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APÉRY-LIKE SEQUENCES AND Z-TRANSFORM

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Abstract. We consider the Zagier's sporadic sequences $\{f_n\}$, that is, certain Apéry-like sequences defined by three-term recurrence relations, to show the participation of the Gauss hypergeometric function and the local Heun functions in the construction of the Z-transform of $\{f_n\}$.

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1. Introduction

The Zagier's sporadic sequences [1] are Apéry-like sequences defined by three-term recurrence relations [2, 3]:

$$(n+1)^2 f_{n+1} = (an^2+an+b)f_n - cn^2 f_{n-1}, \quad f_{-1}=0, \quad f_0=1, \tag{1}$$

for a given a, b and c, thus [2]:

$$\begin{array}{cccc} (a,b,c) & f_n & (a,b,c) & f_n \\ (16,4,0) & \binom{2n}{n}^2 & (7,2,-8) & \sum_{k=0}^{n}\binom{n}{k}^3, \\ (27,6,0) & \binom{2n}{n}\binom{3n}{n} & (10,3,9) & \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}, \\ (64,12,0) & \binom{2n}{n}\binom{4n}{2n} & (12,4,32) & \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}, \\ (432,60,0) & \binom{3n}{n}\binom{6n}{3n} & (17,6,72) & \sum_{k=0}^{n}(-1)^k 8^{n-k}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^3, \end{array} \tag{2}$$

We have interest to construct the Z-transform [4, 5] of sequences defined by (1). In Sec. 2 and 3 we show that $Z\{f_n\}$ for $c=0$ and $c\neq 0$ are in terms of the Gauss hypergeometric function [6-10] and the local Heun functions [11-14], respectively.

2. $Z\{f_n\}$ FOR $c = 0$

The Z-transform has the following properties [4, 5] with $Z\{f_n\} = F(z)$:

$$Z\Big\{n^2 f_{n-1}\Big\} = \frac{1}{z}\Big(F - z\frac{dF}{dz} + z^2\frac{dF^2}{dz^2}\Big), \quad Z\Big\{n(n+1)f_n\Big\} = z^2\frac{d^2F}{dz^2},$$

$$Z\Big\{(n+1)^2 f_{n+1}\Big\} = z^2\Big(\frac{dF}{dz} + z\frac{d^2F}{dz^2}\Big), \tag{3}$$

then (1) and (3) imply the differential equation:

$$z^2(z^2 - az + c)\frac{d^2F}{dz^2} + z(z^2 - c)\frac{dF}{dz} + (c - bz)F = 0, \tag{4}$$

which acquires great simplification when $c = 0$:

$$z^2(z - a)\frac{d^2F}{dz^2} + z^2\frac{dF}{dz} - bF = 0, \tag{5}$$

whose solution is immediate in terms of the Gauss hypergeometric function [6-10]:

$$F(z) = {}_2F_1\Big(\tilde{a}, \tilde{b}; 1; \frac{a}{z}\Big), \quad \tilde{a} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{b}{a}}, \quad \tilde{b} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{b}{a}}, \quad c = 0, \quad f_0 = 1. \tag{6}$$

This expression can be applied to the first column in (2) to obtain the relations:

$$Z\left\{\binom{2n}{n}^2\right\} = {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{16}{z}\Big), \qquad Z\left\{\binom{2n}{n}\binom{3n}{n}\right\} = {}_2F_1\Big(\frac{1}{3}, \frac{2}{3}; 1; \frac{27}{z}\Big),$$

$$Z\left\{\binom{2n}{n}\binom{4n}{2n}\right\} = {}_2F_1\Big(\frac{1}{4}, \frac{3}{4}; 1; \frac{64}{z}\Big), \qquad Z\left\{\binom{3n}{n}\binom{6n}{3n}\right\} = {}_2F_1\Big(\frac{1}{6}, \frac{5}{6}; 1; \frac{432}{z}\Big), \tag{7}$$

3. $Z\{f_n\}$ FOR $c \neq 0$

Here we study (4) when $c = q_1 q_2 \neq 0$, $q_1 \neq q_2$ and $a = q_1 + q_2$, besides we make the change of variable $x = q_1/z$, then (4) adopts the form:

$$\frac{d^2F}{dx^2} + \left[\frac{3x^2 - \frac{2a}{q_2}x + \frac{q_1}{q_2}}{x(x-1)(x - \frac{q_1}{q_2})}\right]\frac{dF}{dx} + \frac{x - \frac{b}{q_2}}{x(x-1)(x - \frac{q_1}{q_2})}F = 0, \tag{8}$$

which has the same structure than the Heun's differential equation [11-14]:

$$\frac{d^2F}{dx^2} + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-r}\right]\frac{dF}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-r)}F = 0, \tag{9}$$

with $\alpha = \beta = \gamma = \delta = \epsilon = 1$, $q = b/q_2$ and $r = q_1/q_2$, therefore, the solution of (4) is given in terms of the local Heun functions:

$$F(z) = Hl\Big(\frac{q_1}{q_2}, \frac{b}{q_2}; 1, 1, 1, 1; \frac{q_1}{z}\Big), \quad c = q_1 q_2 \neq 0, \quad q_1 \neq q_2, \quad a = q_1 + q_2, \tag{10}$$

whose application to the second column in (2) allows deduce the relations:

$$Z\Big\{\sum_{k=0}^{n}\binom{n}{k}^3\Big\}=Hl\Big(-8,-2;1,1,1,1;\frac{8}{z}\Big),\ Z\Big\{\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\Big\}=Hl\Big(9,3;1,1,1,1;\frac{9}{z}\Big),$$

$$Z\Big\{\sum_{k=0}^{n}(-1)^k\,8^{n-k}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^3\Big\}=Hl\Big(\frac{9}{8},\frac{3}{4};1,1,1,1;\frac{9}{z}\Big), \tag{11}$$

$$Z\Big\{\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\Big\}=Hl\Big(2,1;1,1,1,1;\frac{8}{z}\Big).$$

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