ and every sufficiently large odd integer a, the dimension of the $\mathbb{Q}$-vector space spanned by $1, \zeta(3), \zeta(5), \ldots, \zeta(a)$ is at least $\dfrac{1-\varepsilon}{1+\log 2}\log a$.*

By work of Zudilin, at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

Theorem 2.7 (Fischler–Zudilin, [8])**.** *There exist odd integers j_1, j_2 with $5 \leq j_1 \leq 139$ and $5 \leq j_2 \leq 1961$ such that $1, \zeta(3), \zeta(j_1), \zeta(j_2)$ are linearly independent over $\mathbb{Q}$.*

3. Multiple zeta values

The question of algebraic independence of zeta values can be reduced to one of *linear* independence, since the product of two special values is a sum of multiple zeta values:

$$\zeta(s_1)\zeta(s_2) = \sum_{n_1>n_2\geq 1} \frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_2>n_1\geq 1} \frac{1}{n_2^{s_2}n_1^{s_1}} + \sum_{n\geq 1} \frac{1}{n^{s_1+s_2}}.$$

Definition 3.1 (Euler)**.** For $s_1, s_2 \geq 2$,

$$\zeta(s_1,s_2) = \sum_{n_1>n_2\geq 1} n_1^{-s_1}n_2^{-s_2}, \qquad \zeta(s_1)\zeta(s_2) = \zeta(s_1,s_2)+\zeta(s_2,s_1)+\zeta(s_1+s_2).$$

Example 3.2. $\zeta(2)^2 = 2\zeta(2,2) = \zeta(4)$, and $\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$.

For $\underline{s} = (s_1, \ldots, s_k)$ with $s_i \in \mathbb{Z}_{>0}$ and $s_1 \geq 2$, set

$$\zeta(\underline{s}) = \sum_{n_1>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}},$$

recovering ordinary zeta values when $k = 1$. We call k the *depth* and $n = s_1 + \cdots + s_k$ the *weight*.

Since the product of two MZV is again a $\mathbb{Q}$-linear combination of MZV, the $\mathbb{Q}$-vector space $\mathfrak{Z}$ spanned by all $\zeta(\underline{s})$ is in fact a $\mathbb{Q}$-algebra, and the problem of algebraic independence becomes one of linear independence within $\mathfrak{Z}$ — a space whose rich structure is not yet fully understood.

3.1. **Two main conjectures.** Let $\mathfrak{Z}_n \subset \mathfrak{Z}$ be spanned by the $\zeta(\underline{s})$ of weight n, with $\mathfrak{Z}_0 = \mathbb{Q}$, $\mathfrak{Z}_1 = \{0\}$.

Conjecture 3.3 (No relations between weights)**.**

$$\mathfrak{Z} = \bigoplus_{n\geq 0} \mathfrak{Z}_n.$$

Let $d_n = \dim_{\mathbb{Q}} \mathfrak{Z}_n$.

Conjecture 3.4 (Zagier)**.** *For $n \geq 3$, $d_n = d_{n-2}+d_{n-3}$, so that $(d_0, d_1, d_2, \dots) = (1, 0, 1, 1, 1, 2, 2, \dots)$ and*

$$\sum_{n\geq 0} d_n X^n = \frac{1}{1 - X^2 - X^3}.$$

In low weight: $d_0 = 1$ ($k = 0$, the empty product); $d_1 = 0$; $d_2 = 1$, generated by $\zeta(2,1) = \zeta(3)$; and in weight 4,

$$\zeta(3,1) = \tfrac{1}{4}\zeta(4), \qquad \zeta(2,2) = \tfrac{3}{4}\zeta(4), \qquad \zeta(2,1,1) = \zeta(4) = \tfrac{2}{5}\zeta(2)^2.$$

Theorem 3.5 (Gangl [4], a modular relation in weight 12)**.**

$$5197\,\zeta(12) = 19348\,\zeta(9,3) + 103650\,\zeta(7,5) + 116088\,\zeta(5,7).$$

Refining by depth, let $\mathfrak{Z}_n^l$ be the span of MZV of weight n and depth $\leq l$, and $d_{nl} = \dim\bigl(\mathfrak{Z}_n^l/\mathfrak{Z}_n^{l-1}\bigr)$.

Conjecture 3.6 (Broadhurst–Kreimer, [3])**.**

$$\sum_{n\geq 0}\sum_{l\geq 0} d_{nl} X^n Y^l = \frac{1 + \mathbb{E}(X)Y}{1 - \mathbb{O}(X) + \mathbb{S}(X)(Y^2 - Y^4)},$$

where $\mathbb{E}(X) = \dfrac{X^2}{1-X^2}$, $\mathbb{O}(X) = \dfrac{X^3}{1-X^2}$, *and* $\mathbb{S}(X) = \dfrac{X^{12}}{(1-X^4)(1-X^6)} = \sum_{k\geq 0} \dim_{\mathbb{C}}(S_k) X^k$ *is the generating function of dimensions of spaces of cusp forms.*

Setting $Y = 1$ recovers Zagier's conjecture. Hoffman observed that the number of tuples $(s_1, \dots, s_k)$ with each $s_i \in \{2, 3\}$ and $\sum s_i = n$ satisfies exactly Zagier's recursion, suggesting:

Conjecture 3.7 (Hoffman, [5])**.** *A $\mathbb{Q}$-basis of $\mathfrak{Z}_n$ is given by the $\zeta(s_1, \dots, s_k)$ of weight n with each $s_i \in \{2, 3\}$.*

This is compatible with all known relations among MZV for $n \leq 20$, and half of it is now a theorem:

Theorem 3.8 (Brown, [10])**.** *The numbers $\zeta(s_1, \dots, s_k)$ of weight n with each $s_i \in \{2, 3\}$* span *$\mathfrak{Z}_n$ over $\mathbb{Q}$.*

Brown's theorem shows that the algebraic independence of $\zeta(2), \zeta(3), \zeta(5), \dots$ would follow from the two main conjectures above. A combinatorial description of the linear relations among MZV is known, but it is not known whether

it accounts for *all* relations. The outstanding Diophantine problem is to establish nontrivial lower bounds for the d_n: at present, we do not even know how to prove $d_n \geq 2$ for a single value of n.

4. Periods

Definition 4.1 (Kontsevich–Zagier, [6])**.** A *period* is a complex number whose real and imaginary parts are absolutely convergent integrals of rational functions with rational coefficients, over domains of $\mathbb{R}^n$ cut out by finitely many polynomial inequalities with rational coefficients.

Theorem 4.2. *$\zeta(s)$ is a period for every integer $s \geq 2$.*

Proof. Since $\dfrac{1}{1-u_1\cdots u_s} = \sum_{n\geq 1}(u_1\cdots u_s)^{n-1}$ and $\int_0^1 u^{n-1}\,du = \frac{1}{n}$,

$$\int_{[0,1]^s} \frac{du_1\cdots du_s}{1-u_1\cdots u_s} = \sum_{n\geq 1}\Big(\int_0^1 u^{n-1}du\Big)^s = \sum_{n\geq 1}\frac{1}{n^s} = \zeta(s).$$

Equivalently, iterating the identity $\int_0^{t_1}\sum_{n\geq 1} t_2^{n-1}dt_2 = \sum_{n\geq 1} t_1^n/n$ over the simplex $1 > t_1 > \cdots > t_s > 0$ gives

$$\zeta(s) = \int_{1>t_1>\cdots>t_s>0} \frac{dt_1}{t_1}\cdots\frac{dt_{s-1}}{t_{s-1}}\cdot\frac{dt_s}{1-t_s}. \qquad \square$$

Writing $\omega_0 = dt/t$ and $\omega_1 = dt/(1-t)$, the identity above reads $\zeta(s) = \int_0^1 \omega_0^{s-1}\omega_1$, an *iterated integral* in the sense of Chen: for holomorphic 1-forms $\varphi_1,\ldots,\varphi_k$, define inductively

$$\int_0^x \varphi_1\cdots\varphi_k := \int_0^x \varphi_1(t)\int_0^t \varphi_2\cdots\varphi_k.$$

For $\underline{s} = (s_1,\ldots,s_k)$ set $\omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1$; then

Theorem 4.3. *Every MZV is a period:*

$$\zeta(\underline{s}) = \int_0^1 \omega_{\underline{s}} = \int_{1>t_1>\cdots>t_p>0} \omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1, \qquad p = s_1+\cdots+s_k.$$

Example 4.4. $\zeta(2,1) = \displaystyle\int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1}\cdot\frac{dt_2}{1-t_2}\cdot\frac{dt_3}{1-t_3} = \int_0^1 \omega_0\omega_1^2.$

4.1. **Two products, one algebra.** The product of two MZV can be expanded either from the series definition (the *stuffle* product) or from the iterated-integral representation (the *shuffle* product), and these generally give *different-looking* expressions for the same number. Comparing them produces nontrivial linear relations among MZV. For instance, decomposing the product of two simplices $\{1>t_1>t_2>0\}\times\{1>u_1>u_2>0\}$ into six essentially disjoint simplices yields

$$\zeta(2)^2 = 4\zeta(3,1) + 2\zeta(2,2) \qquad \text{(shuffle)},$$

while the stuffle product gives $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$. Comparing the two expressions produces the relation $\zeta(4) = 4\zeta(3,1)$.

4.2. **The algebra of formal periods.** Let $\mathcal{P}$ be the $\mathbb{Q}$-algebra generated by symbols $\mathbb{Z}_{\underline{s}}$ subject to all relations among MZV coming from the stuffle and shuffle products. There is a natural evaluation homomorphism $ev : \mathcal{P} \to \mathbb{R}$, $\mathbb{Z}_{\underline{s}} \mapsto \zeta(\underline{s})$ — one may think of $\mathcal{P}$ as a space of formal "programs" and ev as their execution — and it is expected (but not known) that ev is injective.

A third algebra, the algebra $\mathfrak{M}$ of *motivic* zeta values, is a graded $\mathbb{Q}$-algebra generated by homogeneous elements $\zeta^m(\underline{s})$, equipped with an evaluation map $ev^m : \mathfrak{M} \to \mathbb{R}$ with $ev^m(\zeta^m(\underline{s})) = \zeta(\underline{s})$, fitting into a commutative diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{ev^m} & \mathfrak{Z} \\ \downarrow & \nearrow_{ev} & \\ \mathcal{P} & & \end{array}$$

Brown showed that the $\zeta^m(\underline{s})$ with $s_i \in \{2, 3\}$ form a $\mathbb{Q}$-basis of $\mathfrak{M}$, giving control of its automorphism group, and deduced that the category of mixed Tate motives over $\mathbb{Z}$ is generated by the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

If, as expected, $ev^m : \mathfrak{M} \to \mathbb{R}$ is injective, it would follow that $\pi, \zeta(3), \zeta(5), \ldots$ are all transcendental and algebraically independent. In Cartier's words, this "wild dream" is a goal for around 2040.

5. Conclusion

The study of multiple zeta values lies at a remarkable intersection of classical number theory, Diophantine analysis, algebraic geometry, and motive theory. While foundational milestones—such as Francis Brown's proof that $\zeta(s_1, \ldots, s_k)$ with $s_i \in \{2, 3\}$ span the space of MZVs—have provided deep insights into the motivic structure of these numbers, fundamental Diophantine questions remain strikingly open. Establishing non-trivial lower bounds for $d_n = \dim_{\mathbb{Q}} \mathfrak{Z}_n$, or even proving $d_n \geq 2$ for a single weight n, continues to be an elusive challenge that highlights the boundary of our current analytic techniques.

Looking forward, the primary objective remains establishing the injectivity of the evaluation map $ev^m : \mathfrak{M} \to \mathbb{R}$, which would resolve the longstanding conjecture on the algebraic independence of odd zeta values $\zeta(3), \zeta(5), \ldots$. Future research directions are increasingly focused on extending these frameworks to broader structures, including multiple polylogarithms, elliptic multiple zeta values, and higher-genus motivic periods. As Pierre Cartier famously envisioned, bridging the gap between formal combinatorial relations and genuine transcendental numbers will define the landscape of motivic number theory in the coming decades.

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