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STUDY OF CONVERGENCE AND STABILITY OF A NOVEL THREE-STEP ITERATIVE SCHEME IN BANACH SPACES

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Abstract. A novel three-step iterative scheme is proposed in the settings of uniformly convex Banach space to approximate fixed points of generalized $\alpha-$non expansive mappings and some weak and strong convergence results are established under appropriate assumptions. Further, we study the stability, data dependence and error estimation of the proposed iterative process and its applicability to different problems. We substantiate our findings through numerical experiments, showing better convergence rate and bounded error estimations of our proposed iterative scheme as compared to a number of existing iterative schemes documented in the literature. The practicality of the proposed iterative technique is also demonstrated in solving a nonlinear fractional differential equation.

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1. Introduction

Fixed point theory has emerged as an extensive field of research with multidisciplinary applications in various branches of mathematics, science and engineering. It offers basic ideas, impactful theorems, and adaptable methods for deciphering the behavior of operators, mappings, and dynamical systems. Determining the conditions under which fixed points of mappings or operators exist and are unique is one of the main goals of fixed point theory. In many contexts, a number of fixed-point theorems offer adequate prerequisites for the uniqueness and existence of fixed points. Iterative techniques for numerically approximating fixed points are based on fixed point theory. These techniques are frequently applied to numerical analysis, optimization, and equation solving-problems for which it is difficult to find explicit solutions, see for instance [2, 7, 12, 36, 41] and some references thereof.

Let T be a nonempty closed convex subset of M and let M be a real Banach space. An element $a \in T$ satisfying $Sa = a$ is referred to as a fixed point of a mapping $S : T \to T$. $F(S)$ is the set of all fixed points of the mapping S. The mapping S is said to be contraction if a constant $\gamma \in [0, 1)$ such that $\|Sa - Sb\| \leq \gamma \|a - b\|$ and nonexpansive if $\gamma = 1$.

Numerous researchers have looked into a number of generalizations of nonexpansive mapping in recent years, see for instance [2, 6, 20, 30, 31, 37, 39, 40, 42, 43, 45]. In 2008, Suzuki [39] established a class of mappings broader than the nonexpansive mappings, and achieved certain results about the existence of fixed points for these class of mappings.

A Suzuki generalized nonexpansive mapping is a mapping $S : T \to T$ such that for every $a, b \in T$, we have

$$\frac{1}{2}\|a - Sa\| \le \|a - b\| \Longrightarrow \|Sa - Sb\| \le \|a - b\|.$$

The class of nonexpansive mappings is contained in the class of Suzuki generalized nonexpansive mappings.

Within the context of Banach spaces, Aoyama and Kohsaka [6] established the class of α−nonexpansive mapping and proved some fixed point results for such mappings. Every mapping that is nonexpansive is 0−nonexpansive, and every α−nonexpansive mapping with a non-empty fixed point set is quasi-nonexpansive. A mapping $S : T \to T$ is said to be α− nonexpansive mapping if $\exists$ a real number $\alpha \in [0, 1)$ such that for each $a, b \in T$,

$$\|Sa - Sb\|^2 \le \alpha \|Sa - b\|^2 + \alpha \|Sb - a\|^2 + (1 - 2\alpha)\|a - b\|^2.$$

Recently, Pant and Shukla [31] introduced a new class of mappings called generalized α−nonexpansive mappings, which includes Suzuki generalized nonexpansive mappings as well as α− nonexpansive mappings.

A mapping $S : T \to T$ is said to be generalized α−nonexpansive mapping if $\exists$ a real number $\alpha \in [0, 1)$ such that for each $a, b \in T$,

$$\frac{1}{2}\|a - Sa\| \le \|a - b\|$$
$$\Longrightarrow \|Sa - Sb\| \le \alpha \|Sa - b\| + \alpha \|Sb - a\| + (1 - 2\alpha)\|a - b\|.$$

Clearly, a generalized α−nonexpansive mapping for $\alpha = 0$ reduces to Suzuki generalized nonexpansive mapping [39].

An iterative process plays an important role in approximating the fixed point of the map under consideration. Some of the iterative schemes existing in the literature are: Mann iteration [26], Ishikawa iteration [23], Noor iteration [28], Agarwal et al. iteration [3], Abbas and Nazir iteration [1], SP iteration [32], CR iteration [14], Picard-S iteration [19], Thakur iteration [40], Garodia and Uddin iteration [16], and many others [5, 27].

The following iterative scheme known as the K* iterative process was introduced by Ullah and Arshad [43]:

$$\begin{cases} a_0 \in T, \\ c_n = (1 - \alpha_n)a_n + \alpha_n S a_n, \\ b_n = S\left((1 - \beta_n)c_n + \beta_n S c_n\right), \\ a_{n+1} = S b_n. \end{cases} \quad \forall n \ge 1.$$

They demonstrated that it is faster than the S, Picard-S, Thakur et al. iteration procedures using a numerical example.

In 2021, Ahmad et al.[4] provided the following iteration scheme named the JK [4] iteartive process:

$$\begin{cases} a_0 \in T, \\ c_n = (1 - \alpha_n)a_n + \alpha_n S a_n, \\ b_n = S c_n, \\ a_{n+1} = S\left((1 - \gamma_n)S c_n + \gamma_n S b_n\right). \end{cases} \quad \forall n \ge 1.$$

In [4], it is shown to have better rate of convergence than Thakur et al. [40] and Agarwal [3] iteration processes.

Many real-world problems are nonlinear or involve complex dynamics that are difficult to capture in a single or two steps. Multiple iterations allow the process to account for these complexities, adapting the solution at each step based on new information or calculations. Increasing the number of steps in an iterative process allows for more refinement, error correction, and stability, leading to better and more accurate results. Due to the prevalence of challenging operators in differential equations, optimization, and applied dynamics, iterative schemes serve as essential tools for both theoretical analysis and computational implementation. This motivates us to propose a new three-step iterative process to approximate fixed points of generalized $\alpha-$nonexpansive mappings.

Let $S : T \rightarrow T$ be a mapping defined on a nonempty convex subset T of a Banach space M . Then, taking a_0 as the arbitrary element in T, define a sequence $\{a_n\}_{n=0}^{\infty}$ as:

$$\text{(I)} \qquad \begin{cases} a_0 \in T, \\ c_n = S\left((1-\alpha_n) a_n + \alpha_n S a_n\right), \\ b_n = S\left((1-\beta_n) c_n + \beta_n S c_n\right), \\ a_{n+1} = S\left((1-\gamma_n) S b_n + \gamma_n S c_n\right). \end{cases} \qquad \forall n \geq 1.$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$.

The paper is organised as follows: In Section 2, we collect some basic definitions and essential results. In Section 3, we establish some convergence results for the proposed iterative process (I) for generalized $\alpha-$nonexpansive mapping in uniformly convex Banach spaces and give a numerical example to show that the iterative process (I) converges faster than a number of existing iteration processes. The stability of the proposed iterative technique applied to almost contraction maps is demonstrated in Section 4. We also assess the error of our proposed iterative method and demonstrate that the accumulation of errors remains bounded for nonexpansive mappings in Section 5. In the final Section 6, we apply our findings to solve nonlinear fractional differential equations.

2. Preliminaries

To establish our main results, the definitions, claims, and lemmas listed below will be helpful.

Definition 2.1. [15] A Banach space M is said to be uniformly convex if for $\epsilon > 0$, there exist a positive number δ such that, for $a, b \in M$ with $\|a\| \leq 1$, $\|b\| \leq 1$ and $\|a - b\| \leq \epsilon$, $\|a + b\| \leq 2(1 - \delta)$ holds.

Definition 2.2. [30] A Banach space M is said to satisfy Opial's condition if for any sequence $\{a_n\}$ which converges weakly to $a \in M$ implies $\lim_{n\to\infty} sup \|a_n - a\| < \lim_{n\to\infty} sup \|a_n - b\|, \forall\, b \in T$ with $a \neq b$.

Hilbert spaces and all l^p spaces$(1 < p < \infty)$ are some of the examples of Banach spaces satisfying *Opial's* condition.

Now, we present a few results of generalized $\alpha-$nonexpansive mappings required for our subsequent results.

Proposition 2.3. [31] *Let $S : T \rightarrow T$ be a mapping defined on a nonempty, closed and convex subset T of a Banach space M, then*

(i) *S is a generalized $\alpha-$nonexpansive mapping if S is a Suzuki generalized nonexpansive mapping.*
(ii) *S is a quasi-nonexpansive mapping if it is a generalized $\alpha-$ nonexpansive mapping with a fixed point.*
(iii) *$F(S)$, the set of fixed points of the mapping S is closed if it is a generalized $\alpha-$nonexpansive mapping. Moreover, $F(S)$ is convex if T is strictly convex.*
(iv) *If S is a generalized $\alpha-$nonexpansive mapping, then for each $a, b \in T$*

$$\|a - Sb\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\|a - Sa\| + \|a - b\|.$$

(v) *The sequence $\{a_n\}$ converges weakly to a point a if S has the Opial property and S is a generalized $\alpha-$nonexpansive mapping, and also if $\lim_{n\to\infty} \|Sa_n - a_n\| = 0$, then $a \in F(S)$.*

The following example illustrates that the class of generalized $\alpha-$non expansive mapping is more general than the class of Suzuki generalized nonexpansive mappings.

Example 2.1. *Consider the mapping $S : \mathbb{R} \to \mathbb{R}$ defined by $Sa = \frac{a}{2}$ for $a \leq 0$ and $Sa = \frac{a}{3}$ for $a > 0$. The mapping S satisfies generalized $\alpha-$nonexpansive mapping condition with $\alpha = \frac{1}{3}$. However, it is not a Suzuki generalized nonexpansive mapping.*

Berinde [9] presented the notion of almost contraction or weak contraction mapping. This concept is broader than the Zamfirescu mapping [44], which also contains contraction mapping, Kannan mapping [25], and Chatterjea mapping [13].

Definition 2.4. [9] A mapping $S : T \to T$ is called almost contraction mapping if $\exists\, \gamma \in (0, 1)$ and some constant $L \geq 0$, such that

$$\|Sa - Sb\| \leq \gamma \|a - b\| + L\|a - Sa\|, \quad \forall\, a, b \in T.$$

Example 2.2. *Let $T = [0, 2]$ and the mapping $S : T \to T$ be defined as:*

$$S(a) = \begin{cases} \frac{a}{3}, & if\ a \in [0, 1) \\ \frac{1}{2}, & if\ a \in [1, 2) \end{cases}$$

The mapping S being discontinuous is not a contraction mapping, but it is an almost contraction mapping for $\gamma = \frac{1}{3}$.

Definition 2.5. [17] Let $\{a_n\}$ be a bounded sequence in a nonempty subset T in a Banach space M. Then the type function $r(., \{a_n\}) : T \to [0, \infty)$ is defined by:

$$r(a, \{a_n\}) = \lim_{n\to\infty} sup \|a_n - a\|.$$

$r(\{a_n\}) = inf\{r(a, \{a_n\}) : a \in G\}$ gives the asymptotic radius and the asymptotic center $A(\{a_n\})$ of the sequence $\{a_n\}$ is defined as:

$$A(\{a_n\}) = \{a \in T : r(a, \{a_n\}) = r(\{a_n\})\}.$$

Lemma 2.6. [37] *Let M be a uniformly convex Banach space and $0 \leq p \leq \alpha_n < q < 1$ for every $n \in \mathbb{N}$, if $\{x_n\}$ and $\{y_n\}$ are two sequences in M such that $\lim_{n\to\infty} sup \|x_n\| \leq t$, $\lim_{n\to\infty} sup \|y_n\| \leq t$ and $\lim_{n\to\infty} \left\|\alpha_n x_n + (1-\alpha_n) y_n\right\| = t$ for some $t \geq 0$, then, $\lim_{n\to\infty} \left\|x_n - y_n\right\| = 0$.*

3. Convergence Results

In this section, we prove weak and strong convergence of the iteration scheme (I) for generalized $\alpha-$nonexpansive mapping in the context of uniformly convex Banach space.

Prior to deriving our main results, we first assert and prove in the following lemma that the distance of the iterates from any fixed point converges, which provides the basis for further convergence analysis.

Lemma 3.1. *Given a nonempty, closed and convex subset T of a Banach space M, let $S : T \to T$ be a generalized $\alpha-$nonexpansive mapping with nonempty fixed point set. If $\{a_n\}$ is a sequence given by iterative process (I), then for every $a \in F(S)$, $\lim_{n\to\infty} \|a_n - a\|$.*

Proof. Let $a \in F(S)$, then,

$$\begin{aligned}\|c_n - a\| &= \|S((1-\alpha_n)a_n + \alpha_n S a_n) - a\| \\ &= \|S((1-\alpha_n)a_n + \alpha_n S a_n) - Sa\| \\ &\leq \|(1-\alpha_n)a_n + \alpha_n S a_n - a\|,\end{aligned}$$

by Proposition 2.3(ii)

$$\begin{aligned}\|c_n - a\| &\leq (1-\alpha_n)\|a_n - a\| + \alpha_n \|S a_n - a\| \\ &\leq (1-\alpha_n)\|a_n - a\| + \alpha_n \|a_n - a\|\end{aligned}$$

$$\Rightarrow \|c_n - a\| \leq \|a_n - a\|. \tag{1}$$

Also,

$$\begin{aligned}\|b_n - a\| &= \|S((1-\beta_n)c_n + \beta_n S c_n) - a\| \\ &= \|S((1-\alpha_n)c_n + \alpha_n S c_n) - Sa\| \\ &\leq \|(1-\beta_n)c_n + \beta_n S c_n - a\| \\ &\leq (1-\beta_n)\|c_n - a\| + \beta_n \|S c_n - a\| \\ &\leq (1-\beta_n)\|c_n - a\| + \beta_n \|c_n - a\|\end{aligned}$$

$$\Rightarrow \|b_n - a\| \leq \|c_n - a\|. \tag{2}$$

Also,

$$\begin{aligned}\|a_{n+1} - a\| &= \|S((1-\gamma_n)S b_n + \gamma_n S c_n) - a\| \\ &= \|S((1-\alpha_n)S b_n + \alpha_n S c_n) - Sa\| \\ &\leq \|(1-\gamma_n)S b_n + \gamma_n S c_n - a\| \\ &\leq (1-\gamma_n)\|S b_n - a\| + \gamma_n \|S c_n - a\| \\ &\leq (1-\gamma_n)\|b_n - a\| + \gamma_n \|c_n - a\|.\end{aligned}$$

Using (1) & (2), we get

$$\begin{aligned}\|a_{n+1} - a\| &\leq (1-\gamma_n)\|c_n - a\| + \gamma_n \|c_n - a\| \\ &\leq (1-\gamma_n)\|a_n - a\| + \gamma_n \|a_n - a\| \\ \Longrightarrow & \|a_{n+1} - a\| \leq \|a_n - a\|,\end{aligned}$$

Thus, $\{\|a_n - a\|\}$ is bounded and nonincreasing sequence, which implies $\lim_{n\to\infty} \|a_n - a\|$ exists for all $a \in F(S)$.

□

The following theorem proves that the fixed point set of generalized $\alpha-$nonexpansive mappings is nonempty when the iterative sequence is bounded and asymptotically regular.

Theorem 3.2. *Given a nonempty, closed and convex subset T of a Banach space M, let $S : T \to T$ be a generalized $\alpha-$nonexpansive mapping. Let the sequence $\{a_n\}$ defined by (I), then $F(S) \neq \emptyset$ if and only if $\{a_n\}$ is bounded and $\lim_{n\to\infty} \|S a_n - a_n\| = 0$.*

Proof. Suppose that $F(S) \neq \emptyset$ and $a \in F(S)$, then by Lemma 3.1, $\lim_{n\to\infty} \|a_n - a\|$ exists and $\{a_n\}$ is bounded.
Let

$$\lim_{n\to\infty} \|a_n - a\| = t, \tag{3}$$

Using Lemma 3.1 and above as $\|c_n - a\| \leq \|a_n - a\|$, we get

$$\lim_{n\to\infty} sup \|c_n - a\| \leq \lim_{n\to\infty} sup \|a_n - a\| = t, \tag{4}$$

By Proposition 2.3 (ii), every generalized $\alpha-$nonexpansive mapping with nonempty fixed-point set is a quasi-nonexpansive mapping, which gives

$$\lim_{n\to\infty} sup \|S a_n - a\| \leq \lim_{n\to\infty} sup \|a_n - a\| = t, \tag{5}$$

Again, by the proof of Lemma 3.1, we have

$$\begin{aligned} \|a_{n+1} - a\| &\leq (1-\gamma_n)\|b_n - a\| + \gamma_n \|c_n - a\| \\ &\leq (1-\gamma_n)\|c_n - a\| + \gamma_n \|c_n - a\| \\ &= \|c_n - a\|. \end{aligned}$$

$$\implies \|a_{n+1} - a\| \leq \|c_n - a\|$$

$$i.e.\ t \leq \lim_{n\to\infty} inf \|c_n - a\|, \tag{6}$$

Using (4) & (6), we obtain

$$t = \lim_{n\to\infty} \|c_n - a\|. \tag{7}$$

which implies that,

$$\begin{aligned} t = \lim_{n\to\infty} \|c_n - a\| &= \lim_{n\to\infty} \|S((1-\alpha_n)a_n + \alpha_n S a_n) - a\| \\ &\leq \lim_{n\to\infty} \|(1-\alpha_n)a_n + \alpha_n S a_n - a\| \\ &\leq \lim_{n\to\infty} \|(1-\alpha_n)(a_n - a) + \alpha_n (S a_n - a)\| \\ &\leq \lim_{n\to\infty} \{(1-\alpha_n)\|a_n - a\| + \alpha_n \|S a_n - a\|\} \\ &\leq \lim_{n\to\infty} (1-\alpha_n)\|a_n - a\| + \alpha_n \|a_n - a\| \quad = t, (using\ 3) \end{aligned}$$

Hence,

$$t = \lim_{n\to\infty} \|(1-\alpha_n)(a_n - a) + \alpha_n (S a_n - a)\| \tag{8}$$

Now from (3), (5) and (8) together with Lemma 2.6, we obtain

$$\lim_{n\to\infty} \|S a_n - a_n\| = 0.$$

Conversely, to prove $F(S) \neq \emptyset$, we consider that the sequence $\{a_n\}$ is bounded and $\lim_{n\to\infty} \|S a_n - a_n\| = 0$.
Let $a \in A(T, \{a_n\})$, we have

$$\begin{aligned} r(Sa, \{a_n\}) &= \lim_{n\to\infty} sup \|a_n - Sa\| \\ &\leq \left(\frac{3+\alpha}{1-\alpha}\right) \lim_{n\to\infty} sup \|S a_n - a_n\| + \lim_{n\to\infty} sup \|a_n - a\| \\ &= \lim_{n\to\infty} sup \|a_n - a\| \\ &= r(a, \{a_n\}) . \end{aligned}$$

Consequently, we deduce that $Sa \in A(T, \{a_n\})$ and $A(T, \{a_n\})$ consists of a unique element because M is uniformly convex. Hence, a is a fixed point of S. □

In many iterative schemes, proving only asymptotic regularity is not sufficient to guarantee convergence to a fixed point, since the sequence may still have multiple weak limits. To overcome this, additional geometric conditions of the space, such as uniform convexity together with the Opial property, are employed. The following theorem shows that under these assumptions, the iterative sequence generated by process (I) converges weakly to a fixed point of the mapping.

Theorem 3.3. *Let $S : T \to T$ be a generalized $\alpha-$nonexpansive mapping with $F(S) \neq \emptyset$ defined on a nonempty, closed and convex subset T of a uniformly convex Banach space M with Opial property. Then, the sequence $\{a_n\}$ as given by (I), converges weakly to an element of fixed-point set of the mapping S.*

Proof. By Theorem 3.2, $\{a_n\}$ is bounded and $\lim_{n\to\infty} \|S a_n - a_n\| = 0$. Given that M is a uniformly convex set, there is a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $\{a_{n_i}\}$ weakly converges to some $v_1 \in T$. We have $v_1 \in F(S)$ by Proposition 2.3 (v).

It suffices to show that $\{a_n\}$ converges weakly to v_1, which we will demonstrate using the contradiction technique. Let the sequence $\{a_n\}$ doesnot converges weakly to v_1, then $\exists$ a subsequence $\left\{a_{n_j}\right\}$of $\{a_n\}$ and $v_2 \in T$ such that $\left\{a_{n_j}\right\}$ converges weakly to v_2 and $v_2 \neq v_1$. By Proposition 2.3 (v), $v_2 \in F(S)$.
By Lemma 3.1 and the *Opial* property

$$\begin{aligned} \lim_{n\to\infty} \|a_n - v_1\| &= \lim_{n\to\infty} \left\|a_{n_i} - v_1\right\| < \lim_{i\to\infty} \left\|a_{n_i} - v_2\right\| \\ &= \lim_{n\to\infty} \|a_n - v_2\| = \lim_{j\to\infty} \left\|a_{n_j} - v_2\right\| \\ &< \lim_{j\to\infty} \left\|a_{n_j} - v_1\right\| = \lim_{n\to\infty} \|a_n - v_1\| . \end{aligned}$$

which is a contradiction, so $v_1 = v_2$. Thus, the sequence $\{a_n\}$ converges weakly to $v_1 \in F(S)$. □

Weak convergence only ensures that the iterates approach the fixed point set in a generalized sense, whereas strong convergence requires convergence in norm. The following theorem provides a necessary and sufficient condition for strong convergence in terms of the limiting distance to the fixed point set.

Theorem 3.4. *Let $S : T \to T$ be a generalized $\alpha-$nonexpansive mapping defined on a nonempty, closed and convex subset T of a uniformly convex Banach space M. Then, the sequence $\{a_n\}$ as given by (I) converges strongly to an element of $F(S)$ if and only if $\lim_{n\to\infty} \inf d(a_n, F(S)) = 0$, where $d(a, F(S)) = inf\{\|a - v\|, v \in F(S)\}$.*

Proof. The proof of the necessary condition is obvious. To prove conversely, we assume that $\lim_{n\to\infty} inf d(a_n, F(S)) = 0$. From Lemma 3.1, $\lim_{n\to\infty} \|a_n - a\|$ exists for all $a \in F(S)$, it follows that $\lim_{n\to\infty} \inf d(a_n, F(S))$ exists, but by hypothesis $\lim_{n\to\infty} \inf d(a_n, F(S)) = 0$, thus $\lim_{n\to\infty} d(a_n, F(S)) = 0$. Now we will prove that $\{a_n\}$ is a Cauchy sequence in T. Since, $\lim_{n\to\infty} \inf d(a_n, F(S)) = 0$, then given $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $\forall\, n, k \geq n_0$, we have

$$d(a_n, F(S)) \leq \tfrac{\epsilon}{2}, \quad d(a_k, F(S)) \leq \tfrac{\epsilon}{2}$$

Thus, we have

$$\begin{aligned}\|a_n - a_k\| &\leq \|a_n - a\| + \|a - a_k\|\\ &\leq d(a_n, F(S)) + d(a_k, F(S))\\ &\leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$

which proves that $\{a_n\}$ is a Cauchy sequence in T. Now, as the set T is closed, there exists a point $a \in T$ such that $\lim_{n\to\infty} a_n = a$.
Since, $\lim_{n\to\infty} d(a_n, F(S) = 0 \Longrightarrow \lim_{n\to\infty} d(a, F(S)) = 0$. Hence, $a \in F(S)$ as the fixed-point set of the mapping S is closed. □

We now give an example of a generalized α−nonexpansive mapping and compare our iterative scheme (I) with some of the known iteration processes in terms of their convergence behavior.

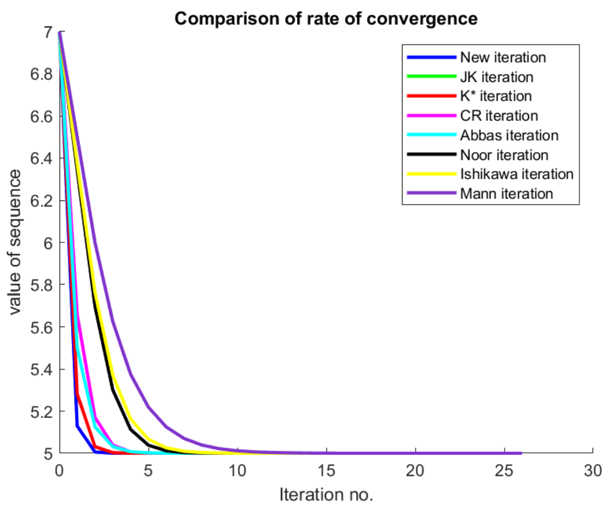

Figure 1. Comparison of rate of convergence of iterative schemes for Example 3.1

Example 3.1. *Given the usual norm* $|.|$ *and the set* $T = [5, 10]$*, define a mapping* $S : T \to T$ *as follows:*

$$Sa = \begin{cases} \frac{a+5}{2}, & if \quad 5 \leq a < 10 \\ 5, & if \quad a = 10 \end{cases}$$

Then, the mapping S is generalized $\alpha-$nonexpansive for $\alpha = \frac{1}{2}$ but for $a = 8.8, b = 10$, the Suzuki generalized nonexpansive mapping condition is not satisfied, $\frac{1}{2}|Sa - Sb| < |a - b|$, but $|Sa - Sb| > |a - b|$.
The fixed point of the above defined mapping is $a = 5$. Taking $\alpha_n = \beta_n = \gamma_n = \delta_n = \frac{n}{n+1}, n \in \mathbb{N}$, and initial value $x_0 = 7$ in New, GU, JK, KF, K*, CR, Abbas, Noor, Ishikawa, Mann iterative schemes, Table 1 shows the comparison of convergence rate of the iterative schemes which is also shown graphically in Figure1.

Table 1. Comparison of rate of convergence of iterative schemes for Example 1

Iteration	New	JK	K*	CR	Abbas	Noor	Ishikawa	Mann
1	7.0000000	7.0000000	7.0000000	7.0000000	7.0000000	7.0000000	7.0000000	7.0000000
2	5.1289063	5.2812500	5.2812500	5.6562500	5.5000000	6.3437500	6.3750000	6.5000000
3	5.0059679	5.0312500	5.0312500	5.1701389	5.1250000	5.6967593	5.7638889	6.0000000
4	5.0002258	5.0030518	5.0030518	5.0382148	5.0312500	5.3007496	5.3700087	5.6250000
5	5.0000075	5.0002747	5.0002747	5.0077958	5.0078125	5.1130819	5.1628038	5.3750000
6	5.0000002	5.0000234	5.0000234	5.0014843	5.0019531	5.0381520	5.0667043	5.2187500
7	5.0000000	5.0000019	5.0000019	5.0002683	5.0004883	5.0117904	5.0258649	5.1250000
8		5.0000002	5.0000002	5.0000466	5.0001221	5.0033880	5.0095983	5.0703125
9		5.0000000	5.0000000	5.0000078	5.0000305	5.0009156	5.0034364	5.0390625
10				5.0000013	5.0000076	5.0002347	5.0011942	5.0214844
11				5.0000002	5.0000019	5.0000575	5.0004046	5.0117188
12				5.0000000	5.0000005	5.0000135	5.0001342	5.0063477
13					5.0000001	5.0000031	5.0000437	5.0034180
14					5.0000000	5.0000007	5.0000140	5.0018311
15						5.0000001	5.0000044	5.0009766
16						5.0000000	5.0000014	5.0005188
17							5.0000004	5.0002747
18							5.0000000	5.0001450
19								5.0000763
20								5.0000401
21								5.0000210
22								5.0000110
23								5.0000057
24								5.0000030
25								5.0000015
26								5.0000008
27								5.0000000

The selection of the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ will not affect the convergence of the iterative scheme (I), as shown by the example that follows.

Example 3.2. *Let $M = (C[0, 1], \|.\|_\infty)$, be the Banach space. The supremum norm on $C^1[0, 1]$ represented by $\|.\|_\infty$ is given by $\|x\|_\infty = \{sup|x(t)| : t \in [0, 1]\}$. Consider the following initial value problem (IVP)*

$$x'(t) = \frac{1}{3}x(t) - t, \quad x(0) = 0$$

Finding the solution of the above IVP is equivalent to identifying a fixed point of an integral operator
$S : C[0, 1] \to C[0, 1]$ defined by:

$$S(x(t)) = x(0) + \int_0^t \left[\frac{1}{3}x(r) - r\right] dr. \tag{9}$$

The operator S being a contraction is also a generalized $\alpha-$nonexpansive mapping, hence by Theorem 3.4 for any choice of real sequences $\{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$, the operator S converges to its unique fixed point $x^*(t) = 3t - 9e^{\frac{t}{3}} + 9$

for the initial function $x(0) = 0$. Parametric influence on difference between exact and approximate solution is shown in Table 2, (1.456(-16) stands for 1.456×10^{-16}).

Table 2. Influence of parameters on error between exact and approximate solution for Example 3.2

	Error=$\lvert x^*(t) - x_{n+1}(t)\rvert$				
t	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
$\alpha_n=\beta_n=\gamma_n=\frac{1}{n+1}$	1.456(-16)	5.456(-16)	4.548(-16)	1.232(-15)	1.532(-15)
$\alpha_n=\beta_n=\gamma_n=\frac{1}{\sqrt[3]{n}+1}$	1.456(-16)	5.456(-16)	4.548(-16)	1.232(-15)	1.532(-15)
$\alpha_n=\beta_n=\gamma_n=\frac{1}{n^4+1}$	1.456(-16)	5.456(-16)	4.548(-16)	1.232(-15)	1.532(-15)
$\alpha_n=\beta_n=\gamma_n=\frac{n}{n+1}$	1.456(-16)	5.456(-16)	4.542(-16)	1.246(-15)	1.376(-15)
$\alpha_n=\beta_n=\gamma_n=\frac{1}{3}$	1.456(-16)	5.458(-16)	4.421(-16)	1.459(-15)	6.066(-16)
$\alpha_n=\beta_n=\gamma_n=\frac{n}{\sqrt[10]{n}+1}$	1.456(-16)	5.456(-16)	4.542(-16)	1.246(-15)	1.376(-15)
$\alpha_n=\frac{n}{n+1}, \beta_n=\frac{1}{n^3+1}, \gamma_n=\frac{1}{n^2+1}$	1.456(-16)	5.467(-16)	3.919(-16)	2.353(-15)	1.534(-15)

	Error=$\lvert x^*(t) - x_{n+1}(t)\rvert$				
t	**0.6**	**0.7**	**0.8**	**0.9**	**1**
$\alpha_n=\beta_n=\gamma_n=\frac{1}{n+1}$	1.620(-16)	8.172(-16)	3.579(-16)	2.680(-15)	8.397(-15)
$\alpha_n=\beta_n=\gamma_n=\frac{1}{\sqrt[3]{n}+1}$	1.620(-16)	8.171(-16)	3.578(-16)	2.680(-15)	8.397(-15)
$\alpha_n=\beta_n=\gamma_n=\frac{1}{n^4+1}$	1.620(-16)	8.172(-16)	3.578(-16)	2.680(-15)	8.397(-15)
$\alpha_n=\beta_n=\gamma_n=\frac{n}{n+1}$	1.326(-15)	7.172(-15)	9.845(-16)	9.366(-17)	7.961(-16)
$\alpha_n=\beta_n=\gamma_n=\frac{1}{3}$	1.423(-16)	6.914(-16)	9.845(-16)	9.372(-17)	7.959(-16)
$\alpha_n=\beta_n=\gamma_n=\frac{n}{\sqrt[10]{n}+1}$	1.326(-15)	7.172(-15)	9.845(-16)	9.366(-17)	7.962(-16)
$\alpha_n=\frac{n}{n+1}, \beta_n=\frac{1}{n^3+1}, \gamma_n=\frac{1}{n^2+1}$	1.423(-16)	6.914(-16)	9.845(-16)	9.375(-17)	7.958(-16)

We can see from the Table 2 that the error in the exact and approximate solution generated by the iterative algorithm (I) for the integral operator S in (9) is independent of the choice of the parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$.

The stability of iterative schemes helps in selecting appropriate methods for different types of problems and ensuring reliable and accurate numerical solutions. In real-world applications, where computing errors (due to rounding off, numerical approximations of functions, integrals, derivatives) are unavoidable, numerical stability is critical, and attaining convergence despite these errors is vital to the dependability of the iterative process. This is discussed in the following section.

4. Stability Result

The notion of numerical stability is presented, characterizing the fixed-point iterative process as numerically stable if, in comparison to the theoretical sequence $\{a_n\}$, the approximation sequence $\{b_n\}$ converges to the fixed point of the mapping S at each stage. The concept of stability of fixed point iterative procedures has been widely studied by [8, 10, 20, 22, 21, 35, 38, 34] and many others.

On the basis of the following idea, Harder [20] defined stability results in metric spaces.

Definition 4.1. [20] Consider a sequence $\{a_n\}$ in T, with $S : T \to T$, the iterative process $a_{n+1} = f(S, a_n)$, that converges to fixed point a, is said to be S-stable, if for $\epsilon_n = \|a_{n+1} - f(S, a_n)\|$,
$\forall n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \epsilon_n = 0 \Longleftrightarrow \lim_{n\to\infty} a_n = a.$$

The following Lemma given by Berinde [8] shall be used in the subsequent result.

Lemma 4.2. [8] *If δ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{t_n\}_{n=0}^{\infty}$ satisfying $t_{n+1,} \leq \delta t_n + \epsilon_n$, $n = 0, 1, 2, \ldots$ we have, $lim_{n\to\infty} t_n = 0$*

In practical computations, iterative schemes are unavoidably affected by numerical errors and perturbations. The notion of S-stability guarantees that such perturbations do not alter the convergence of the process, thereby ensuring its robustness. The following theorem establishes that the iterative process (I) possesses S-stability in case of an almost contraction mapping.

Theorem 4.3. *Let S be an almost contraction mapping defined on a closed convex subset T of a Banach space M and the iterative sequence $\{a_n\}$ as defined by (I) with sequences $\alpha_n, \beta_n, \gamma_n \in (0, 1)$, satisfying the condition $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the iterative process (I) is S-stable.*

Proof. Given that $\{t_n\} \subset M$ be an arbitrary sequence in T and $\{a_n\}$ is a sequence iteratively generated by (I) such that $a_{n+1} = f(S, a_n)$ which converges to a unique point a, define $\epsilon_n = \|t_{n+1} - f(S, t_n)\|$. In order to prove the stability of the mapping S, we have to show that $\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = a$.

From (I), we have

$$\begin{aligned}
\|c_n - a\| &= \|S((1-\alpha_n)a_n + \alpha_n S a_n) - a\| \\
&= \|Sa - S((1-\alpha_n)a_n + \alpha_n S a_n)\| \\
&\leq \gamma \|a - ((1-\alpha_n)a_n + \alpha_n S a_n)\| + L\|a - Sa\| \\
&= \gamma \|((1-\alpha_n)a_n + \alpha_n S a_n) - a\| \\
&\leq \gamma(1-\alpha_n)\|a_n - a\| + \gamma\alpha_n \|Sa_n - a\| \\
&= \gamma(1-\alpha_n)\|a_n - a\| + \gamma\alpha_n \|Sa_n - Sa\| \\
&\leq \gamma(1-\alpha_n)\|a_n - a\| + \gamma\alpha_n(\gamma\|a_n - a\| + L\|a_n - Sa_n\|) \\
&= \gamma(1-\alpha_n)\|a_n - a\| + \gamma^2\alpha_n\|a_n - a\| \\
&= \gamma(1-(1-\gamma)\alpha_n)\|a_n - a\|,
\end{aligned}$$

Or

$$\|c_n - a\| \leq \gamma(1-(1-\gamma)\alpha_n)\|a_n - a\|. \tag{10}$$

Proceeding as above, we get

$$\|b_n - a\| \leq \gamma(1-(1-\gamma)\beta_n)\|c_n - a\|$$

On using (10), this gives

$$\|b_n - a\| \leq \gamma^2(1-(1-\gamma)\alpha_n)(1-(1-\gamma)\beta_n)\|a_n - a\| \tag{11}$$

$$\begin{aligned}
\|a_{n+1}| &= \|S((1-\gamma_n)Sb_n + \gamma_n Sc_n) - a\| \\
&= \|Sa - S((1-\gamma_n)Sb_n + \gamma_n Sc_n)\| \\
&\leq \gamma \|a - ((1-\gamma_n)Sb_n + \gamma_n Sc_n)\| + L\|a - Sa\| \\
&\leq \gamma(1-\gamma_n)\|Sb_n - a\| + \gamma\gamma_n \|Sc_n - a\| \\
&= \gamma(1-\gamma_n)\|Sa - Sb_n\| + \gamma\gamma_n \|Sa - Sc_n\| \\
&\leq \gamma(1-\gamma_n)[\gamma\|a-b_n\| + L\|a - Sa\|] + \gamma\gamma_n[\gamma\|a - c_n\| + L\|a - Sa\|] \\
&= \gamma^2(1-\gamma_n)\|b_n - a\| + \gamma^2\gamma_n\|c_n - a\| \\
&= \gamma^4(1-\gamma_n)(1-(1-\gamma)\alpha_n)(1-(1-\gamma)\beta_n)\|a_n - a\| \\
&+ \gamma^3\gamma_n(1-(1-\gamma)\alpha_n)\|a_n - a\|
\end{aligned}$$

(12) $\therefore \|a_{n+1}-a\| \leq \gamma^3(1-(1-\gamma)\alpha_n)[\gamma(1-\gamma_n)(1-(1-\gamma)\beta_n) + \gamma_n]\|a_n - a\|$

Let $\lim_{n\to\infty} \epsilon_n = 0$, then

$$\begin{aligned}
\|t_{n+1} - a\| &= \|t_{n+1} - f(S,t_n) + f(S,t_n) - a\| \\
&\leq \|t_{n+1} - f(S,t_n)\| + \|f(S,t_n) - a\| \\
&= \epsilon_n + \|f(S,t_n) - a\| \\
&\leq \gamma^3(1-(1-\gamma)\alpha_n)[\gamma(1-\gamma_n)(1-(1-\gamma)\beta_n) + \gamma_n]\|t_n - a\| + \epsilon_n
\end{aligned}$$

Also, since $\gamma \in (0,1)$ and $\beta_n \in (0,1)$ for all $n \in \mathbb{N}$, $\gamma(1-(1-\gamma)\beta_n) < 1$, let $\rho = \gamma(1-(1-\gamma)\beta_n)$ then

$$\begin{aligned}
[\gamma(1-\gamma_n)(1-(1-\gamma)\beta_n) + \gamma_n] &= \gamma(1-(1-\gamma)\beta_n) - \gamma_n(\gamma(1-(1-\gamma)\beta_n) - 1) \\
&= \rho - \gamma_n(\rho - 1) = \rho(1-\gamma_n) - \gamma_n < 1.
\end{aligned}$$

(13) $$[\gamma(1-\gamma_n)(1-(1-\gamma)\beta_n) + \gamma_n] < 1$$

and also $\lim_{n\to\infty} \epsilon_n = 0$, so, by using Lemma 4.2, we get

$$\lim_{n\to\infty} t_n = a.$$

Conversely, suppose that $\lim_{n\to\infty} t_n = a$, proceeding similarly as before, we obtain

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - f(S,t_n)\| \\
&= \|t_{n+1} - a + a - f(S,t_n)\| \\
&\leq \|t_{n+1} - a\| + \|f(S,t_n) - a\| \epsilon_n \\
&\leq \|t_{n+1} - a\| + \gamma^3(1-(1-\gamma)\alpha_n)[\gamma(1-\gamma_n)(1-(1-\gamma)\beta_n) + \gamma_n]\|a_n - a\|.
\end{aligned}$$

Using the hypothesis, $\lim_{n\to\infty} t_n = a$, we get $\lim_{n\to\infty} \epsilon_n = 0$. Hence, the iterative process (I) is S-stable. □

The following example verifies the above result.

Example 4.1. *Let $T = [0,2]$ and the mapping $S : T \to T$ be defined as*

$$S(a) = \begin{cases} \frac{a}{3}, & if \quad a \in [0,1) \\ \frac{1}{2}, & if \quad a \in [1,2) \end{cases}$$

Clearly, the fixed point of the mapping S is 0. The mapping S being discontinuous at $a = 1$, is not a contraction mapping, but S is an almost contraction mapping, for $\gamma = \frac{1}{3}$ and for $L \geq 0$.
Case (i) When $a, b \in [0, 1)$

$$\|Sa - Sb\| - \gamma \|a - b\| - L\|a - Sa\| = \frac{1}{3}|a - b| - \frac{1}{3}|a - b| - L\left|a - \frac{a}{3}\right|$$

$$= -L\left|\frac{2a}{3}\right| \leq 0.$$

Case (ii) When $a, b \in [1, 2)$

$$\|Sa - Sb\| = \left|\frac{1}{2} - \frac{1}{2}\right| = 0.$$

$$\gamma \|a - b\| + L\|a - Sa\| = \frac{1}{3}|a - b| + L\left|a - \frac{1}{2}\right| \geq 0 = \|Sa - Sb\|$$

So, in both the above cases $\|Sa - Sb\| \leq \gamma \|a - b\| + L\|a - Sa\|$, i.e. S is an almost contraction mapping. Now, we show that the iterative scheme (I) is is S-stable with respect to S.
Let $\alpha_n = \beta_n = \gamma_n = \frac{1}{n+3}$ and $a_0 \in [0, 1]$, then we have

$$c_n = \frac{1}{3}\left(1 - \frac{1}{n+3} + \frac{1}{3(n+3)}\right)a_n = \frac{1}{3}\left(1 - \frac{2}{3(n+3)}\right)a_n,$$

$$\begin{aligned} b_n &= \frac{1}{3}\left(\left(1 - \frac{1}{n+3}\right)\left(\frac{1}{3}\left(1 - \frac{2}{3(n+3)}\right)\right) + \frac{1}{n+3}\left(\frac{1}{9}\left(1 - \frac{2}{3(n+3)}\right)\right)\right)a_n \\ &= \frac{1}{9}\left(1 - \frac{2}{3(n+3)}\right)^2 a_n, \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \frac{1}{3}\left(\left(1 - \frac{1}{n+3}\right)\left(\frac{1}{27}\left(1 - \frac{2}{3(n+3)}\right)^2\right) + \frac{1}{n+3}\left(\frac{1}{9}\left(1 - \frac{2}{3(n+3)}\right)\right)\right)a_n \\ &= \frac{1}{81}\left(1 + \frac{2}{3(n+3)} - \frac{10}{9(n+3)^2}\right)a_n \\ &= \left(\frac{1}{81} + \frac{2}{243(n+3)} - \frac{10}{729(n+3)^2}\right)a_n. \end{aligned}$$

Let $\delta = \frac{1}{81} + \frac{2}{243(n+3)} - \frac{10}{729(n+3)^2}$. Obviously, $\delta \in (0, 1)$, so by Lemma 4.2 $\lim_{n\to\infty} a_n = a$.
Let $t_n = \frac{1}{n+2}$

$$\begin{aligned} \epsilon_n &= \|t_{n+1} - f(S, t_n)\| \\ &= \left|t_{n+1} - \left(\frac{1}{81} + \frac{2}{243(n+3)} - \frac{10}{729(n+3)^2}\right)t_n\right| \\ &= \left|\frac{1}{n+3} - \left(\frac{1}{3^4(n+3)} + \frac{2}{3^6(n+3)^2} - \frac{10}{3^7(n+3)^3}\right)\right|. \end{aligned}$$

Clearly, $\lim_{n\to\infty} \epsilon_n = 0$. Hence, the iterative scheme (I) is stable with respect to S.

The next illustration shows that the iterative method (I) is S-stable irrespective of the selection of real sequences $\alpha_n, \beta_n, \gamma_n \in [0, 1]$.

Example 4.2. *Let $M = (C[0,1], \|.\|_\infty)$, be the Banach space. The supremum norm on $C^1[0,1]$ represented by $\|.\|_\infty$ is given by $\|x\|_\infty = \{sup\,|x(t)| : t \in [0,1]\}$. Consider the boundary value problem (BVP)*

$$x''(t) + tx(t) = 6t + t^4, \qquad t \in [0,1]$$

subject to

$$x(0) = 0, \quad x(1) = 1.$$

The exact solution to the (BVP) defined above is $x(t) = t^3$. Establishing the existence of solutions for the BVP is analogous to finding a continuous solution to an integral equation, as follows:

$$S(x(t)) = x(t) + \int_0^1 G(t,s) f(s,x,x')\,ds,$$

where,

$$f(t,x,x') = x''(t) + tx(t) - 6t - t^4,$$

and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t, \\ t(1-s), & t \le s \le 1. \end{cases}$$

The mapping $S : C^1[0,1] \to C^1[0,1]$ defined above being a contraction mapping, is also an almost contraction mapping. Subsequently, according to Theorem 3.4, the iterative algorithm (I) associated with S converges to the exact solution $x(t) = t^3$, and is S-stable irrespective of the selection of real sequences $\alpha_n, \beta_n, \gamma_n \in [0,1]$, by Theorem 4.3. Next, we present the subsequent cases:

Case 1. Let $y_n(t) = t^3 + \frac{t}{n^{10}+1}$, $t \in [0,1]$, $n \in \mathbb{N}$. Clearly, $y_n(t) \to t^3 = x(t) = S(x(t))$, as $n \to \infty$. Let $\alpha_n, \beta_n, \gamma_n = \frac{1}{n+1}$ for any $n \in \mathbb{N}$ and set $\epsilon_n = \|y_{n+1} - f(S, y_n)\|_\infty$, where S is generated by the sequence (I). Then it can be observed from Table 3 and Figure 2 that $\epsilon_n \to 0$ when $n \to \infty$, where $n = 20(5)45$.

Case 2. Let $y_n(t) = \frac{n^2 t^3}{n^2+1}$, $t \in [0,1]$, $n \in \mathbb{N}$. Clearly, $y_n(t) \to t^3 = x(t) = S(x(t))$, as $n \to \infty$. Let $\alpha_n, \beta_n, \gamma_n = \frac{1}{n+1}$ for any $n \in \mathbb{N}$ and set $\epsilon_n = \|y_{n+1} - f(S, y_n)\|_\infty$, where S is generated by the sequence (I). Then it can be observed from Table 4 and Figure 2 that $\epsilon_n \to 0$ when $n \to \infty$, where $n = 50(50)300$.

TABLE 3. Values of $\epsilon_n(t) = \|y_{n+1} - f(S, y_n)\|$ for $n = 20(5)45$ (**Case 1**)

t	$\epsilon_{20}(t)$	$\epsilon_{25}(t)$	$\epsilon_{30}(t)$	$\epsilon_{35}(t)$	$\epsilon_{40}(t)$	$\epsilon_{45}(t)$
0.1	3.9461(-15)	3.5713(-16)	4.9656(-17)	9.3241(-18)	2.1684(-18)	4.3368(-19)
0.2	7.8877(-15)	7.1297(-16)	9.7144(-17)	1.5612(-17)	1.7347(-18)	1.7347(-18)
0.3	1.1827(-14)	1.0721(-15)	1.5612(-16)	3.1225(-17)	1.0408(-17)	6.9388(-18)
0.4	1.5723(-14)	1.4155(-15)	1.8041(-16)	1.3877(-17)	1.3877(-17)	1.3877(-17)
0.5	1.9595(-14)	1.7763(-15)	2.498(-16)	5.5511(-17)	2.7755(-17)	2.7755(-17)
0.6	2.3453(-14)	2.1649(-15)	3.3306(-16)	8.3266(-17)	2.7755(-17)	2.7755(-17)
0.7	2.7255(-14)	2.5535(-15)	4.4408(-16)	1.6653(-16)	1.1102(-16)	1.1102(-16)
0.8	3.0642(-14)	2.6645(-15)	2.2204(-16)	1.1102(-16)	1.1102(-16)	1.1102(-16)
0.9	3.4194(-14)	3.1086(-15)	3.3306(-16)	1.1102(-16)	1.1102(-16)	1.1102(-16)
1.0	3.7747(-14)	3.3306(-15)	6.6613(-16)	2.22204(-16)	2.2204(-16)	2.2204(-16)

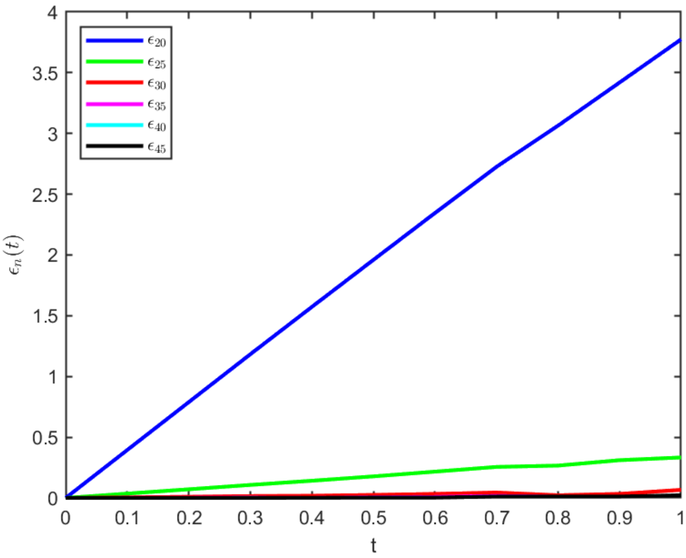

FIGURE 2. Convergence behavior of $\epsilon_n(t)$ for $n = 20\,(5)\,45$

TABLE 4. Values of $\epsilon_n(t) = \|y_{n+1} - f(S, y_n)\|$ for $n = 50\,(50)\,300$ (**Case 2**)

t	$\epsilon_{50}(t)$	$\epsilon_{100}(t)$	$\epsilon_{150}(t)$	$\epsilon_{200}(t)$	$\epsilon_{250}(t)$	$\epsilon_{300}(t)$
0.1	5.8886(-6)	1.2386(-6)	5.1450(-7)	2.7913(-7)	1.7466(-7)	1.1944(-7)
0.2	1.1654(-5)	2.4503(-6)	1.0176(-6)	5.5204(-7)	3.454(-7)	2.3620(-7)
0.3	1.7053(-5)	3.5768(-6)	1.4839(-6)	8.0453(-7)	5.0322(-7)	3.4403(-7)
0.4	2.1739(-5)	4.5320(-6)	1.8753(-6)	1.0153(-6)	6.3443(-7)	4.3345(-7)
0.5	2.5297(-5)	5.2128(-6)	2.1461(-6)	1.1585(-6)	7.2262(-7)	4.9306(-7)
0.6	2.7302(-5)	5.5139(-6)	2.2496(-6)	1.2081(-6)	7.5104(-7)	5.1126(-7)
0.7	2.7391(-5)	5.3471(-6)	2.1471(-6)	1.1424(-6)	7.0590(-7)	4.7848(-7)
0.8	2.5354(-5)	4.6663(-6)	1.8189(-6)	9.5063(-7)	5.8040(-7)	3.9005(-7)
0.9	2.1245(-5)	3.4960(-6)	1.2780(-6)	6.4055(-7)	3.7971(-7)	2.4967(-7)
1.0	1.5520(-5)	1.9700(-6)	5.8667(-7)	2.4813(-7)	1.2723(-7)	7.3704(-8)

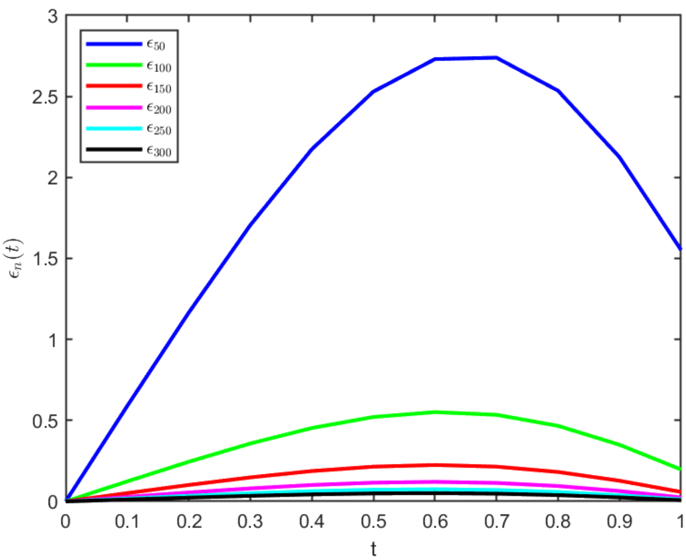

FIGURE 3. Convergence behavior of $\epsilon_n(t)$ for $n = 50\,(50)\,300$

The error estimation in iterative procedures is essential for evaluating the accuracy and convergence of the methods. In the next section, we will prove that the direct error of our proposed iterative process (I) is controllable and bounded.

5. Error Estimation Result

After establishing the convergence properties of the iteration process (I), we proceed to investigate its error behavior. Specifically, we derive estimates showing that the accumulation of errors in the iterates is bounded, confirming the robustness of the scheme in practical computations.
The following notations are used in the sequel.

Notation	**Meaning / Definition**
$\overline{a_n}, \overline{b_n}, \overline{c_n}$	Exact values of a_n, b_n, c_n
Sa_n, Sb_n, Sc_n	Approximated values under map S generated by (I)-iterative scheme
$\overline{Sa_n}, \overline{Sb_n}, \overline{Sc_n}$	Exact values under map S
$u_n = Sa_n - \overline{Sa_n}$	Error in Sa_n
$v_n = Sb_n - \overline{Sb_n}$	Error in Sb_n
$w_n = Sc_n - \overline{Sc_n}$	Error in Sc_n
M_u, M_v, M_w	Absolute error limits of $\{Sa_n\}_{n=0}^{\infty}$, $\{Sb_n\}_{n=0}^{\infty}$ and $\{Sc_n\}_{n=0}^{\infty}$
$M = \max\{M_u, M_v, M_w\}$	Uniform error bound, $M_u = sup_{n\in\mathbb{N}} \|u_n\|$, $M_v = sup_{n\in\mathbb{N}} \|v_n\|$, $M_w = sup_{n\in\mathbb{N}} \|w_n\|$
$E_n^{(1)} = \|a_{n+1} - \overline{a_{n+1}}\|$	Error in a_{n+1}
$E_n^{(2)} = \|b_n - \overline{b_n}\|$	Error in b_n
$E_n^{(3)} = \|c_n - \overline{c_n}\|$	Error in c_n
N	Error bound ensuring accumulation is bounded

Table 5. Notation used in the error estimation result of the iterative process (I).

The main idea proceeds in several steps:

Step 1 (Notation and Formulation): Introduce the error terms u_n, v_n and w_n, uniformly bounded by the positive constant M.
Step 2 (Stepwise error bounds): Derive the inequalities of successive errors in terms of previous errors.
Step 3 (Bounding the iterates): Use the bounds on u_n, v_n and w_n, to compute deviations of all iterates.
Step 4 (Control condition): Derive the control condition preserving boundedness.

Let $S : T \to T$ be a nonexpansive mapping defined on a set T of a Banach space M and considering the iterative scheme (I) for every $n \in \mathbb{N}$, define the errors of Sa_n, Sb_n and Sc_n as follows:

$$u_n = Sa_n - \overline{Sa_n}, \quad v_n = Sb_n - \overline{Sb_n}, \quad w_n = Sc_n - \overline{Sc_n} \tag{14}$$

where $\overline{Sa_n}, \overline{Sb_n}$ and $\overline{Sc_n}$ are the precise values of Sa_n, Sb_n and Sc_n, respectively. It is implied by the error theory that the sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are bounded. Assume that

$$M = max\{M_u, M_v, M_w\}, \tag{15}$$

where $M_u = sup_{n\in\mathbb{N}} \|u_n\|$, $M_v = sup_{n\in\mathbb{N}} \|v_n\|$ and $M_w = sup_{n\in\mathbb{N}} \|w_n\|$ are the absolute error limits of $\{S a_n\}_{n=0}^{\infty}$, $\{S b_n\}_{n=0}^{\infty}$ and $\{S c_n\}_{n=0}^{\infty}$, respectively, and the iterative process (I) have accumulated errors due to u_n, v_n and w_n. As a result, we can set $\forall n \geq 1$:

$$\begin{cases} \overline{a_0} \in T, \\ \overline{c_n} = \overline{S}\left((1-\alpha_n)\overline{a_n} + \alpha_n \overline{S a_n}\right), \\ \overline{b_n} = \overline{S}\left((1-\beta_n)\overline{c_n} + \beta_n \overline{S c_n}\right), \\ \overline{a_{n+1}} = \overline{S}\left((1-\gamma_n)\overline{S b_n} + \gamma_n \overline{S c_n}\right), \end{cases} \tag{16}$$

where the precise values of a_n, b_n and c_n are, respectively, $\overline{a_n}, \overline{b_n}$ and $\overline{c_n}$. The subsequent (n+1) steps are impacted by each iteration error.
At the initial step, we have:

$$\|a_0\| = \|\overline{a_0}\| \tag{17}$$

Now from (16),

$$\begin{aligned} \|c_0 - \overline{c_0}\| &= \left\|S\left((1-\alpha_0)a_0 + \alpha_0 S a_0\right) - \overline{S}\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right)\right\| \\ &= \left\| \begin{array}{c} S\left((1-\alpha_0)a_0 + \alpha_0 S a_0\right) - S\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right) \\ + S\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right) - \overline{S}\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right) \end{array} \right\| \\ &\leq \begin{array}{c} \left\|S\left((1-\alpha_0)a_0 + \alpha_0 S a_0\right) - S\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right)\right\| \\ + \left\| S\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right) - \overline{S}\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right)\right\| \end{array} \\ &\leq \left\|S\left((1-\alpha_0)a_0 + \alpha_0 S a_0\right) - S\left((1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right)\right\| + \epsilon \end{aligned}$$

As S is nonexpansive, we have

$$\begin{aligned} \|c_0 - \overline{c_0}\| &= \left\|(1-\alpha_0)a_0 + \alpha_0 S a_0 - (1-\alpha_0)\overline{a_0} + \alpha_0 \overline{S a_0}\right\| + \epsilon \\ &= (1-\alpha_0)\|a_0 - \overline{a_0}\| + \alpha_0 \left\|S a_0 - \overline{S a_0}\right\| + \epsilon \end{aligned}$$

From (14) & (17), we get

$$\|c_0 - \overline{c_0}\| = \alpha_0 \|u_0\| + \epsilon \tag{18}$$

Again from (16),

$$\begin{aligned} \left\|b_0 - \overline{b_0}\right\| &= \left\|S\left((1-\beta_0)c_0 + \beta_0 S c_0\right) - \overline{S}\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right)\right\| \\ &= \left\| \begin{array}{c} S\left((1-\beta_0)c_0 + \beta_0 S c_0\right) - S\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right) \\ + S\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right) - \overline{S}\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right) \end{array} \right\| \\ &\leq \left\|S\left((1-\beta_0)c_0 + \beta_0 S c_0\right) - S\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right)\right\| \\ &+ \left\| S\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c}\right) - \overline{S}\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right)\right\| \\ &\leq \left\|S\left((1-\beta_0)c_0 + \beta_0 S c_0\right) - S\left((1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right)\right\| + \epsilon \end{aligned}$$

As S is nonexpansive, we have

$$\left\|b_0 - \overline{b_0}\right\| = \left\|(1-\beta_0)c_0 + \beta_0 S c_0 - (1-\beta_0)\overline{c_0} + \beta_0 \overline{S c_0}\right\| + \epsilon$$

$$= (1-\beta_0)\,\|c_0-\overline{c_0}\| + \beta_0 \left\|S\,c_0 - \overline{S\,c_0}\right\| + \epsilon$$

from (18),

$$\begin{aligned}\left\|b_0-\overline{b_0}\right\| &= (1-\beta_0)\,[\alpha_0\,\|u_0\|+\epsilon] + \beta_0\,\|w_0\| + \epsilon \\ &= \alpha_0\,(1-\beta_0)\,\|u_0\| + \beta_0\,\|w_0\| + \epsilon.\end{aligned}$$

(19)

In a similar way, for the errors in the first step of a, b, c from (16) using (18) & (19), we have

$$\begin{aligned}\|a_1-\overline{a_1}\| &= \left\|S\,((1-\gamma_0)\,S\,b_0+\gamma_0 S\,c_0) - \overline{S}\left((1-\gamma_0)\,\overline{S\,b_0}+\gamma_0\overline{S\,c_0}\right)\right\| \\ &= \left\|\begin{array}{c} S\,((1-\gamma_0)\,S\,b_0+\gamma_0 S\,c_0) - S\left((1-\gamma_0)\,\overline{S\,b_0}+\gamma_0\overline{S\,c_0}\right) \\ +\,S\left((1-\gamma_0)\,\overline{S\,b_0}+\ \gamma_0\overline{S\,c_0}\right) - \overline{S}\left((1-\gamma_0)\,\overline{S\,b_0}+\gamma_0\overline{S\,c_0}\right)\end{array}\right\| \\ &\le (1-\gamma_0)\left\|S\,b_0-\overline{S\,b_0}\right\| + \gamma_0\left\|S\,c_0-\overline{S\,c_0}\right\| + \epsilon \\ &= (1-\gamma_0)\left\|S\,b_0 - S\,\overline{b_0} + S\,\overline{b_0} - \overline{S\,b_0}\right\| + \gamma_0\left\|S\,c_0 - S\,\overline{c_0} + S\,\overline{c_0} - \overline{S\,c_0}\right\| + \epsilon \\ &\le (1-\gamma_0)\left\|b_0-\overline{b_0}\right\| + \gamma_0\,\|c_0-\overline{c_0}\| + \epsilon \\ &= (1-\gamma_0)\,[\alpha_0\,(1-\beta_0)\,\|u_0\| + \beta_0\,\|w_0\| + \epsilon] + \gamma_0\,[\alpha_0\,\|u_0\| + \epsilon] + \epsilon \\ &= \alpha_0\,[(1-\beta_0)\,(1-\gamma_0)+\gamma_0]\,\|u_0\| + (1-\gamma_0)\beta_0\,\|w_0\| + \epsilon.\end{aligned}$$

$$\begin{aligned}\|c_1-\overline{c_1}\| &= \left\|(1-\alpha_1)\,a_1 + \alpha_1 S\,a_1 - (1-\alpha_1)\,\overline{a_1} + \alpha_1\overline{S\,a_1}\right\| + \epsilon \\ &= (1-\alpha_1)\,\|a_1-\overline{a_1}\| + \alpha_1\left\|S\,a_1 - \overline{S\,a_1}\right\| + \epsilon \\ &= (1-\alpha_1)\,[\alpha_0\,[(1-\beta_0)\,(1-\gamma_0)+\gamma_0]\,\|u_0\| + (1-\gamma_0)\beta_0\,\|w_0\| + \epsilon] \\ &+ \alpha_1\,\|u_1\| + \epsilon \\ &= \alpha_0\,(1-\alpha_1)\,[(1-\beta_0)\,(1-\gamma_0)+\gamma_0]\,\|u_0\| \\ &+ (1-\alpha_1)\,(1-\gamma_0)\beta_0\,\|w_0\| + \alpha_1\,\|u_1\| + \epsilon.\end{aligned}$$

$$\begin{aligned}\left\|b_1-\overline{b_1}\right\| &= \left\|(1-\beta_1)\,c_1 + \beta_1 S\,c_1 - (1-\beta_1)\,\overline{c_1} + \beta_1\overline{S\,c_1}\right\| + \epsilon \\ &= (1-\beta_1)\,\|c_1-\overline{c_1}\| + \beta_1\left\|S\,c_1-\overline{S\,c_1}\right\| + \epsilon \\ &= (1-\beta_1)\left[\begin{array}{c}\alpha_0\,(1-\alpha_1)\,[(1-\beta_0)\,(1-\gamma_0)+\gamma_0]\,\|u_0\| \\ +\,(1-\alpha_1)\,(1-\gamma_0)\beta_0\,\|w_0\| + \alpha_1\,\|u_1\| + \epsilon\end{array}\right] \\ &+ \beta_1\,\|w_1\| + \epsilon \\ &= \alpha_0\,(1-\alpha_1)\,(1-\beta_1)\,[(1-\beta_0)\,(1-\gamma_0)+\gamma_0]\,\|u_0\| \\ &+ (1-\alpha_1)\,(1-\gamma_0)\,(1-\beta_1)\beta_0\,\|w_0\| + \alpha_1\,(1-\beta_1)\,\|u_1\| \\ &+ \beta_1\,\|w_1\| + \epsilon.\end{aligned}$$

For the errors in the second step of a, b, c we have,

$$\begin{aligned}\|a_2-\overline{a_2}\| &= \alpha_0\,(1-\alpha_1)\,[(1-\beta_0)\,(1-\gamma_0)+\gamma_0]\,[(1-\beta_1)\,(1-\gamma_1)+\gamma_1]\,\|u_0\| \\ &+ (1-\alpha_1)\,(1-\gamma_0)\,(1-\beta_1)\beta_0\,[(1-\beta_1)\,(1-\gamma_1)+\gamma_1]\,\|w_0\| \\ &+ \alpha_1\,[(1-\beta_1)\,(1-\gamma_1)+\gamma_1]\,\|u_1\| + (1-\gamma_1)\beta_1\,\|w_1\| + \epsilon.\end{aligned}$$

$$\begin{aligned}\|c_2-\overline{c_2}\| &= \alpha_0(1-\alpha_1)(1-\alpha_2)\Big[\ [(1-\beta_0)(1-\gamma_0)+\gamma_0][(1-\beta_1)(1-\gamma_1)+\gamma_1]\ \Big]\|u_0\|\\ &+(1-\alpha_1)(1-\alpha_2)(1-\gamma_0)\beta_0\Big[\ (1-\beta_1)(1-\gamma_1)+\gamma_1\ \Big]\|w_0\|\\ &+\alpha_1(1-\alpha_2)\Big[\ (1-\beta_1)(1-\gamma_1)+\gamma_1\ \Big]\|u_1\|\\ &+(1-\alpha_2)(1-\gamma_1)\beta_1\|w_1\|+\alpha_2\|u_2\|+\epsilon.\end{aligned}$$

Also, we have

$$\begin{aligned}\left\|b_2-\overline{b_2}\right\| &= \alpha_0(1-\alpha_1)(1-\alpha_2)(1-\beta_2)[(1-\beta_0)(1-\gamma_0)+\gamma_0]\\ &\times[(1-\beta_1)(1-\gamma_1)+\gamma_1]\|u_0\|+(1-\alpha_1)(1-\alpha_2)(1-\beta_2)(1-\gamma_0)\beta_0\\ &\times[(1-\beta_1)(1-\gamma_1)+\gamma_1]\|w_0\|+\alpha_1(1-\alpha_2)(1-\beta_2)\\ &\times[(1-\beta_1)(1-\gamma_1)+\gamma_1]\|u_1\|+(1-\alpha_2)(1-\beta_2)(1-\gamma_1)\beta_1\|w_1\|\\ &+\alpha_2(1-\beta_2)\|u_2\|+\beta_2\|w_2\|+\epsilon.\end{aligned}$$

And

$$\begin{aligned}\|a_3-\overline{a_3}\| &= \alpha_0(1-\alpha_1)(1-\alpha_2)\\ &\times\begin{bmatrix}(1-\beta_0)(1-\gamma_0)\\+\gamma_0\end{bmatrix}\begin{bmatrix}(1-\beta_1)(1-\gamma_1)\\+\gamma_1\end{bmatrix}\begin{bmatrix}(1-\beta_2)(1-\gamma_2)\\+\gamma_2\end{bmatrix}\|u_0\|\\ &+(1-\alpha_1)(1-\alpha_2)(1-\gamma_0)(1-\beta_1)\beta_0\\ &\times\begin{bmatrix}(1-\beta_1)(1-\gamma_1)\\+\gamma_1\end{bmatrix}\begin{bmatrix}(1-\beta_2)(1-\gamma_2)\\+\gamma_2\end{bmatrix}\|w_0\|\\ &+\alpha_1(1-\alpha_2)\begin{bmatrix}(1-\beta_1)(1-\gamma_1)\\+\gamma_1\end{bmatrix}\begin{bmatrix}(1-\beta_2)(1-\gamma_2)\\+\gamma_2\end{bmatrix}\|u_1\|\\ &+(1-\gamma_1)\beta_1\|w_1\|+\epsilon.\end{aligned}$$

Repeating the above process, we get:

$$\|a_{n+1}-\overline{a_{n+1}}\| = \sum_{k=0}^{n}\begin{matrix}\left\{\begin{matrix}[(1-\beta_k)(1-\gamma_k)+\gamma_k]\,\alpha_k\|u_k\|\\ +\beta_k(1-\gamma_k)\|w_k\|\end{matrix}\right\}\\ \left\{\prod_{j=k+1}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\}\end{matrix}+\epsilon.$$

$$\begin{aligned}\left\|b_n-\overline{b_n}\right\| &= \beta_n\|w_n\|+(1-\beta_n)\alpha_n\|u_n\|\\ &+(1-\alpha_n)(1-\beta_n)\sum_{k=0}^{n}\begin{bmatrix}[(1-\beta_k)(1-\gamma_k)+\gamma_k]\,\alpha_k\|u_k\|\\ +\beta_k(1-\gamma_k)\|w_k\|\end{bmatrix}\\ &\times\prod_{j=k+1}^{n}\left\{\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right](1-\alpha_j)\right\}+\epsilon.\end{aligned}$$

$$\left\|b_n-\overline{b_n}\right\| = (1-\beta_n)\alpha_n\|u_n\|+\beta_n\|w_n\|+(1-\alpha_n)(1-\beta_n)\|a_n-\overline{a_n}\|+\epsilon.$$

$$\begin{aligned}\|c_n-\overline{c_n}\| &= \alpha_n\|u_n\|\\ &+(1-\alpha_n)\sum_{k=0}^{n}\begin{bmatrix}[(1-\beta_k)(1-\gamma_k)+\gamma_k]\,\alpha_k\|u_k\|\\ +\beta_k(1-\gamma_k)\|w_k\|\end{bmatrix}\\ &\times\prod_{j=k+1}^{n}\left\{\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right](1-\alpha_j)\right\}+\epsilon.\end{aligned}$$

$$= \alpha_n \|u_n\| + (1-\alpha_n)\|a_n - \overline{a_n}\| + \epsilon.$$

Define:

$$E_n^{(1)} := \|a_{n+1} - \overline{a_{n+1}}\| = \sum_{k=0}^{n} \begin{matrix} \left\{ \begin{matrix} [(1-\beta_k)(1-\gamma_k)+\gamma_k]\,\alpha_k\|u_k\| \\ +\beta_k(1-\gamma_k)\|w_k\| \end{matrix} \right\} \\ \left\{\prod_{j=k+1}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} \end{matrix} + \epsilon.$$

$$E_n^{(2)} := \left\|b_n - \overline{b_n}\right\| = (1-\beta_n)\,\alpha_n\|u_n\| + \beta_n\|w_n\| + (1-\alpha_n)(1-\beta_n)\,E_{n-1}^{(1)} + \epsilon.$$

$$E_n^{(3)} := \|c_n - \overline{c_n}\| = \alpha_n\|u_n\| + (1-\alpha_n)\,E_{n-1}^{(1)} + \epsilon.$$

The error increased to $(n+1)$ iterations in the iterative scheme (14), is represented by the values of $E_n^{(1)}, E_n^{(2)}$ and $E_n^{(3)}$.

Theorem 5.1. *Let $T, S, M, E_n^{(1)}, E_n^{(2)}$ and $E_n^{(3)}$ be defined as above and ϵ be a positive fixed real number:*

(1) If $\sum_{j=0}^{\infty}\alpha_j = +\infty$ or $\sum_{j=0}^{\infty}\beta_j = +\infty$ or $\sum_{j=0}^{\infty}\gamma_j = +\infty$, then the error estimation of (I) is bounded and cannot be greater than N;

(2) Random errors of (I) are controlled if $\sum_{j=0}^{\infty}\left[\beta_j\left(1-\alpha_j\right)\left(1-\gamma_j\right)+\alpha_j\right] < +\infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty}\gamma_n = 0$.

Proof. The well-known fact that $\sum_{i=0}^{\infty}\beta_i = +\infty$ implies $\prod_{i=0}^{\infty}(1-\beta_i) = 0$ can be used, along with the previously mentioned inequalities, to obtain:

$$\left\|E_n^{(1)}\right\| = \left\| \begin{matrix} \left\{ \begin{matrix} [(1-\beta_0)(1-\gamma_0)+\gamma_0]\,\alpha_0 u_0 \\ +\beta_0(1-\gamma_0)\,w_0 \end{matrix} \right\} \\ \times\left\{\prod_{j=1}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} \\ +\left\{ \begin{matrix} [(1-\beta_1)(1-\gamma_1)+\gamma_1]\,\alpha_1 u_1 \\ +\beta_1(1-\gamma_1)\,w_1 \end{matrix} \right\} \\ \times\left\{\prod_{j=2}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} \\ +\cdots+\{[(1-\beta_n)(1-\gamma_n)+\gamma_n]\,\alpha_n u_n + \beta_n(1-\gamma_n)\,w_n\} + \epsilon \end{matrix} \right\|,$$

$$\leq \left\| \left\{ \begin{matrix} [(1-\beta_0)(1-\gamma_0)+\gamma_0]\,\alpha_0 u_0 \\ +\beta_0(1-\gamma_0)\,w_0 \end{matrix} \right\} \left\{ \prod_{j=1}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right) \right\} \right\|$$

$$+ \left\| \left\{ \begin{matrix} [(1-\beta_1)(1-\gamma_1)+\gamma_1]\,\alpha_1 u_1 \\ +\beta_1(1-\gamma_1)\,w_1 \end{matrix} \right\} \left\{ \prod_{j=2}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right) \right\} \right\|$$

$$+\cdots+\left\|\{[(1-\beta_n)(1-\gamma_n)+\gamma_n]\,\alpha_n u_n + \beta_n(1-\gamma_n)\,w_n\}\right\| + \epsilon,$$

$$\leq \left\{ \begin{matrix} [(1-\beta_0)(1-\gamma_0)+\gamma_0]\,\alpha_0\|u_0\| \\ +\beta_0(1-\gamma_0)\ \|w_0\| \end{matrix} \right\} \left\{ \prod_{j=1}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right) \right\}$$

$$+ \left\{ \begin{matrix} [(1-\beta_1)(1-\gamma_1)+\gamma_1]\,\alpha_1\|u_1\| \\ +\beta_1(1-\gamma_1)\|w_1\| \end{matrix} \right\} \left\{ \prod_{j=2}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right) \right\}$$

$$+\cdots+\{[(1-\beta_n)(1-\gamma_n)+\gamma_n]\,\alpha_n\|u_n\| + \beta_n(1-\gamma_n)\|w_n\|\} + \epsilon,$$

$$\left\|E_n^{(1)}\right\| \leq M\left[\begin{array}{c} \left\{\prod_{j=0}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} \\ +\left\{\begin{array}{c}\left[\left(1-\beta_0\right)\left(1-\gamma_0\right)+\gamma_0\right]\alpha_0 \\ +\beta_0\left(1-\gamma_0\right)\end{array}\right\} \\ \times\left\{\prod_{j=1}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} \\ +\left\{\begin{array}{c}\left[\left(1-\beta_1\right)\left(1-\gamma_1\right)+\gamma_1\right]\alpha_1 \\ +\beta_1\left(1-\gamma_1\right)\end{array}\right\} \\ \times\left\{\prod_{j=2}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} \\ +\cdots+\left\{\left[\left(1-\beta_n\right)\left(1-\gamma_n\right)+\gamma_n\right]\alpha_n+\beta_n\left(1-\gamma_n\right)\right\} \\ -\left\{\prod_{j=0}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} \end{array}\right]+\epsilon,$$

$$\leq M\left[1-\left\{\prod_{j=0}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\}\right]+\epsilon$$

$$= M\left[1-\prod_{j=0}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\prod_{j=0}^{n}\left(1-\alpha_j\right)\right]+\epsilon$$

$$= M\left[1-\prod_{j=0}^{n}\left[1-\left(\beta_j\left(1-\gamma_j\right)\right)\right]\prod_{j=0}^{n}\left(1-\alpha_j\right)\right]+\epsilon = M+\epsilon = N.$$

$$\begin{aligned}\left\|E_n^{(2)}\right\| &= \left\|\beta_n w_n+\left(1-\beta_n\right)\alpha_n u_n+\left(1-\alpha_n\right)\left(1-\beta_n\right)E_{n-1}^{(1)}+\epsilon\right\| \\ &\leq \beta_n\left\|w_n\right\|+\left(1-\beta_n\right)\alpha_n\left\|u_n\right\|+\left(1-\alpha_n\right)\left(1-\beta_n\right)\left\|E_{n-1}^{(1)}\right\|+\epsilon \\ &\leq M\left[\beta_n+\left(1-\beta_n\right)\alpha_n+\left(1-\alpha_n\right)\left(1-\beta_n\right)\right]+\epsilon = M+\epsilon = N.\end{aligned}$$

$$\begin{aligned}\left\|E_n^{(3)}\right\| &= \left\|\alpha_n u_n+\left(1-\alpha_n\right)E_{n-1}^{(1)}+\epsilon\right\| \\ &\leq \alpha_n\left\|u_n\right\|+\left(1-\alpha_n\right)\left\|E_{n-1}^{(1)}\right\|+\epsilon \\ &\leq M\left[\alpha_n+\left(1-\alpha_n\right)\right]+\epsilon = M+\epsilon = N.\end{aligned}$$

Hence, we have $\max_{n\in\mathbb{N}}\left[\left\|E_n^{(1)}\right\|,\left\|E_n^{(2)}\right\|,\left\|E_n^{(3)}\right\|\right] \leq N$.

Also, $\sum_{j=0}^{\infty}\left[\beta_j\left(1-\alpha_j\right)\left(1-\gamma_j\right)+\alpha_j\right] < +\infty$ implies that

$$\prod_{j=0}^{n}\left[1-\left[\beta_j\left(1-\alpha_j\right)\left(1-\gamma_j\right)+\alpha_j\right]\right] = \prod_{j=0}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right) \in (0,1),$$

Let $1-\left\{\prod_{j=0}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\} = l \in (0,1)$. We have,

$$\left\|E_n^{(1)}\right\| \leq M\left[1-\left\{\prod_{j=0}^{n}\left[\left(1-\beta_j\right)\left(1-\gamma_j\right)+\gamma_j\right]\left(1-\alpha_j\right)\right\}\right]+\epsilon,$$

Using the conditions, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0 \implies \lim_{n\to\infty} (\beta_n + \alpha_n - \alpha_n\beta_n) = 0$ $\implies \exists\, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, we have $\beta_n + \alpha_n - \alpha_n\beta_n \leq \frac{1}{(1-l)}$, so we have

$$\begin{aligned}
\left\|E_n^{(2)}\right\| &\leq \beta_n \|w_n\| + (1-\beta_n)\,\alpha_n \|u_n\| + (1-\alpha_n)(1-\beta_n)\left\|E_{n-1}^{(1)}\right\| + \epsilon \\
&\leq M\left[\beta_n + (1-\beta_n)\,\alpha_n + (1-\alpha_n)(1-\beta_n)\,l\right] + \epsilon, \\
&= M\left[l + (\beta_n + \alpha_n - \alpha_n\beta_n)(1-l)\right] + \epsilon, \\
&\leq M\left[l + \frac{1}{(1-l)}(1-l)\right] + \epsilon, \\
&= 2lM + \epsilon.
\end{aligned}$$

Also, since, $\lim_{n\to\infty} \beta_n = 0 \Rightarrow \exists\, n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, we have

$$\begin{aligned}
\left\|E_n^{(3)}\right\| &\leq \alpha_n \|u_n\| + (1-\alpha_n)\left\|E_{n-1}^{(1)}\right\| + \epsilon \\
&\leq \alpha_n M + (1-\alpha_n)\,Ml + \epsilon, \\
&= \alpha_n M (1-l) + Ml + \epsilon, \\
&\leq \frac{1}{(1-l)} M (1-l) + Ml + \epsilon, \\
&= 2lM + \epsilon.
\end{aligned}$$

Thus, we deduce that the sequential parameters $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ $\forall n \geq n_0$ can be suitably chosen to control the errors $\left\|E_n^{(1)}\right\|$, $\left\|E_n^{(2)}\right\|$and $\left\|E_n^{(3)}\right\|$. □

The ensuing examples shows that the direct error in the iterative process given by (I) is independent of the choice of parameters and is also controlled and bounded.

Example 5.1. *Given the usual norm* $|.|$ *and the set* $T = (-\infty, 2) \cup (2, \infty)$*, let* $S : T \to T$ *be defined as follows:*

$$S(a) = \frac{7}{a-2}$$

Then, S *is a nonexpansive map having* $a = -1.8571$ *as its fixed point.*

Example 5.2. *Let the set* $T = \left[\frac{1}{2}, \frac{5}{2}\right)$ *be equipped with the usual norm* $|.|$ *and let* $S : T \to T$ *be defined as:*

$$S(a) = \frac{1}{2} cos(a) + \frac{3}{2}$$

Then, the mapping S *has* $a = 1.52359$ *as its fixed point.*

In Table 6, we have compared convergence behaviour of the iterative scheme (I) to some of the well-known iterative schemes for the mappings defined in Example 5.1 (initial point $a_0 = 0$) and Example 5.2 (initial point $a_0 = 20$) by taking the stopping criteria as 10^{-14} for different values of the parameters $\alpha_n, \beta_n, \gamma_n$.

TABLE 6. Comparison of convergence behaviour of various iterative schemes for Example 5.1& 5.2)

Iterative scheme	Ishikawa	Noor	Abbas	Agarwal	KF	Thakur	New
Example 5.1							
$\alpha_n = \beta_n = \gamma_n = \frac{1}{n+1}$	26	27	26	37	18	25	**17**
$\frac{1}{\sqrt{n+1}}$	156	152	26	45	37	22	**11**
$\frac{1}{\sqrt[3]{n+1}}$	62	51	26	33	18	20	**10**
$\frac{1}{\sqrt[5]{n+1}}$	39	28	26	22	15	16	**9**
Example 5.2							
$\alpha_n = \beta_n = \gamma_n = \frac{n}{n+1}$	30	15	26	16	13	25	**10**
$\frac{1}{\sqrt{n+1}}$	144	129	26	43	11	22	**10**
$\frac{1}{\sqrt[3]{n+1}}$	65	60	26	36	30	21	**10**
$\frac{1}{\sqrt[5]{n+1}}$	40	31	26	23	22	21	**9**

Table 7 and Table 8 compares the errors between two consecutive iterations (E.C.= $\|a_{k+1} - a_k\|$) and residual errors (R.E.= $\|S a_k - a_k\|$) in the corresponding function obtained by our proposed iterative scheme (I) and other well-known iterative schemes (for the number of iterations required by our iterative method (I) to achieve the fixed point) for various values of the parameters $\alpha_n, \beta_n, \gamma_n$.

TABLE 7. Comparisons of errors (E.C. & R.E.) for various iterative schemes(Example 5.1)

Parameters	$\alpha_n = \beta_n = \gamma_n = \frac{1}{n+1}$		$\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt{n+1}}$		$\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt[3]{n+1}}$		$\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt[5]{n+1}}$	
Iterative Steps	**17**		**11**		**11**		**9**	
Iterative Scheme	E.C.	R.E.	E.C.	R.E.	E.C.	R.E.	E.C.	R.E.
Ishikawa	2.190(-9)	8.9914(-3)	8.9914(-3)	1.008(00)	1.219(-3)	1.491(-3)	1.078(-3)	7.930(-4)
Noor	2.617(-7)	7.1045(-3)	7.1045(-)	9.505(-1)	3.510(-5)	3.684(-5)	5.351(-6)	2.436(-6)
Abbas	2.152(-9)	1.0094(-6)	1.0094(-6)	1.522(-6)	6.699(-5)	2.925(-5)	2.937(-4)	1.282(-4)
Agarwal	7.398(-5)	1.1780(-4)	1.1780(-4)	2.059(-3)	1.080(-5)	3.820(-6)	6.550(-7)	1.409(-7)
KF	7.327(-15)	1.1969(-5)	1.1969(-5)	8.929(-7)	4.051(-8)	7.602(-9)	9.947(-9)	1.052(-9)
Thakur	5.509(-9)	3.3338(-8)	3.3338(-8)	1.201(-6)	1.220(-7)	3.185(-8)	1.768(-8)	2.315(-9)
New	4.663(-15)	1.048(-13)	1.048(-13)	2.220(-16)	9.770(-15)	2.220(-16)	2.198(-14)	2.220(-16)

TABLE 8. Comparisons of errors (E.C. & R.E.) for various iterative schemes(Example 5.2)

Parameters	$\alpha_n = \beta_n = \gamma_n = \frac{n}{n+1}$		$\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt{n+1}}$		$\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt[3]{n+1}}$		$\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt[5]{n+1}}$	
Iterative Steps	**10**		**10**		**10**		**9**	
Iterative Scheme	E.C.	R.E.	E.C.	R.E.	E.C.	R.E.	E.C.	R.E.
Ishikawa	5.0847(-4)	2.6385(-4)	4.004(-3)	8.374(-3)	2.0531(-3)	2.6797(-3)	8.1925(-4)	6.4629(-4)
Noor	3.3352(-9)	7.6652(-11)	7.901(-4)	1.569(-3)	9.0244(-4)	1.0123(-3)	3.5576(-5)	1.7278(-5)
Abbas	4.0468(-6)	2.0166(-6)	6.699(-5)	2.925(-5)	4.0468(-6)	2.0166(-6)	1.6223(-5)	8.0847(-6)
Agarwal	7.1410(-11)	1.2851(-11)	5.322(-4)	2.243(-4)	2.2029(-5)	8.3463(-6)	1.4481(-7)	3.4435(-8)
KF	1.4843(-10)	2.369(-12)	9.193(-14)	8.660(-15)	3.4472(-6)	1.1470(-6)	2.3086(-7)	5.2883(-8)
Thakur	6.3488(-7)	2.6193(-7)	2.252(-7)	7.839(-8)	2.0977(-7)	5.8323(-8)	2.4774(-6)	7.2732(-7)
New	2.665(-15)	2.220(-16)	2.442(-15)	2.220(-16)	1.621(-14)	4.441(-16)	8.693(-14)	1.110(-15)

It is evident from Tables 6,7 and 8 that our proposed iterative scheme (I) has higher rate of convergence than the other iterative schemes and controlled error accumulation over a range of parameter selections.
Error bounds bridge theory and practice by ensuring that numerical results are not only convergent but also dependable in real-world applications. The availability of explicit error bounds enhances the practical value of the method, since they allow practitioners to trust the computed outcomes even in the presence of unavoidable numerical approximations which is crucial in sensitive areas such as numerical simulation, engineering design and optimization.

6. Application to nonlinear fractional differential equation

Numerous disciplines, including physics, engineering, biology, finance and control theory have used fractional calculus. Among other things fractal or long-range memory behaviour, anomalous diffusion and viscoelasticity are all modelled using fractional differential equations [29, 33].

Fractional differential equations (FDEs) involve derivatives of fractional order instead of integer order derivatives such as first-order or second-order derivatives and hence they expand upon ordinary and partial differential equations.

The fractional derivative of a function of order $\vartheta > 0$ of a function $g(x)$ is given by

$$D^{\vartheta} g(x) = \frac{1}{\Gamma(n-\vartheta)} \frac{d^n}{dx^n} \int_c^x \frac{g(t)}{(x-t)^{\vartheta-n+1}} dt$$

where $\Gamma(n-\vartheta)$ is the gamma function, $n-1 < \vartheta < n$ for a positive integer n and n is the smallest integer greater than ϑ.
The existence of the solution for a class of nonlinear boundary value problem of fractional differential equations with integral boundary conditions studied extensively in [11] is given by:

$$\begin{cases} D^{\vartheta} g(x) + f(x, g(x)) = 0 \qquad (0 \le x \le 1) \\ g(0) = g'(0) = 0, \;\; g(1) = \lambda \int_0^1 g(s) ds \end{cases} \tag{20}$$

where $2 < \vartheta \le 3, \lambda > 0, \lambda \neq \vartheta, D^{\vartheta}$ is the Reimann-Lioville fractional derivatiove and f is a continous function.

The Green's function linked to (20) is defined as follows:

$$G(x,s) = \begin{cases} \frac{(x^{\vartheta-1}(1-s)^{\vartheta-1}(\vartheta-\lambda+\lambda s)-(\vartheta-\lambda)(x-s)^{\vartheta-1}}{\Gamma(\vartheta)(\vartheta-\lambda)}, & 0 \le s \le x \le 1 \\ \frac{x^{\vartheta-1}(1-s)^{\vartheta-1}(\vartheta-\lambda+\lambda s)}{\Gamma(\vartheta)(\vartheta-\lambda)}, & 0 \le x \le s \le 1 \end{cases}$$

Numerous authors have examined the use of fixed point results in the solution of linear and nonlinear fractional differential equations.
Many authors have examined the application of fixed point results in the solution of linear and nonlinear fractional differential equations [18, 24, 42]
We will now apply the iterative approach (I) to estimate the solution of the FDE (20).

Theorem 6.1. *Consider $M = C[0,1]$ be a Banach space with the supremum norm and $\{a_n\}$ be the sequence defined by the iterative scheme (I) for the integral operator*

$S : M \to M$

$$S\left(g\left(x\right)\right) = \int_0^1 G\left(x,s\right) f\left(s, g(s)\right) ds, \qquad for\ all\ x \in [0,1]\ and\ g \in M. \tag{21}$$

Also, assume that f is a Lipschitz function with respect to the second variable, i.e., $|f(x,g) - f(x,h)| \leq |g-h|$, for all $x \in [0,1]$ and $g,h \in M$. Then the iterative sequence (I) converges to the solution of the problem (20).

Proof. We know that $g \in M$ is a solution of the FDE (20) if and only if it is a solution of the integral equation

$$g\left(x\right) = \int_0^1 G\left(x,s\right) f\left(s, g(s)\right) ds,$$

which is same as finding the fixed point of the integral operator S defined by (20).

Let $g,h \in M$and $x \in [0,1]$. Then,

$$\begin{aligned}
\left|S\left(g\left(x\right)\right) S\left(h\left(x\right)\right)\right| &\leq \int_0^1 G\left(x,s\right) \left|f\left(s,g(s)\right) - f\left(s,h(s)\right)\right| ds \\
&\leq \int_0^1 G\left(x,s\right) \left|g(s) - h(s)\right| ds \\
&\leq \left\|g-h\right\| \left(\sup_{x\in[0,1]} \int_0^1 G\left(x,s\right) \right) ds \\
&\leq \left\|g-h\right\|,
\end{aligned}$$

which gives

$$\left|S\left(g\left(x\right)\right) - S\left(h\left(x\right)\right)\right| \leq \left\|g-h\right\|, \quad \forall g,h \in M$$

i.e., S is a nonexpansive mapping and so a generalized $0-$nonexpansive mapping, and hence, by Theorem 3.4, the iterative sequence (I) converges to the solution of (20). □

7. Conclusion

A new iterative process (I) is obtained for approximating fixed points of generalized $\alpha-$nonexpansive mappings. In the setting of uniformly convex Banach space, strong and weak convergence to the fixed point of generalized $\alpha-$nonexpansive mapping are proved. The stability results supported by subsequent examples are given for almost contraction mappings. Additionally, we have proved that the direct error of the proposed iterative process (I) is controllable, bounded and independent of parametric selections backed by numerical illustrations. We also applied our results to find the solution of fractional differential equations.

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