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FIXED-POINT THEOREMS FOR PROINOV-SUZUKI CONTRACTION IN QUASI PARTIAL b-METRIC SPACE

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Abstract. In this article, we investigate asymptotically regular Proinov-Suzuki contraction maps and proved some fixed point results for these maps in a quasi-partial b-metric space. Most of the maps we've utilized here aren't always continuous on their domains. To show the veracity of our findings, we offer a few illustrative examples as part of the generalization of the Banach contraction theorem.

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1. Introduction

The renowned Banach contraction principle [4] in fixed point theory is relatively easy to apply if all that is needed is a complete metric space. Some researchers subsequently extended and generalized the idea in various ways; for a list, see [6, 7, 19]. The concept of quasi-contraction was first put forth by Ćirić in 1972 [9], and he subsequently offered a significant finding that generalized the Banach contraction principle. In 2006, Proinov [26] proved that fixed points for asymptotically regular mappings exist and are unique. Rhoades explored several conceptualizations of contractive mappings in 1977. In 2018, Ravindra *et al.* [27] investigated generalized Meir-Keeler type contractions and discontinuity at a fixed point. Later in 2019, [27] also investigated Proinov contraction and discontinuity at a fixed point. Recently, Karapinar *et al.* [22] gave a note on the Gornicki-Proinov type contraction. Roldan Lopez *et al.* [29] proved Proinov-type fixed-point results in non-Archimedean fuzzy metric spaces. Karapinar *et al.* [21] proved an extended Proinov X-contractions in metric spaces and fuzzy metric spaces satisfying the property NC by avoiding the monotone condition.

A contraction mapping has to be continuous at all points in the given space to satisfy the Banach contraction theorem. Other researchers, however, have attempted to look for the reverse. Whether it is possible to formulate a mapping that satisfies the Banach contraction theorem without this constraint of continuity was addressed by Suzuki's [31] response to the query in 2008. By proving the existence and uniqueness of fixed points for F-Kannan mappings in metric spaces with an application to the integral equation, nonlinear fractional differential equation, and ordinary differential equation for damped forced oscillations, Wangwe and Kumar [33] generalized the Banach contraction theorem. Gautam *et al.* [13] evaluated non-unique fixed points via Kannan F-Contraction on Quasi-Partial b-Metric Space. Also, Gautam *et al.* [14] proved some fixed point results for interpolative Chatterjea-type Suzuki contractions in quasi-partial b-metric spaces. Wangwe and Kumar [33, 35] also proved fixed point theorems of similar type contractions.

Several researchers have generalized Banach's contraction theorem to find a fixed point without taking into account the continuity of a certain contraction map (one can see [1, 11, 12, 23, 33]). On the other hand, other researchers tried to make the idea of metric space more widespread. The idea of b-metric space was first suggested by Bakhtin [5] in 1989. By including a variable in the definition of a standard metric space. In this paper, we prove the existence and uniqueness of a common fixed point for a pair of asymptotically regular Proinov-Suzuki contractions f and g in the setting of quasi-partial b-metric space. The results of this paper extend specifically the results obtained in [2, 8, 25, 27, 28, 31, 34]. An illustrative example and an application to the integral equation are given.

2. Preliminaries

In this section, we present the necessary definitions, lemmas, and theorems that serve as foundational tools for establishing the main results.

The triangle inequality axiom was modified in this idea.

Definition 2.1. [5] *A b-metric space on a nonempty set X is a function $d : X \times X \to [0, \infty)$, such that for all $u, v, w \in X$ and for some real number $s \geq 1$, it satisfies the following:*
(i) If $d(u, v) = 0$, Then, $u = v$,
(ii) $d(u, v) = d(v, u)$,
(iii) $d(u, v) \leqslant s[d(u, w) + d(w, v)]$.

The pair (X, d) is called a b-metric space with a coefficient $s \geq 1$. Every metric space is a b-metric space with $s = 1$, but the converse is not true in general as is shown by the following example:

Example 2.2. [30] *Let $X = \{0, 1, 2\}$ and $d : X \times X \to \mathbb{R}^+$ defined by*

$$\begin{aligned} &d(0,0) = d(1,1) = d(2,2) = 0, \\ &d(1,0) = d(0,1) = d(2,1) = d(1,2) = 1, \\ &d(0,2) = d(2,0) = h, \end{aligned}$$

where h is given a real number such that $h \geq 2$. It is easy to check that for all $u, v, w \in X$

$$d(u, v) \leq \frac{h}{2}[d(u, w) + d(w, v)].$$

Therefore, (X, d) is a b-metric space with a coefficient $s = \frac{h}{2}$. The ordinary triangle inequality does not hold if $h > 2$ and so (X, d) is not a metric space.

Example 2.3. [17] *Let (X, d) be a metric space and*

$$\rho(u, v) = \{d(u, v)^p\},$$

where $p > 1$ is a real number. Then, ρ is a b-metric with $s = 2^{p-1}$.

For more examples of a b-metric, one can see [5].

The notion of b-metric space was re-evaluated in 1993 by Czerwik [10] as follows:

Theorem 2.4. [10] *Let (X,d) be a complete b-metric space and $f : X \to X$ satisfy*

$$d[f(u), f(v)] \leq \psi[d(u,v)],$$

$u, v \in X,$ *where $\psi : R^+ \to R^+$ is increasing function such that*

$$\lim_{n\to\infty} \psi^n(t) = 0;$$

for each fixed $t > 0$. Then, f has exactly one fixed point c and

$$\lim_{n\to\infty} d[f^n(u), c] = 0.$$

In 1994 Metthews [24], introduced the notion of partial metric on a non-empty set by arguing that the self distance of a point need not be zero.

Definition 2.5. [24] Let $X \neq \emptyset$. A partial metric is a function $p : X \times X \to R^+$satisfying the following:
(i) $p(u,v) = p(v,u)$,
(ii) If $0 \leqslant p(u,u) = p(u,v) = p(v,v)$, Then, $u = v$,
(iii) $p(u,v) + p(w,w) \leqslant p(u,w) + p(w,v)$,
for all $u, v, w \in X$.

Then, a pair (X,p) is called partial metric space. It is clear that, If $p(u,v) = 0$, Then, $u = v$ however, if $u = v$, Then, $p(u,v)$ may not be zero.

Example 2.6. [3] *Let $X = R^+$ and $p : X \times X \to R^+$ given by $p(u,v) = max\{u,v\}$ for all $u, v \in R^+$. Then, (R^+, p) is a partial metric space.*

Example 2.7. [3] *Let $X = \{[\alpha,\beta] : \alpha, \beta \in \mathbb{R}, \alpha \leq \beta\}$. Then, $p([\alpha,\beta],[\delta,\gamma]) = max\{\beta,\gamma\} - min\{\alpha,\delta\}$ defines a partial metric p on X.*

Partial b-metric space was introduced by Mustafa *et al.* [32] in 2013 as the result of the failure of metric functions in computer science. It was Then, defined as follows:

Definition 2.8. [32] *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $b : X \times X \to [0,1)$ is a partial b-metric if for all $u, v, w \in X$ the following conditions are satisfied:*
(i) $u = v$ if and only if $b(u,u) = b(u,v) = b(v,v)$,
(ii) $b(u,u) \leq b(u,v)$,
(iii) $b(u,v) = b(v,u)$,
(iv) $b(u,v) \leq s[b(u,w) + b(w,v) - b(w,w)] + (\frac{1-s}{2})(b(u,u) + b(v,v))$.

The following example exhibits the properties of a partial b-metric space.

Example 2.9. [32] *Let $X = \mathbb{R}$ be a set of real numbers. Consider the metric space (X,d) where d is the Euclidean distance metric*

$$d(u,v) = |u - v|,$$

for all $u, v \in X$. Define $b = (u-v)^2 + 5$ for all $u, v \in X$. Then, b is a partial b-metric on X with $s = 2$, but not a partial metric on X. To see this, let $u = 1, v = 4$ and $w = 2$, Then,

$$b(1,4) = (1-4)^2 + 5 \nleq b(2,4) - b(2,2) = 6 + 9 - 5 = 10.$$

Also, b is not a b-metric since $b(u,u) \neq 0$ for all $u \in X$.

We can express partial metrics into b-metric as shown in the proposition below:

Proposition 2.10. *Every partial b-metrics define a b-metric where*

$$d(u,v) = 2b(u,v) - b(u,u) - b(v,v),$$

for all $u, v \in X$.

In 2013 Karapinar *et al.* [20] defined quasi-partial metrics and investigated on existence and uniqueness of fixed points. The definition of the quasi-partial metric is as follows:

Definition 2.11. [20] *A quasi-partial metric on a non-empty set X is a function $q_p : X \times X \to [0, \infty)$, satisfying the following:*
(i) *If $q_p(u,u) = q_p(u,v) = q_p(v,v)$ Then, $u = v$,*
(ii) $q_p(u,u) \leqslant qp(u,v)$,
(iii) $q_p(u,u) \leqslant q_p(v,u)$,
(iv) $q_p(u,v) + q_p(w,w) \leqslant q_p(u,w) + q_p(w,v)$.

The pair (X, qp) is called a quasi-partial metric space.
In 2015 Gupta *et al.* [15], introduced the notion of quasi partial b -metric space as follows:

Definition 2.12. [15] *A quasi-partial b-metric on a nonempty set X is a function $q_{p_b} : X \times X \to [0, \infty)$ such that for some real number $s \geq 1$, it satisfies the following axioms:*
(QPB1) *If $q_{p_b}(u,u) = q_{p_b}(u,v) = q_{p_b}(v,v)$ Then, $u = v$,*
(QPB2) $q_{p_b}(u,u) \leqslant q_{p_b}(u,v)$,
(QPB3) $q_{p_b}(u,u) \leqslant q_{p_b}(v,u)$,
(QPB4) $q_{p_b}(u,v) + q_{p_b}(w,w) \leqslant s[q_{p_b}(u,w) + q_{p_b}(w,v)]$.

Example 2.13. [15] Let $X = \mathbb{R}$ be the set of all real numbers. Define $q_{p_b} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$q_{p_b}(u,v) = |u - v| + |u|,$$

for all $u, v \in X$.
Then, it is clear that (X, q_{p_b}) is a quasi partial b-metric on space X.

Example 2.14. [15] *Let $X = [0, 1]$. Define*

$$q_{p_b}(u,v) = |u - v| + u.$$

So (X, q_{p_b}) is a quasi partial b-metric space with $s \geq 1$.

Example 2.15. [15] *Let $X = [1, \infty)$. Define*

$$q_{p_b}(u,v) = \ln(uv).$$

Then, (X, q_{p_b}) is a quasi partial b-metric space.

Example 2.16. [15] *Let $X = [0, \frac{\pi}{4}]$ and define $q_{p_b} : X \times X \to \mathbb{R}^+$ as*

$$q_{p_b}(u,v) = \sin u + \sin v.$$

Then, (X, q_{p_b}) is a quasi partial b-metric space.

We give the quasi-partial b-metric topology.
The following lemma states the relationship between quasi-partial metrics and quasi-partial b metrics as follows:

Lemma 2.17. [15] *Every quasi-partial space is quasi partial b-metric space, but the converse doesn't need to be true.*

The following definition states the conditions for a sequence in quasi-partial b-metric space to be convergent, Cauchy and Complete.

Definition 2.18. [15] *Let (X, q_{p_b}) be a quasi-partial b-metric. Then*

(i) *A sequence $\{u_n\} \subset X$ converges to $u \in X$ if and only if*

$$q_{p_b}(u, u) = \lim_{n\to\infty} q_{p_b}(u, u_n) = \lim_{n\to\infty} q_{p_b}(u_n, u).$$

(ii) *A sequence $\{u_n\} \subset X$ is called a Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} q_{p_b}(u_n, u_m)$$

and

$$\lim_{n,m\to\infty} q_{p_b}(u_m, u_n)$$

exist (and are finite).

(iii) *The quasi-partial b-metric space (X, q_{p_b}) is said to be complete if every Cauchy sequence $\{u_n\} \subset X$ converges with respect to $\tau_{q_{p_b}}$ to a point $u \in X$ such that*

$$q_{p_b}(u, u) = \lim_{n,m\to\infty} q_{p_b}(u_m, u_n) = \lim_{n,m\to\infty} q_{p_b}(u_n, u_m).$$

(iv) *A mapping $f : X \to X$ is said to be continuous at $u_0 \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(u_0, \delta)) \subset B(f(u_0), \epsilon)$.*

Lemma 2.19. [15] *Let (X, q_{p_b}) be a quasi-partial b-metric space. Then, the following hold:*

(i) if $q_{p_b}(\sigma, \varsigma) = 0$, Then, $\sigma = \varsigma$.
(ii) if $\sigma \neq \varsigma$, Then, $q_{p_b}(\sigma, \varsigma) > 0$ and $q_{p_b}(\varsigma, \sigma) > 0$

Definition 2.20. [16] *Let (X, q_{p_b}) be quasi partial b-metric spaces. Then, for $u_0 \in X$, $\epsilon > 0$, the q_{p_b}-ball with centre u_0 and radius ϵ is defined as*

$$Bq_{p_b}(u_0, \epsilon) \quad = \quad \{y \in X : Bq_{p_b}(u_0, y) < \epsilon \text{ and } Bq_{p_b}(y, u_0) < \epsilon\},$$

Definition 2.21. [16] *A quasi partial b-metric spaces (X, q_{p_b}) is said to be compact quasi partial b-metric if its q_{p_b}-complete and quasi b-totally bounded.*

Further, we provide some preliminary results.

In 1969, Browder [7] defined an asymptotically regular map as follows:

Definition 2.22. [7] *Let (X, d) be a complete metric space. A mapping $f : X \to X$ is said to be asymptotically regular at some point $c \in X$ if*

$$\lim_{n\to\infty} d(f^n u, f^{n+1} u) = 0.$$

The mapping f is said to be asymptotically regular on X if for all $u \in X$,

$$\lim_{n\to\infty} d(f^n u, f^{n+1} u) = 0.$$

Proinov [26] gave the new type of contractions known as asymptotically regular contractions as a generalization of the Banach contraction principle by giving two basic conditions as follows:

Theorem 2.23. [26] *Suppose (X, d) is a complete metric space and $f : X \to X$ is a continuous and asymptotically regular mapping such that:*
(i) $d(fu, fv) \leqslant \psi(\Omega(u, v))$ for all $u, v \in X$,
(ii) $d(fu, fv) < (\Omega(u, v)), u$
whenever $\Omega(u, v) \neq 0$, $\psi : R^+ \to R^+$ is a function such that: for any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < t < \delta$ implies $\psi(t) \leqslant \epsilon$. Here, R^+ is a set of all non-negative real numbers, and

$$\Omega(u, v) = d(u, v) + \eta[d(u, fu) + d(v, fv)], \eta \geq 0$$

Then, there exists a fixed point $z \in X$ for f. A mapping satisfying the two above conditions is called a Proinov- contraction. The Proinov contraction is more general than the quasi-contraction.

Example 2.24. [26] *Let $X = \{1, 2, 3\}$ be equipped with the usual metric d. Suppose $f : X \to X$ is a mapping defined as$f(1) = 1, f(2) = 3, f(3) = 1$. Then, the mapping f does not satisfy Theorem 2.23. However, for*

$$\psi(t) = \frac{2t}{1 + \eta}$$

and $\eta > 1$, the mapping f satisfies the conditions in Theorem 2.23.

Suzuki [31] gave a contraction which does not require a map to hold at all points on a given space as follows:

Theorem 2.25. [31] *Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping such that for all $u, v \in X$,*

$$\Phi(k)d(u, fu) \leq d(u, v) \Longrightarrow d(fu, fv) \leqslant kd(u, v)$$

where $\Phi : [0, 1) \to (\frac{1}{2}, 1)$ is a non-increasing function defined by:

$$\Phi(k) = \begin{cases} 1, & \text{if} \quad 0 \leq k \leq \frac{(\sqrt{5}-1)}{2}, \\ (1-k)k^{-2}, & \text{if} \quad \dfrac{(\sqrt{5}-1)}{2} \leq k \leq 2^{\frac{-1}{2}}, \\ (1+k)^{-1}, & \text{if} \quad 2^{\frac{-1}{2}} \leq k < 1. \end{cases}$$

Then, there exists a unique fixed point $z \in X$. A mapping f satisfying Theorem 2.25 is called a Suzuki contraction. The following example shows the generality of Theorem 2.25 over Theorem 2.23.

Example 2.26. [31] *Let $X = \{(1, 1), (4, 1), (1, 4), (4, 5), (5, 4)\}$ with a metric d defined by*

$$d((u_1, u_2), (v_1, v_2)) = |u_1 - v_1| + |u_2 - v_2|.$$

Define a mapping

$$f(u_1, u_2) = \begin{cases} (u_1, 1), & \text{if } u_1 \geq u_2, \\ (1, u_2), & \text{if } u_1 > u_2. \end{cases}$$

Then, the map f satisfies all the hypotheses of Theorem 2.25 and $(1, 1)$ is the unique fixed point of f. However, for $u = (4, 5)$ and $v = (5, 4)$

$$d(fu, fv) = 6 > 2 = d(u, v).$$

Thus f does not satisfy the assumptions in Theorem 2.25 for any $k \in [0, 1)$.

Remark 2.27. [31]
(i) A mapping satisfying Theorem 2.25 need not to be continuous.
(ii) A metric space X is complete if and only if every Suzuki contraction mapping on X has a fixed point.

In 2001, Khamis and Kirk [18]introduced metric compactness as follows:

Theorem 2.28. [18] *A subset D of a metric space X is compact if and only if any sequence $\{u_n\}$ of points of D has a subsequence $\{u_{n_k}\}$ which converges to a point of D.*

3. Main Results

We establish our first main results by generalizing Theorem 2.23 in the following definition:

Definition 3.1. Let (X, q_{p_b}) be a complete quasi partial b-metric space. A pair of maps $f, g : X \to X$, is called Proinov-Suzuki contraction, if for all $u, v \in X$,

$$\frac{1}{2} q_{p_b}(gu, fu) \leq q_{p_b}(gu, fv) \Longrightarrow q_{p_b}(gu, fv) \leq \psi(\Omega(u, v)), \tag{1}$$

where

$$\Omega(u, v) = q_{p_b}(u, v) + \eta[q_{p_b}(u, gu) + q_{p_b}(v, fv)]$$

and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semi-continuous function from right such that $\psi(t) < t$, for all $t > 0$.

Now we present our first main theorem.

Theorem 3.2. *Let (X, q_{p_b}) be a complete quasi partial b-metric space and let $f, g : X \to X$ be a pair of continuous asymptotically regular Proinov-Suzuki contractions. Then, f and g have a unique common fixed point. Further, if $\eta = 1$ Then, f and g need not be continuous.*

Proof. Let $u_0 \in X$ be arbitrary and define a sequence $\{u_n\}_{n=1}^{\infty}$ in way that

$$u_1 = g(u_0), u_2 = f(u_1), u_3 = g(u_2), ...u_{2n} = f(u_{2n-1}), u_{2n+1} = g(u_{2n})$$

Since, f and g are asymptotically regular maps, i.e

$$\lim_{n\to\infty} q_{p_b}(f(u_{2n-1}), g(u_{2n})) = \lim_{n\to\infty} q_{p_b}(u_{2n}, u_{2n+1}) = 0,$$

implies that there exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that for all $n \geq k$,

$$q_{p_b}(u_{2n}, u_{2n+1}) \leq \epsilon. \tag{2}$$

Let $u = u_{2n-1}$ and $v = u_{2n}$ in (1), we have

$$\begin{aligned} \frac{1}{2} q_{p_b}(gu_{2n-1}, fu_{2n-1}) \quad &\leq \quad q_{p_b}(gu_{2n-1}, fu_{2n}) \Longrightarrow \\ & \quad\;\; q_{p_b}(gu_{2n-1}, fu_{2n}) \leq \psi(\Omega(u_{2n-1}, u_{2n})), \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Omega(u_{2n-1}, u_{2n}) &= q_{p_b}(u_{2n-1}, u_{2n}) + \eta[q_{p_b}(u_{2n-1}, gu_{2n-1}) + q_{p_b}(u_{2n}, fu_{2n})], \\ &\leq q_{p_b}(u_{2n-1}, u_{2n}) + \eta[q_{p_b}(u_{2n-1}, u_{2n}) + q_{p_b}(u_{2n}, u_{2n+1})], \\ &\leq q_{p_b}(u_{2n-1}, u_{2n}) + \eta q_{p_b}(u_{2n-1}, u_{2n}) + \eta q_{p_b}(u_{2n}, u_{2n+1}), \\ &\leq (1+\eta) q_{p_b}(u_{2n-1}, u_{2n}) + \eta q_{p_b}(u_{2n}, u_{2n+1}). \end{aligned} \tag{4}$$

Using (4) in (3), we get

$$\begin{aligned}\frac{1}{2}q_{p_b}(u_{2n},u_{2n}) &\leq q_{p_b}(gu_{2n-1},fu_{2n})\\ \Longrightarrow q_{p_b}(u_{2n},u_{2n+1}) &\leq \psi((1+\eta)qp_b(u_{2n-1},u_{2n})+\eta q_{p_b}(u_{2n},u_{2n+1})).\end{aligned}$$

By the continuity of ψ we have

$$\begin{aligned}q_{p_b}(u_{2n},u_{2n+1}) &< (1+\eta)q_{p_b}(u_{2n-1},u_{2n})+\eta q_{p_b}(u_{2n},u_{2n+1}),\\ (1-\eta)q_{p_b}(u_{2n},u_{2n+1}) &< (1+\eta)q_{p_b}(u_{2n-1},u_{2n}),\\ q_{p_b}(u_{2n},u_{2n+1}) &< \frac{1+\eta}{1-\eta}q_{p_b}(u_{2n-1},u_{2n}) \leq \psi\Big(\frac{1+\eta}{1-\eta}q_{p_b}(u_{2n-1},u_{2n})\Big),\end{aligned}$$

Let $\delta=\frac{1+\eta}{1-\eta}$ in the above inequality, gives

$$q_{p_b}(u_{2n},u_{2n+1}) < \delta q_{p_b}(u_{2n-1},u_{2n}) \leq \psi(\delta q_{p_b}(u_{2n-1},u_{2n})).$$

Thus, we get $\{q_{p_b}(u_{2n-1},u_{2n})\}$ is a non-decreasing sequence and

$$q_{p_b}(u_{2n-1},u_{2n}) = 0.$$

Due to the non-decreasing nature of function ψ, we obtain

$$\begin{aligned}q_{p_b}(u_{2n},u_{2n+1}) &\leq \psi\delta q_{p_b}(u_{2n-1},u_{2n}) \leq \psi^2(\delta^2 q_{p_b}(u_{2n-2},u_{2n-1}))\dots\\ &\leq \dots \leq \psi^n(\delta^n q_{p_b}(u_{2n-2},u_{2n-1})).\end{aligned}$$

To show $\{u_n\}$ is a Cauchy sequence, it is sufficient to show that $\{u_{2n}\}$ is Cauchy sequence. Suppose now that $\{u_{2n}\}$is not Cauchy sequence. Then, there is an $\epsilon>0$ such that for each integer

$$2k \leq 2n_k < 2m_k$$

such that

$$q_{p_b}(u_{2n_k},u_{2m_k}) < \epsilon. \tag{5}$$

Let, for each integer $2k$, $k\in\mathbb{N}$, $2m_k$ be the least integer exceeding $2n_k$ satisfying 5, that is,

$$q_{p_b}(u_{2n_k},u_{2m_k-2}) \leq \epsilon$$

and

$$q_{p_b}(u_{2n_k},u_{2m_k}) > \epsilon.$$

Using the triangle inequality, we get for $s\geq 1$

$$\begin{aligned}\epsilon < q_{p_b}(u_{2n_k},u_{2m_k}) &\leq s[q_{p_b}(u_{2n_k},u_{2m_k-2})+q_{p_b}(u_{2m_k-2},u_{2m_k})-\\ &\quad q_{p_b}(u_{2m_k-2},u_{2m_k-2})]\\ &\leq s[\epsilon+q_{p_b}(u_{2m_k-2},u_{2m_k})-q_{p_b}(u_{2m_k-2},u_{2m_k-2})].\end{aligned}$$

Since f and g are Proinov-Suzuki contractions, Equation 1 implies

$$\begin{aligned}&q_{p_b}(gu_{2n_k},fu_{2m_k}) \leq \psi(q_{p_b}(u_{2n_k},u_{2m_k})+\eta[q_{p_b}(u_{2n_k},fu_{2n_k})\\ &+q_{p_b}(u_{2m_k},gu_{2m_k})]).\end{aligned}$$

In the same way, assume that Let $u=u_{2n}$ and $v=u_{2n-1}$ in (1), we have

$$\begin{aligned}\frac{1}{2}q_{p_b}(gu_{2n},fu_{2n}) &\leq q_{p_b}(gu_{2n},fu_{2n-1}) \Longrightarrow\\ &\quad q_{p_b}(gu_{2n-1},fu_{2n}) \leq \psi(\Omega(u_{2n},u_{2n-1})),\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\Omega(u_{2n}, u_{2n-1}) &= q_{p_b}(u_{2n}, u_{2n-1}) + \eta[q_{p_b}(u_{2n}, gu_{2n}) + q_{p_b}(u_{2n-1}, fu_{2n-1})], \\
&\leq q_{p_b}(u_{2n}, u_{2n-1}) + \eta[q_{p_b}(u_{2n}, u_{2n+1}) + q_{p_b}(u_{2n-1}, u_{2n})], \\
&\leq q_{p_b}(u_{2n}, u_{2n-1}) + \eta q_{p_b}(u_{2n}, u_{2n+1}) + \eta q_{p_b}(u_{2n-1}, u_{2n}), \\
&\leq (1+\eta) q_{p_b}(u_{2n}, u_{2n-1}) + \eta q_{p_b}(u_{2n}, u_{2n+1}). \qquad (7)
\end{aligned}$$

Using (4) in (3), we get

$$\begin{aligned}
\frac{1}{2} q_{p_b}(u_{2n+1}, u_{2n+1}) &\leq q_{p_b}(gu_{2n}, fu_{2n-1}) \\
\Longrightarrow q_{p_b}(u_{2n+1}, u_{2n}) &\leq \psi((1+\eta) q_{p_b}(u_{2n}, u_{2n-1}) + \eta q_{p_b}(u_{2n}, u_{2n+1})).
\end{aligned}$$

By the continuity of ψ we have

$$\begin{aligned}
q_{p_b}(u_{2n+1}, u_{2n}) &< (1+\eta) q_{p_b}(u_{2n}, u_{2n-1}) + \eta q_{p_b}(u_{2n}, u_{2n+1}), \\
(1-\eta) q_{p_b}(u_{2n+1}, u_{2n}) &< (1+\eta) q_{p_b}(u_{2n}, u_{2n-1}), \\
q_{p_b}(u_{2n+1}, u_{2n}) &< \frac{1+\eta}{1-\eta} q_{p_b}(u_{2n}, u_{2n-1}) \leq \psi\Big(\frac{1+\eta}{1-\eta}_{p_b}(u_{2n}, u_{2n-1})\Big).
\end{aligned}$$

Let $\delta = \frac{1+\eta}{1-\eta}$ in the above inequality, gives

$$q_{p_b}(u_{2n+1}, u_{2n}) < \delta q_{p_b}(u_{2n}, u_{2n-1}) \leq \psi(\delta q_{p_b}(u_{2n}, u_{2n-1})).$$

Thus, we get $\{q_{p_b}(u_{2n}, u_{2n-1})\}$ is a non-decreasing sequence and

$$q_{p_b}(u_{2n}, u_{2n-1}) = 0.$$

Due to the non-decreasing nature of function ψ, we obtain

$$\begin{aligned}
q_{p_b}(u_{2n+1}, u_{2n}) &\leq \psi\delta q_{p_b}(u_{2n}, u_{2n-1}) \leq \psi^2(\delta^2 q_{p_b}(u_{2n-1}, u_{2n-2}))\ldots \\
&\leq \cdots \leq \psi^n(\delta^n q_{p_b}(u_{2n-1}, u_{2n-2})).
\end{aligned}$$

Letting $n \to \infty$, gives $\epsilon \leq \psi(\epsilon) < \epsilon$, a contradiction unless $\epsilon = 0$.

For $m > n$, using, $(QPB4)$ we obtain

$$\begin{aligned}
q_{p_b}(u_{2n}, u_{2m}) &\leq s[q_{p_b}(u_{2n}, u_{2n-1}) + q_{p_b}(u_{2n-1}, u_{2m})] - q_{p_b}(u_{2n-1}, u_{2n-1}), \\
&\leq s q_{p_b}(u_{2n}, u_{2n-1}) + s q_{p_b}(u_{2n-1}, u_{2m}) - q_{p_b}(u_{2n-1}, u_{2n-1}), \\
&\leq s q_{p_b}(u_{2n}, u_{2n-1}) + s^2 q_{p_b}(u_{2n-1}, u_{2n-2}) + \ldots \\
&\quad + s^{m-n-1} q_{p_b}(u_{2n}, u_{2n-1}), \\
&\leq s\psi\delta^{2n-1} + s^2\psi^2\delta^{2n-2} + s^3\psi^3\delta^{2n-3} + \ldots \\
&\quad + s^{m-n-1}\psi^n\delta^{2n-1} q_{p_b}(u_{2n}, u_{2n}), \\
&\leq s\psi\delta^{2n-1}[1 + s\psi\delta^{-1} + s^2\psi^2\delta^{-2} + \ldots \\
&\quad + s^{m-n-2}\psi^{m-n-1}\delta^{m-n+1}] q_{p_b}(u_1, u_0), \\
&\leq \frac{s\psi\delta^{2n-1}}{1 - s\psi\delta^{-1}} q_{p_b}(u_1, u_0).
\end{aligned}$$

Letting $n \to \infty$, we have $q_{p_b}(u_{2n}, u_{2m}) = 0$. Thus $\{u_{2n}\}$ is a Cauchy sequence.

Since X is complete, $\{u_n\}$ converges to a point $z \in X$. If f and g are continuous. Then, z is a common fixed point of f and g. Now suppose that

$\eta = 1$ and f and g are discontinuous. Then, for any $n \in \mathbb{N}$ either

$$\frac{1}{2} q_{p_b}(u_n, u_{n+1}) \leq q_{p_b}(u_n, w) \tag{8}$$

or

$$\frac{1}{2} q_{p_b}(u_{n+1}, u_{n+2}) \leq q_{p_b}(u_{n+1}, w). \tag{9}$$

Assume on the contrary, that for some $n > k$,

$$q_{p_b}(u_n, w) < \frac{1}{2} q_{p_b}(u_n, u_{n+1})$$

and

$$q_{p_b}(u_{n+1}, w) < \frac{1}{2} qp_b(u_{n+1}, u_{n+2}).$$

Then, by the triangle inequality, we have for $s \geq 1$

$$\begin{aligned} q_{p_b}(u_{n+1}, u_{n+2}) - q_{p_b}(u_{n+2}, u_{n+2}) &\leq [q_{p_b}(u_n, w) + q_{p_b}(u_{n+1}, w) \\ &\quad -q_{p_b}(u_{n+2}, u_{n+2})], \\ &< s[\frac{1}{2} q_{p_b}(u_n, u_{n+1}) + \frac{1}{2} q_{p_b}(u_{n+1}, u_{n+2}) \\ &\quad -q_{p_b}(u_{n+2}, u_{n+2})], \\ &= s[q_{p_b}(u_n, u_{n+1}) - q_{p_b}(u_{n+2}, u_{n+2})], \end{aligned}$$

a contradiction unless $s > 1$ in which Equation (8) holds.

In the case

$$\frac{1}{2} q_{p_b}(u_n, u_{n+1}) \leq q_{p_b}(u_n, w),$$

by (1), we shall have

$$\begin{aligned} q_{p_b}(u_{n+1}, w) &= q_{p_b}(gu_n, fw) \leq \psi \Omega(gu_n, fw) \\ &= \psi(q_{p_b}(u_n, w) + q_{p_b}(u_n, fu_n) + q_{p_b}(w, gw)). \end{aligned}$$

Letting $n \to \infty$, gives

$$q_{p_b}(w, fw) \leq \psi(q_{p_b}(w, fw)) < q_{p_b}(w, fw),$$

a contradiction unless w is a common fixed point of f and g. We get the same conclusion in other cases. □

Now, we will consider another class of mappings:

Definition 3.3. *Let (X, q_{p_b}) be a compact quasi partial b-metric space and $f : X \to X$ be a mapping such that for all $u, v \in X$,*

$$q_{p_b}(u, fv) < \xi(q_{p_b}(u, v) + \phi[(q_{p_b}(u, fu) + q_{p_b}(f^n u, f^{n+1} u)]), \tag{10}$$

where $n \in \mathbb{N}$, $\xi \in (0, 1)$ and $\phi \in [0, 1)$.

The above class of mappings contains many important classes of mappings. A number of contractions listed in [28] *are particular cases of the following mapping:*

Definition 3.4. *Let (X, q_{p_b}) be a compact quasi partial b-metric space and let $f : X \to X$ be mapping such that for all $u, v \in X$*

$$\begin{aligned} q_{p_b}(fu, fv) < k \quad max\{&q_{p_b}(u, v), \frac{q_{p_b}(u, fu) + q_{p_b}(v, fv)}{2}, q_{p_b}(u, fv), q_{p_b}(v, fu), \\ &q_{p_b}(f^2 u, u), q_{p_b}(f^{n+1} u, fu), q_{p_b}(f^{n+1} u, v), q_{p_b}(f^{n+1} u, fv)\} \end{aligned}$$

where $k \in [0,1)$ is fixed.

Now, we show that a mapping satisfying Definition 3.4 also satisfies Definition 3.3.

Proposition 3.5. *Let (X, q_{p_b}) be a compact quasi partial b-metric space and let $f : X \to X$ be a map which satisfies Definition 3.3. Then, the map also satisfies Definition 3.4 .*

Proof. We consider the following cases:
Case (i) $q_{p_b}(fu, fv) < kq_{p_b}(u, v)$. Using $(QPB4)$ and $s \geq 1$, we get

$$\begin{aligned} q_{p_b}(u, fv) &< s[q_{p_b}(u, fu) + q_{p_b}(fu, fv) - q_{p_b}(fu, fu)], \\ &< kq_{p_b}(u, v) + q_{p_b}(u, fu). \end{aligned} \tag{11}$$

Case (ii) $q_{p_b}(u, fv) < k(\frac{q_{p_b}(u,fu)+q_{p_b}(v,fv)}{2})$. Then,

$$\begin{aligned} q_{p_b}(u, fv) &< q_{p_b}(u, fu) + q_{p_b}(fu, fv), \\ &< \tfrac{3}{2}q_{p_b}(u, fu) + \tfrac{1}{2}kq_{p_b}(u, v) + \tfrac{1}{2}q_{p_b}(u, fv), \end{aligned} \tag{12}$$

it implies that

$$q_{p_b}(u, fv) < kq_{p_b}(u, v) + 3q_{p_b}(u, fu).$$

Case (iii)

$$q_{p_b}(fx, fy) < kq_{p_b}(x, fy).$$

Then,

$$q_{p_b}(u, fv) < qq_{p_b}(u, fu) + kq_{p_b}(u, fv),$$

and

$$q_{p_b}(u, fv) < kq_{p_b}(u, v) + \frac{1}{1-k}q_{p_b}(u, fu).$$

Case (iv)

$$q_{p_b}(fu, fv) < kq_{p_b}(v, fu),$$

implies that

$$q_{p_b}(u, fv) < q_{p_b}(u, fu) + kq_{p_b}(v, fu).$$

Then

$$\begin{aligned} q_{p_b}(u, fv) &< q_{p_b}(u, fu) + kq_{p_b}(v, fv), \\ &< q_{p_b}(u, fu) + k[q_{p_b}(u, v) + q_{p_b}(u, fu)], \\ &< kq_{p_b}(u, v) + (1 + k)q_{p_b}(u, fu). \end{aligned}$$

Case (v)

$$q_{p_b}(u, fv) < kq_{p_b}(f^{n+1}u, u).$$

Then, implies

$$\begin{aligned} q_{p_b}(u, fv) &< q_{p_b}(u, fu) + kq_{p_b}(f^{n+1}u, u), \\ &< q_{p_b}(u, fu) + kq_{p_b}(f^{n+1}u, fu) + kq_{p_b}(u, fu), \\ &< (1 + k)q_{p_b}(u, fu) + q_{p_b}(f^{n+1}, u). \end{aligned}$$

Case (vi)

$$q_{p_b}(fu, fv) < kq_{p_b}(f^{n+1}u, fu).$$

Then,

$$q_{p_b}(u, fv) < q_{p_b}(u, fu) + q_{p_b}(f^{n+1}u, fu)).$$

Case (vii)

$$q_{p_b}(fu, fv) < kq_{p_b}(f^{n+1}u, v).$$

Then

$$\begin{aligned} q_{p_b}(u, fv) &< q_{p_b}(u, fu) + kq_{p_b}(f^{n+1}u, fu) + kq_{p_b}(fu, v), \\ &< kq_{p_b}(u, v) + (1+k)[q_{p_b}(u, fu) + q_{p_b}(f^{n+1}u, fu)]. \end{aligned}$$

Case (viii)

$$q_{p_b}(fu, fu) < kq_{p_b}(f^{n+1}u, fv).$$

Then

$$\begin{aligned} q_{p_b}(u, fv) &< q_{p_b}(u, fu) + kq_{p_b}(f^{n+1}u, fv) \\ &< q_{p_b}(u, fu) + kq_{p_b}(f^2u, fu) + kq_{p_b}(fu, u) + kq_{p_b}(u, fv), \end{aligned} \tag{13}$$

and $q_{p_b}(u, fv) < k(q_{p_b}(u, v) + \frac{2}{1-k}[q_{p_b}(u, fu) + q_{p_b}(f^{n+1}u, fu)])$.

Thus, f satisfies Definition 3.3, with $\xi = k$, $\phi = \max\{3, \frac{2}{1-k}\}$ and $n = 2$. □

Theorem 3.6. *Suppose (X, q_{p_b}) is a complete and compact quasi partial b-metric space and $f : X \to X$ is an asymptotically regular map satisfying condition in the Definition 3.3. Then, there exists a unique fixed point $\alpha \in X$ for f and for any $u_0 \in X$ we have*

$$\lim_{n\to\infty} fu_0 = \alpha.$$

Proof. Let $u_0 \in X$ and define a sequence that satisfies following condition:

$$u_n = f^n(u_0) = f(u_{n-1})$$

and for all $n \in \mathbb{N}$.

Let $u = u_{n-1}$ and $v = u_n$ in (10) we get

$$\begin{aligned} q_{p_b}(u_{n-1}, fu_n) &< \xi(q_{p_b}(u_{n-1}, u_n) + \phi[(q_{p_b}(u_{n-1}, fu_{n-1}) + \\ &\quad q_{p_b}(f^n u_{n-1}, f^{n+1}u_{n-1})]), \\ q_{p_b}(u_{n-1}, u_{n+1}) &< \xi(q_{p_b}(u_{n-1}, u_n) + \phi[(q_{p_b}(u_{n-1}, u_n) + q_{p_b}(u_n, u_{n+1})]), \\ &< \xi q_{p_b}(u_{n-1}, u_n) + \xi\phi(q_{p_b}(u_{n-1}, u_n) + \xi\phi q_{p_b}(u_n, u_{n+1}). \end{aligned} \tag{14}$$

$$\begin{aligned} q_{p_b}(u_{n-1}, fu_n) &< s[q_{p_b}(u_{n-1}, fu_{n-1}) + q_{p_b}(fu_{n-1}, fu_n)] - q_{p_b}(fu_{n-1}, fu_{n-1}), \\ q_{p_b}(u_{n-1}, u_{n+1}) &< sq_{p_b}(u_{n-1}, fu_{n-1}) + sq_{p_b}(fu_{n-1}, fu_n), \\ &< sq_{p_b}(u_{n-1}, fu_{n-1}) + sq_{p_b}(fu_{n-1}, fu_n), \\ &< sq_{p_b}(u_{n-1}, u_n) + sq_{p_b}(u_n, u_{n+1}). \end{aligned} \tag{15}$$

Using (15) in (14), gives

$$\begin{aligned} sq_{p_b}(u_{n-1}, u_n) + sq_{p_b}(u_n, u_{n+1}) &< \xi q_{p_b}(u_{n-1}, u_n) + \xi\phi(q_{p_b}(u_{n-1}, u_n) + \\ &\quad \xi\phi q_{p_b}(u_n, u_{n+1}), \\ (s - \xi\phi)q_{p_b}(u_n, u_{n+1}) &< (\xi(1+\phi) - s)q_{p_b}(u_{n-1}, u_n), \\ q_{p_b}(u_n, u_{n+1}) &< \frac{(\xi(1+\phi) - s)}{(s - \xi\phi)} q_{p_b}(u_{n-1}, u_n). \end{aligned}$$

Repeating the above steps by mathematical induction, we obtain

$$q_{p_b}(u_n, u_{n+1}) \quad < \quad \left(\frac{(\xi(1+\phi)-s)}{(s-\xi\phi)}\right)^n q_{p_b}(u_{n-1}, u_n).$$

Letting $\frac{(\xi(1+\phi)-s)}{(s-\xi\phi)} = \vartheta$, we have

$$q_{p_b}(u_n, u_{n+1}) \quad < \quad \vartheta^n q_{p_b}(u_{n-1}, u_n). \tag{16}$$

Hence $q_{p_b}(u_n, u_{n+1}) = 0 \to 0$ as $n \to \infty$.

For any $n, m > 0$, by triangle inequality and from Definition 3.1., We have for $s \geq 1$,

$$\begin{aligned}
q_{p_b}(u_n, u_{n+m}) &\leq s[q_{p_b}(u_n, u_{n+1}) + q_{p_b}(u_{n+1}, u_{n+m})] - q_{p_b}(u_{n+1}, u_{n+1}), \\
&\leq s q_{p_b}(u_n, u_{n+1}) + s q_{p_b}(u_{n+1}, u_n) - q_{p_b}(u_{n+1}, u_{n+1}), \\
&\leq s q_{p_b}(u_n, u_{n+1}) + s^2 q_{p_b}(u_{n+1}, u_{n+2}) + \cdots + s^{n-m-1} q_{p_b}(u_{m-1}, u_m), \\
&\leq (s\vartheta + s^2\vartheta^2 + s^3\vartheta^3 + s^4\vartheta^4 + \cdots + \dots) q_{p_b}(u_0, u_m), \\
&\leq s^n\vartheta^n(1 + s\vartheta + s^2\vartheta^2 + s^3\vartheta^3 + \cdots + \cdots + s^{n-1}\vartheta^{n-1}) q_{p_b}(u_0, u_m), \\
&\leq \frac{s^n\vartheta^n}{1 - s\vartheta} q_{p_b}(u_0, u_m).
\end{aligned}$$

By the asymptotic regularity of f, we obtain $q_{p_b}(u_n, u_{n+m}) \to 0$ as $n \to \infty$. This shows that $\{u_n\}$ is a Cauchy sequence.

Similarly, Let $u = u_n$ and $v = u_{n-1}$ in (10) we get

$$\begin{aligned}
q_{p_b}(u_n, fu_{n-1}) &< \xi(q_{p_b}(u_n, u_{n-1}) + \phi[(q_{p_b}(u_n, fu_n) + q_{p_b}(f^n u_n, f^{n+1} u_n)]), \\
q_{p_b}(u_n, u_n) &< \xi(q_{p_b}(u_n, u_{n-1}) + \phi[(q_{p_b}(u_n, u_{n+1}) + q_{p_b}(u_n, u_{n+1})]), \\
q_{p_b}(u_n, u_n) &< \xi q_{p_b}(u_n, u_{n-1}) + 2\xi\phi q_{p_b}(u_n, u_{n+1}), \\
-2\xi\phi q_{p_b}(u_n, u_{n+1}) &< \xi q_{p_b}(u_n, u_{n-1}), \\
2\xi\phi q_{p_b}(u_{n+1}, u_n) &< \xi q_{p_b}(u_n, u_{n-1}), \\
q_{p_b}(u_{n+1}, u_n) &< \frac{\xi}{2\xi\phi} q_{p_b}(u_n, u_{n-1}). \qquad (17)
\end{aligned}$$

Repeating the above steps by mathematical induction, we obtain

$$q_{p_b}(u_{n+1}, u_n) \quad < \quad \left(\frac{\xi}{2\xi\phi}\right)^n q_{p_b}(u_n, u_{n-1}).$$

Letting $\frac{\xi}{2\xi\phi} = \theta$, we have

$$q_{p_b}(u_{n+1}, u_n) \quad < \quad \theta^n q_{p_b}(u_n, u_{n-1}). \tag{18}$$

Hence $q_{p_b}(u_{n+1}, u_n) = 0 \to 0$ as $n \to \infty$.

For any $n, m > 0$, by triangle inequality and from Definition 3.1., We have for $s \geq 1$,

$$\begin{aligned}
q_{p_b}(u_{n+m}, u_n) &\leq s[q_{p_b}(u_n, u_{n-1}) + q_{p_b}(u_{n-1}, u_{n+m})] - q_{p_b}(u_{n-1}, u_{n-1}),\\
&\leq sq_{p_b}(u_n, u_{n-1}) + sq_{p_b}(u_{n-1}, u_{n+m}) - q_{p_b}(u_{n-1}, u_{n-1}),\\
&\leq q_{p_b}(u_n, u_{n-1}) + s^2 q_{p_b}(u_{n-1}, u_{n-2}) + \cdots + s^{m-n-1} qp_b(u_{n-1}, u_n),\\
&\leq (s\theta^{n-1} + s^2\theta^{n-2} + s^3\theta^{n-3} + s^4\theta^{n-4} + \dots\\
&+ s^{m-n-1}\delta^m) q_{p_b}(u_m, u_0),\\
&\leq s\theta^{n-1}(1 + s\theta^{n-1} + s^2\theta^{n-2} + s^3\theta^{n-3} + \dots\\
&+ s^{m-n-2}\delta^{m-n+1}) q_{p_b}(u_m, u_0),\\
&\leq \frac{s\theta^{n-1}}{1 - s\theta^{-1}} q_{p_b}(u_m, u_0).
\end{aligned}$$

By the asymptotic regularity of f, we obtain $q_{p_b}(u_{n+m}, u_n) \to 0$ as $n \to \infty$. This shows that $\{u_n\}$ is a Cauchy sequence.

Since (X, q_{p_b}) is complete quasi partial b-metric space, there exists $\alpha \in X$ such that

$$\lim_{n\to\infty} fu_n = \alpha.$$

Next, we show that α is a fixed point of f. From Definition 3.3 and $(QPB4)$, it follows that

$$\begin{aligned}
q_{p_b}(\alpha, fu_n) &\leq s[q_{p_b}(\alpha, fu_{n-1}) + q_{p_b}(fu_{n-1}, fu_n)] - q_{p_b}(fu_{n-1}, fu_{n-1}),\\
q_{p_b}(\alpha, fu_n) &\leq sq_{p_b}(\alpha, fu_{n-1}) + sq_{p_b}(fu_{n-1}, fu_n),\\
q_{p_b}(\alpha, fu_n) &\leq sq_{p_b}(\alpha, fu_{n-1}) + s[\xi(q_{p_b}(u_{n-1}, u_n) + \phi[q_{p_b}(u_{n-1}, fu_{n-1}) +\\
&\quad q_{p_b}(f^n u_{n-1}, f^{n+1}u_{n-1})])],\\
\\
q_{p_b}(\alpha, fu_n) &\leq sq_{p_b}(\alpha, fu_{n-1}) + s\xi(q_{p_b}(u_{n-1}, u_n) + s\phi[q_{p_b}(u_{n-1}, fu_{n-1}) +\\
&\quad s\phi q_{p_b}(fu_{n-1}, fu_n)]),\\
q_{p_b}(\alpha, f\alpha) &\leq sq_{p_b}(\alpha, f\alpha) + s\xi(q_{p_b}(\alpha, \alpha) + s\phi[q_{p_b}(\alpha, f\alpha) +\\
&\quad s\phi q_{p_b}(f\alpha, f\alpha)]),\\
q_{p_b}(\alpha, f\alpha) &\leq (s + s\xi + s\phi + 2s\phi) q_{p_b}(f\alpha, f\alpha),\\
q_{p_b}(\alpha, f\alpha) &\leq 0,\\
q_{p_b}(\alpha, f\alpha) &= 0,
\end{aligned}$$

it implies that $u_n \to f(\alpha)$ as $n \to \infty$ and $f(\alpha) = \alpha$.

Suppose β is another fixed point of f. From Definition 3.3, we have

$$\begin{aligned}
0 < q_{p_b}(\alpha, \beta) &= q_{p_b}(\alpha, f\beta) \leq \xi(q_{p_b}(\alpha, \beta) + \phi[q_{p_b}(\alpha, f\alpha) + qp_b(f^n\alpha, f^{n+1}\alpha)])\\
&= q_{p_b}(\alpha, \beta) < \xi qp_b(\alpha, \beta).\\
&= (1 - \xi) q_{p_b}(\alpha, \beta) < 0,\\
&= q_{p_b}(\alpha, \beta) < 0,
\end{aligned}$$

a contradiction unless $\alpha = \beta$. This proves the uniqueness of fixed points.

Further, for any $u \in X$, we have

$$\begin{aligned} q_{p_b}(f^n u, \alpha) &= q_{p_b}(f^n u, f^n u) \\ &\le \xi q_{p_b}(f^n u, \alpha) + \phi[q_{p_b}(f^n u, f^{n+1}u) + q_{p_b}(f^n u, f^{n+1}u)]. \end{aligned}$$

This implies that

$$(1-\xi) q_{p_b}(f^n u, \alpha) \le \phi[q_{p_b}(f^n u, f^{n+1}u) + q_{p_b}(f^n u, f^{n+1}u)] \to 0,$$

as $n \to \infty$. This shows that

$$\lim_{n\to\infty} f^n u = \alpha.$$

By compactness properties of X, let (X, q_{p_b}, s) be quasi-partial b-metric space Then, for any $u_0 \in X$ and $\epsilon > 0$, and if $v \in Bq_{p_b}(u_0, \frac{\epsilon}{2s})$ Then, there exists a $\delta > 0$ such that

$$Bq_{p_b}(v, \delta) \quad \subset \quad Bq_{p_b}(u_0, \epsilon).$$

Let

$$\delta \quad = \quad \frac{\epsilon}{s} - q_{p_b}(u_0, v) - q_{p_b}(v, u_0),$$

and $\lim_{n\to\infty} f^n u = \alpha$, let α_{n_k} be a convergent subsequence of α_n, Then, we have

$$\begin{aligned} \alpha_{n_k} &\in Bq_{p_b}(v, \alpha_{n_k}) < \delta, \\ \alpha_{n_k} &\in Bq_{p_b}(\alpha_{n_k}, v) < \delta, \end{aligned}$$

and

$$\begin{aligned} q_{p_b}(u_0, v) + q_{p_b}(v, \alpha_{n_k}) &< \frac{\epsilon}{s}, \\ q_{p_b}(v, u_0) + q_{p_b}(\alpha_{n_k}, v) &< \frac{\epsilon}{s}. \end{aligned}$$

Using $(QPB4)$, we get

$$\begin{aligned} q_{p_b}(u_0, \alpha_{n_k}) &\le s[q_{p_b}(u_0, v) + q_{p_b}(v, \alpha_{n_k})] - q_{p_b}(v, v), \\ q_{p_b}(u_0, \alpha_{n_k}) &< s \times \frac{\epsilon}{s} - q_{p_b}(v, v), \\ q_{p_b}(u_0, \alpha_{n_k}) &< \epsilon. \end{aligned}$$

Similarly

$$\begin{aligned} q_{p_b}(\alpha_{n_k}, u_0) &\le s[q_{p_b}(v, u_0) + q_{p_b}(\alpha_{n_k}, v)] - q_{p_b}(v, v), \\ q_{p_b}(\alpha_{n_k}, u_0) &< s \times \frac{\epsilon}{s} - q_{p_b}(v, v), \\ q_{p_b}(u_0, \alpha_{n_k}) &< \epsilon. \end{aligned}$$

Therefore $\alpha_{n_k} \in Bq_{p_b}(u_0, \epsilon)$, where Bq_{p_b} is the families of all quasi-balls in X. □

We provide the following corollary to demonstrate the usefulness of Theorem 3.2.

Corollary 3.7. *Let (X, q_{p_b}) be a complete quasi partial b-metric space and let $f, g : X \to X$ be a pair of Proinov-Suzuki contractions such that*

$$\frac{1}{2} q_{p_b}(gu, fu) \le q_{p_b}(gu, fv) \Longrightarrow q_{p_b}(gu, fv) \le \psi(q_{p_b}(u, v)). \tag{19}$$

Then, there exists a unique common fixed point $w \in X$ for f and g.

Proof. Let $u_0 \in X$ be arbitrary and define a sequence $\{u_n\}_{n=1}^{\infty}$ in way that

$$u_1 = g(u_0), u_2 = f(u_1), u_3 = g(u_2), ...u_{2n} = f(x_{2n-1}), u_{2n+1} = g(u_{2n})$$

we get

$$\begin{aligned} &q_{p_b}(u_n, u_{n+1}) = q_{p_b}(gu_{n-1}, fu_n) \leq \psi(q_{p_b}(u_{n-1}, u_n)) \\ &< q_{p_b}(u_{n-1}, u_n). \end{aligned}$$

Hence, the sequence $q_{p_b}(u_n, u_{n+1})$ is decreasing and bounded below, so it has a limit l . Suppose $l > 0$, Then, the above inequality, we get

$$q_{p_b}(u_n, u_{n+1}) = q_{p_b}(gu_{n-1}, fu_n) \leq \psi(q_{p_b}(u_{n-1}, u_n)).$$

So that $l = \lim_{t \to l+} sup\ \psi(t) \leq \psi(l) < l$, which is a contradiction. Therefore

$$\lim_{n \to \infty} q_{p_b}(u_n, u_{n+1}) = 0$$

and f and g are asymptotically regular.

Rest of the proof may be completed following the proof of Theorem 3.2.

Now, we provide the following example for illustration of Theorem 3.2.

Example 3.8. *Let $X = [0, \infty)$ with a quasi partial b-metric*

$$q_{p_b}(u, v) = |u - v| + |u|.$$

Define $f, g : X \to X$ such that

$$f(u) = \begin{cases} \frac{2u}{5} & for \quad u \in [0, 2], \\ 4 & for \quad u > 2. \end{cases}$$

and

$$g(u) = \begin{cases} \frac{u}{5} & for \quad u \in [0, 2], \\ 2 & for \quad u > 2. \end{cases}$$

For, $u, v \in [0, 2]$ and $u, v > 2$

$$\frac{1}{2} q_{p_b}(gu, fu) = \frac{1}{2}\{|\frac{u}{5} - \frac{2u}{5}| \ + |\frac{u}{5}|\} = \frac{1}{5}|u|$$

$$\leq q_{p_b}(gu, fv) = \frac{1}{5}\{|u - 2v| + |u|\}$$

implies that, For $\eta = 1$ and $\psi(t) = \frac{1}{t}$ where $t > 0$

$$q_{p_b}(gx, fy) = \frac{1}{5}\{|u - 2v| + |u|\},$$

$$\begin{aligned} &\leq \psi(\Omega(u, v)) \\ &= q_{p_b}(u, v) + q_{p_b}(u, gu) + qp_b(v, fv), \\ &= (|u - v| + |u|) + (|u - \frac{u}{5}| + |u|) + (|v - \frac{2v}{5}| + |v|). \end{aligned}$$

Clearly, f and g are discontinuous. It is evident that all the hypotheses of Theorem 3.2 hold with the unique common fixed point namely 0.

□

4. Application to Nonlinear-Integral Equation

In this section, we present an application of our results to non-linear integral equations. Now, we consider the following Fredholm integral equation:

$$u(t) = \tau(t) + \lambda \int_a^b \kappa(t,s)\xi(s,u(s))ds, t \in [a,b], \lambda \geq 0. \tag{12}$$

where $u(t)$ is the unknown solution, $\kappa(t,s)$ is a kernel, $\xi(s,u(s))$ is a non-linear smooth function and λ is a constant.

Theorem 4.1. *Let $X = C[\alpha,\beta]$ be a compact quasi-partial b-metric space of continuous functions on $[\alpha,\beta]$ with a quasi-partial b-metric defined by*

$$q_{p_b}(u,v) = sup\{|u(t) - v(t)| + |u(t)|\}.$$

Suppose that the following assumptions are true:

(i) $\tau : [\alpha,\beta] \to \mathbb{R}$ *is a continuous function;*
(ii) $\xi : [\alpha,\beta] \times X \to X$ *is continuous,* $\xi(t,u) \geq 0$ *and there is a constant* $\mu \geq 0$ *such that for all* $u, v \in X$,
$|\xi(t,u) - \xi(t,v)| + |\xi(t,u)| < \mu(|u(t) - v(t)| + |u(t)|)$;
(iii) $\kappa : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ *is a continuous for all* $(t,u) \in [\alpha,\beta] \times [\alpha,\beta]$ *such that* $\kappa(t,u) \geq 0$ *and* $\int_0^1 \kappa(t,s)ds \leq \kappa$;
(iv) $\xi = \lambda\kappa\mu < 1$;
(v) f *is a map defined by:*

$$u(t) = f(u(t)) = \tau(t) + \lambda \int_a^b \kappa(t,s)\xi(s,u(s))ds,$$

and f is an asymptotically regular. Then, the nonlinear integral equation has a unique solution in X. Moreover, for each $u_0 \in X$, the Picard sequence $\{u_n\}_{n=1}^{\infty}$ defined by

$$u_1(t) = f(u_0(t)) = \tau(t) + \lambda \int_a^b \kappa(t,s)\xi(s,u_0(s))ds,$$

$$u_2(t) = f(u_1(t)) = \tau(t) + \lambda \int_a^b \kappa(t,s)\xi(s,u_1(s))ds,$$

....

$$u_n(t) = f(u_{n-1}(t)) = \tau(t) + \lambda \int_a^b \kappa(t,s)\xi(s,u_{n-1}(s)))ds,$$

$\forall n \in \mathbb{N} \cup \{0\}$ *converges to a unique solution of the integral equation.*

For any $u, v \in X$,

$$\begin{aligned}
|u(t)-f(v(t))|+|u(t)| &= |[u(t)-\tau(t)-\lambda\int_a^b \kappa(t,s)\xi(s,v(s))ds]|+|u(t)|,\\
&= |[u(t)-\tau(t)-\lambda\int_a^b \kappa(t,s)\xi(s,u(s))ds]+[\tau(t)\\
&\quad +\lambda\int_a^b \kappa(t,s)\xi(s,u(s))ds-\tau(t)\\
&\quad -\lambda\int_a^b \kappa(t,s)\xi(s,v(s))ds]|+|u(t)|,\\
&< (|u(t)-f(t)|+|u(t)|)+\lambda|\int_a^b \kappa(t,s)\xi(s,u(s))ds\\
&\quad -\int_a^b \kappa(t,s)\xi(s,v(s))ds|+\lambda|\int_a^b \kappa(t,s)\xi(s,u(s))ds|,\\
&< (|u(t)-f(u(t))|+|u(t)|)+\lambda\int_a^b \kappa(t,s)ds|\xi(s,u(s)-\\
&\quad \xi(s,v(s))|+\lambda\int_a^b \kappa(t,s)ds|\xi(s,u(t))|,\\
&= (|u(t)-f(u(t))|)+|u(t)|\\
&\quad +\lambda\int_a^b \kappa(t,s)ds[|\xi(s,u(s))-\xi(s,v(s))|+\\
&\quad |M(s,u(s))|]
\end{aligned}$$

$$< (|u(t)-f(u(t)|+|u(t)|)+\lambda\int_a^b \kappa(t,s)ds\mu(|u(t)-v(t)|+|u(t)|). \tag{13}$$

Now, taking the Supremum in both sides of (13), we obtain the following:

$$q_{p_b}(u(t),f(v(t)))<q_{p_b}(u(t),f(u(t)))+\lambda\kappa\mu q_{p_b}(u(t),v(t)),$$

and finally, we obtain the following relation:

$$q_{p_b}(u(t),f(v(t)))<q_{p_b}(u(t),f(u(t)))+\xi q_{p_b}(u(t),v(t)).$$

Thus, the mapping f satisfies the condition in the Definition 3.3 and all the hypotheses of Theorem 4.1 hold. Therefore, Equation (12) has a unique solution.

Example 4.2. *Let us consider the following Fredholm integral equation:*

$$u(t)=[cos(\frac{\pi}{3}t)-\frac{630}{713}t]+\frac{1}{2}\int_0^1 (ts^2)(u(s)-\frac{s}{2})ds$$

$t \in [0,1]$ *it can be seen that the Fredholm integral equation is a particular case of with*

$$\tau(t)=cos(\frac{\pi}{3}t)-\frac{630}{713}t;\kappa(t,s)=t^2s$$

and $\xi(t,u(t))=u(t)-\frac{t}{2}$. *For any* $u, v \in \mathbb{R}$ *and for* $t\in[0,1]$, *we have*

$$|\xi(t,u)-\xi(t,v)|+|\xi(t,u)|<|u-v|+|u|,$$

it can be easily seen that τ is a continuous function and $t \in [0,1]$

$$\int_0^1 \kappa(t,s)ds = \int_0^1 t^2 s ds = \frac{t^2}{2} \leq \frac{1}{2}.$$

Further, $\mu = 1, \lambda = \frac{1}{2}, \kappa = \frac{1}{2}$ with $\xi = \lambda\kappa L = \frac{1}{4} < 1$. Therefore, all the assumptions of the Theorem (4.1) are satisfied. Hence, there exists a solution of the Fredholm integral equation in quasi-partial b-metric space X.

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