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# An inverse spectral problem algorithm for a class of block band operators and related vector continued fractions.

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**Abstract**

For one class of operators generated by infinite band matrices with operator entries (block band operators) we present a solution algorithm of the inverse spectral problem which amounts to reconstruction of the operator from the moment sequence of its Weyl matrix. We show how this algorithm can me applied for expansion of a system of operator-valued functions into a vector continued fraction with operator coefficients (which can be regarded as a kind of generalization of the $J-$ fraction) and indicate its connection with the Jacobi-Perron algorithm.

*Key Words:* Jacobi operators, Band operators, Inverse spectral problem, Continued fractions, Jacobi-Perron algorithm
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## 1 Introduction

The study of Jacobi operators,by which are usually understood the minimal closed operators generated by infinite Jacobi matrices

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

such that

$$a_i > 0,\ b_i \in \mathbb{R}, \quad i \in \mathbb{Z}_+$$

and acting in the Hilbert space of complex quadratic summable sequences $l^2[0,\infty)$ is of interest to the operator theory itself (they can be regarded as discrete versions of the Sturm-Liouville operators, see e. g. [6]) as well as because of their various applications to the moment problems [1], orthogonal polynomials and Padé approximations [12], analytic theory of continued fractions [26] and integration of nonlinear dynamical systems [25]. We denote such operators by $\mathcal{J}$, also, let $\{e_k\}_{k=0}^{\infty}$ be the standard orthonormal basis in $l^2[0,\infty)$. For an operator $\mathcal{J}$ we denote by $E_\lambda$ its spectral function (for the definition of a spectral function of symmetric operator see

[1], Chapter 4 or [2], Appendix 1). The Weyl function of the operator $\mathcal{J}$ is then defined as follows:

$$m(z) = m(z, \mathcal{J}) := (\mathcal{R}_z e_0, e_0) = \int_{-\infty}^{\infty} \frac{d(E_\lambda e_0, e_0)}{\lambda - z}, \tag{1}$$

where $\mathcal{R}_z = (z\mathcal{I} - \mathcal{J})^{-1}$ is the resolvent operator, $\mathcal{I}$ - identity operator and $z \in \mathbb{C}$ (the latter implication follows from the spectral theorem). According to inverse problem theory, the inverse spectral problem for $\mathcal{J}$ amounts to its reconstruction (i. e. reconstruction of the elements of $J$) from $E_\lambda$ (in some sources, see e. g. [7], the function $\rho(\lambda) := (E_\lambda e_0, e_0)$ is taken in place of $E_\lambda$). As follows from the definition of $m(z)$ and uniqueness of the Stieltjes integral transform, in terms of this inverse spectral problem, $m(z)$ and $E_\lambda$ (or $\rho(\lambda)$) are two equivalent objects. For the moments $s_k := \int_{-\infty}^{\infty} \lambda^k d(E_\lambda e_0, e_0),\ k \in \mathbb{Z}_+$ we have:

$$s_k = \int_{-\infty}^{\infty} \lambda^k d(E_\lambda e_0, e_0) = (\mathcal{J}^k e_0, e_0) = (J^k)_{0,0}.$$

These moments are related to the asymptotic expansion of the resolvent of the operator $J$ at infinity, namely, for sufficiently large $z$ we have $(zE - \mathcal{J})^{-1} \sim \frac{E}{z} + \frac{J}{z^2} + \frac{J^2}{z^3} + \dots$. (if $\mathcal{J}$ is bounded then the Neumann formula for the resolvent gives the exact equality). Thus, as follows from the definition of $m(z)$, we can call the sequence $S = \{s_k\}_{k=0}^{\infty}$ the moment sequence of the Weyl function $m(z)$. It follows from the Hamburger theorem, see e. g. [1] Chapter 2, that $S$ is a positive sequence, i. e. for all $i$

$$\Delta_i > 0; \tag{2}$$

where $\Delta_i$ are determinants of the Hankel matrices $H_i = (s_{j+l})_{j,l=0}^{i}$. The elements of $J$ can be reconstructed from $S$, see [1], Chapter 1:

$$a_i = \frac{\sqrt{\Delta_{i+1}\Delta_{i-1}}}{\Delta_i}, \quad b_i = \frac{D_i}{\Delta_i} - \frac{D_{i-1}}{\Delta_{i-1}}, \tag{3}$$

where

$$D_i = \det \begin{pmatrix} s_0 & \dots & s_{i-1} & s_{i+1} \\ \vdots & & \vdots & \vdots \\ s_i & \dots & s_{2i-1} & s_{2i+1} \end{pmatrix}, \quad \Delta_{-1} = 1, D_{-1} = 0, D_0 = s_1.$$

Also, as follows from the Hamburger theorem, the condition (2) plus the normalization condition $s_0 = 0$ are sufficient for a sequence $S = \{s_k\}_{k=0}^{\infty}, s_k \in \mathbb{R}$ to be the moment sequence of the Weyl function of a certain Jacobi operator $\mathcal{J}$. It is generaly accepted that each inverse problem method should include two parts: solution algorithm and solvability criterion in terms of the inverse problem data. As we see, the inverse spectral problem method for Jacobi operators $\mathcal{J}$ can be completely described in terms of the moment sequence $S$ of their Weyl matrices.

The band operators, by which we mean the ones generated by infinite matrices with a finite number of non-zero diagonals (band matrices) may be regarded as a natural generalization of Jacobi operators. The range of their application is close to the one of the operators $\mathcal{J}$ [10, 11, 3, 4, 19]. It turns out that if the corresponding band matrix contains mort than three nonzero diagonals, then one function $(\mathcal{R}_z e_0, e_0)$ is not enough for the reconstruction of the band operator, so one should take the set of functions defined in a similar way by using its resolvent and the vectors of $\{e_k\}$. This set is forming the Weyl matrix of the band operator, and the size of such matrix depends on the number of nonzero diagonals of the generating band matrix (more details are in the next section). As in the case of operators $\mathcal{J}$, using the asymptotic expansion of the resolvent of the band operator at infinity, we can define the (vector or matrix) moment sequence of this Weyl matrix and, as in the case of Jacobi operators, give a description of the inverse problem method for the band operators in terms of such moment sequence [27, 13, 17, 19].

Block band operators, i. e. the ones generated by infinite band matrices with operator (or matrix) entries represent a further generalization of Jacobi operators. Their spectral theory is much less developed than that of the latter ones. As to the inverse spectral problem method for such operators, partly because of non-commutativity of the block matrix entries, both inverse problem algorithm and solvability criterion for the "scalar" band operators have to be revised to fit this case. The latter was done in [15, 18]. The inverse spectral problem algorithms for some particular classes of block band operators were offered in [23, 14, 18, 21]. In this paper (Section 3), we present an algorithm for another family of block band operators (we introduce them in Section 2), which is close, but different from the one contained in [21]. The latter algorithm provides a link between the block band operators and the vector continued fractions with operator coefficients. It can be argued that this link has its roots in the theory of Jacobi operators. Namely, a known result of their theory is that the Weyl function of $\mathcal{J}$ admits an expansion into the following infinite continued fraction [24] called for short the $J-$ fraction:

$$m(z) \sim \cfrac{1}{z - b_0 - \cfrac{a_0^2}{z - b_1 - \cfrac{a_1^2}{\cfrac{\ddots}{z - b_n - \cfrac{a_n^2}{\ddots}}}}}. \tag{4}$$

In view of the above, we can conclude that solving the inverse spectral problem for $\mathcal{J}$ is equivalent to getting an algorithm for expansion of $m(z)$ into such $J-$ fraction and, since $S$ is the moment sequence of $m(z)$, this can be done by applying (3). Then, as shown in [10, 11], for one class of band operators, their Weyl matrix (which in that case contains just one row) admits an expansion into a vector continued fraction of the certain structure (somewhat similar to that of the above $J-$ fraction). Also, the inverse spectral problem algorithm for such band operators can be regarded as a modified version of the Jacobi-Perron algorithm (see Section 4) applied to a system of functions holomorphic at infinity [12]. Thus, formula (4) admits a vector generalization. These results of [10, 11] were taken into account in [21], where it was shown that they admit a generalization to the case of block band operators. The second goal of our paper is to show a similar connection between the block band operators introduced in Section 2 and another family of vector continued fractions with operator coefficients than the one considered in [21], and to show connection between the algorithm of Section 3 and the Jacobi-Perron algorithm.

## 2 Block band operators

Here we provide the details about the block band operators needed for our further considerations, see also [15, 18].

Let $H$ be a separable Hilbert space, $r, q \in \mathbb{N}$. Consider an infinite block matrix

$$A = (A_{i,j})_{i,j=0}^{\infty} = \begin{pmatrix} A_{0,0} & \dots & A_{0,r} & O & O & \dots & \dots & \dots \\ A_{1,0} & A_{1,1} & \dots & A_{1,r+1} & O & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & \\ A_{q,0} & A_{q,1} & A_{q,2} & \dots & \dots & A_{q,q+r} & O & \dots \\ O & A_{q+1,1} & A_{q+1,2} & A_{q+1,3} & \dots & \dots & A_{q+1,q+r+1} & \dots \\ O & O & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$A_{i,j} \in L(H), \quad A_{i,i+r},\ A_{i+q,i} \quad \text{are invertible,} \quad O \ \text{- zero operator.} \tag{5}$$

We will assume that $A_{i,i+r} = E$. As we see, the matrix $A$ contains $q+r+1$ nontrivial (i. e. not necessarily zero) diagonals ("bands").

The matrix $A$ generates a linear operator in the Hilbert space $l^2(H)$ of sequences $u = (u_0, u_1, \dots)$, $u_i \in H$ with inner product $(u, v) := \sum_{i=0}^{\infty}(v_i, u_i)_H$, where it is defined on the vectors $u$ via matrix multiplication. Assume that

$$\sup_{i,j\geq 0} \|A_{i,j}\|_{L(H)} \leq C < \infty,$$

then this operator is bounded; we denote it by $\mathcal{A}$. Let $\mathcal{R}(z) := (z\mathcal{I} - \mathcal{J})^{-1}$ be the resolvent of $\mathcal{A}$, where $\mathcal{I}$ is the identity operator in $l^2(H)$, $z \in \mathbb{C}$. On the resolvent set of $\mathcal{A}$, the operator $\mathcal{R}(z)$ has the block matrix representation $R := (R_{i,j})_{i,j=0}^{\infty}$, $R_{i,j} = R_{i,j}(z) \in L(H)$, see [16] for more details (including the description of the elements $R_{i,j}$). Next, the Weyl matrix of $\mathcal{A}$ is defined as follows:

$$\mathfrak{M} = \mathfrak{M}(\mathcal{A}, z) = \begin{pmatrix} \mathfrak{M}^{1,1} & \dots & \mathfrak{M}^{1,q} \\ \vdots & & \vdots \\ \mathfrak{M}^{r,1} & \dots & \mathfrak{M}^{r,q} \end{pmatrix} := \begin{pmatrix} R_{0,0} & \dots & R_{0,q-1} \\ \vdots & & \vdots \\ R_{r-1,0} & \dots & R_{r-1,q-1} \end{pmatrix} \tag{6}$$

Turning back to Jacobi operators $\mathcal{J}$, we can take the matrix representation $R$ for their resolvent and find from (1) that $m(z) = R_{0,0}$. Thus, we can conclude that the above Weyl matrix (6) can be regarded as a generalization of the Weyl function $m(z)$ to the case of block band operators.

From now on, we will assume that $q = 1$, (the case $r = 1$ was considered in [14, 21]), so $\mathfrak{M}(\mathcal{A}, z)$ is a column matrix of size $r \times 1$.

As follows from the Neumann series representation for the resolvent of $\mathcal{A}$:

$$\mathcal{R}(z) = \sum_{k=0}^{\infty} \frac{\mathcal{A}^k}{z^{k+1}}, \tag{7}$$

valid for $|z| > r(\mathcal{A})$ (the spectral radius of $\mathcal{A}$) and (6), for such $z$ the elements of the Weyl matrix of $\mathcal{A}$ can be written as follows

$$\mathfrak{M}^{i,1} = \sum_{k=0}^{\infty} \frac{S_k^i}{z^{k+1}}, \quad \text{where} \quad S_k^i = (A^k)_{i-1,0}, \quad i = 1, \dots, r. \tag{8}$$

(here $(A^k)_{i-1,0}$ means the element of the matrix $(A^k)$ with indices $_{i-1,0}$). Thus, $\mathfrak{M}^{i,1}$ are operator-valued functions holomorthic at infinity. The sequence $S = S(\mathfrak{M}, \mathcal{A}) = (S_k)_{k=0}^{\infty}$, $S_k := \begin{pmatrix} S_k^1 \\ \vdots \\ S_k^r \end{pmatrix}$ is called the moment sequence of $\mathfrak{M}$ [15]; it was also shown in [15] that the inverse spectral problem for $\mathcal{A}$ (with an arbitrary $q$) admits the following formalization: given $S(\mathfrak{M}, \mathcal{A})$, find the elements of $A$ and its solution procedure (without the detailed algorithm) was presented. As to the algorithm itself, for the scalar operators $\mathcal{A}$ it can be found e. g. in [17]. For the case of $r = 1$ it is contained in [21]. The general solvability criterion for the block band operators is contained in [15], see also [18]. Here we give its version for our case $q = 1$.

**Theorem 1.** *The sequence* $S := \begin{pmatrix} S_k^1 \\ \vdots \\ S_k^r \end{pmatrix}_{k=0}^{\infty}$, $S_k \in L(H)$ *is the moment sequence of the Weyl matrix of operator $\mathcal{A}$, which admits a matrix representation $A$ (5) with $q = 1$ (and respectively the series $\Phi_0^n(z)$ are the formal power expansions for the elements of the Weyl matrix of $\mathcal{A}$ at infinity) iff the following conditions are satisfied*

1. $S_0^i = \delta_{i,1} E, \quad i = 1, \ldots, r$ *- normalization condition.*
2. *For every $k \in \mathbb{Z}_+$ the operator $\Delta_k(S)$, generated by the block matrix $(\alpha_{i,j})_{i,j=0}^k$ such that*
$$\alpha_{i,j} = S_{i+j}^{n_1}, \quad n_1 = (j \bmod r) + 1; \tag{9}$$
*and acting in the Hilbert space $H^k := \{(u_i)_{i=0}^k,\ u_i \in H\}$ is invertible;*

If the matrix $A$ (5) satisfies additionally the condition

$$A_{i,k} = O \quad \text{for} \quad i \geq 0, \quad k = \max\{0, i - q + 1\}, \ldots, i + r - 1, \tag{10}$$

then we say that the corresponding block band operator $\mathcal{A}$ has a sparse structure (or it is a sparse block band operator). For $r = 1$ or $q = 1$, the sparse block band operators are applicable through the Lax pair formalism to integration of non-Abelian Bogoyavlensky lattices, see [14] for the details. In both such cases, the sparsity criterion for $\mathcal{A}$ can be easily formulated in terms of $S(\mathfrak{M}, \mathcal{A})$. In particular, for $q = 1$ the condition (10) is equivalent to

$$S_k^n = O, \quad n = 1, \ldots, r, \qquad k \neq n - 1 + (r + 1)k'. \tag{11}$$

Using the Theorem 1, we can conclude, that if the sequence $S$ satisfies the conditions of this theorem plus the condition (11), this is the moment sequence of the Weyl matrix of a sparse operator $\mathcal{A}$ for the case $q = 1$ and, conversely, each such sparse block band operator $\mathcal{A}$ has the moment sequence of its Weyl matrix satisfying the conditions of Theorem 1 and (11).

Note that in the general case, getting a sparsity criterion for the operators $\mathcal{A}$ remains an unclosed issue (though for the scalar operators $\mathcal{A}$ it is contained in [17]).

## 3 Inverse problem algorithm

In this subsection, we consider in detail an algorithm for reconstruction of the elements of $A$ from $S(\mathfrak{M}, \mathcal{A})$; for $r = 1$ in the matrix case such algorithm is contained in [18], but for an arbitrary $r$ the situation becomes more complicated.

The key role in this algorithm is played by the matrix $A^{(m)} = (A_{i+m,j+m})_{i,j=0}^{\infty}$, $m \in \mathbb{Z}_+$, which is obtained from $A = A^{(0)}$ by removing first $m$ rows and columns of the latter. For each $m$ it has the same structure as $A$ and therefore all of the above findings are applicable to $A^{(m)}$, in particular, we can define its Weyl matrix and, similarly to (8), the moment sequence $S^m(\mathfrak{M},\mathcal{A})$ of $A^{(m)}$:

$$S^m(\mathfrak{M},\mathcal{A}) = \begin{pmatrix} S_k^{1,m} \\ \vdots \\ S_k^{r,m} \end{pmatrix}_{k=0}^{\infty}, \ (S(\mathfrak{M},\mathcal{A}) = S^0(\mathfrak{M},\mathcal{A})),\ S_k^{n,m} = (A^{(m)\,k})_{n-1,0}, \quad (12)$$

where $n = 1,\dots,r$. Due to the same structure of the matrices $A$ and $A^m$, the sequence $S^m(\mathfrak{M},\mathcal{A})$ satisfies the conditions of Theorem 1 for each $m$. Obviously, $A_{m,m} = S_1^{1,m}$, $A_{m+1,m} = S_1^{2,m}$ (for $r > 1$) and the operators $A_{m,m+n}$ (all of them, together with zero operators, form the first row of $A^{(m)}$) can also be expressed via the elements of $S^m(\mathfrak{M},\mathcal{A})$ (we provide the corresponding formulas below). Hence, the procedure of getting $S^{m+1}(\mathfrak{M},\mathcal{A})$ out of $S^m(\mathfrak{M},\mathcal{A})$ completes the whole algorithm and obtaining such a procedure is our nearest task.

**Lemma 1.** *For the elements $S_k^{n,m}$ the following relations are hold*

$$S_{k+1}^{n,m} = \sum_{p=0}^{k-1} S_{p+1}^{n-1,m+1} A_{m+1,m} S_{k-p-1}^{1,m} + S_1^{n,m} S_k^{1,m}, \quad n = 2,\dots,r; \quad (13)$$

$$S_{k+1}^{1,m} = \sum_{p=0}^{k-1}\sum_{i=1}^{r} A_{m,m+i} S_p^{n,m+1} A_{m+1,m} S_{k-p-1}^{1,m} + S_1^{1,m} S_k^{1,m}; \quad k \ge 1. \quad (14)$$

*Proof.* For $S_{k+1}^{n,m}$, $n = 1,\dots,r;\ m \ge 0$, we have by their definition

$$S_{k+1}^{n,m} = (A^{(m)\,k+1})_{n-1,0} = \sum_{I_{k+1}} (A^{(m)})_{n-1,i_1}(A^{(m)})_{i_1 i_2}\cdots(A^{(m)})_{i_{k-1}i_k}(A^{(m)})_{i_k,i_{k+1}}, \quad (15)$$

where $I_{k+1} := (i_1,\dots,i_{k+1})$ are sets of nonnegative indices such that

1. $i_{k+1} = 0$;
2. $i_1,\dots,i_k \ge 0$;
3. $|i_k - i_{k-1}| \le r$,

wherein (since the matrix $A^{(m)}$ has the same structure as $A$)

$$(A^{(m)})_{i,j} = O \quad \text{for} \quad i - j > 1 \quad \text{or} \quad j - i > r. \quad (16)$$

First consider the case $n = 2,\dots,r$. By its definition, each set $I_{k+1}$ admits the following representation:

$$I_{k+1} = \bigcup_{l=1}^{k+1} I_{k+1,l}; \quad I_{k+1,l} = \{(i_1,\dots,i_{k+1}) \in I_{k+1},\, i_l = 0, \quad i_{l'} > 0 \text{ as } \quad 1 \le l' < l\},$$

so we have

$$S_{k+1}^{n,m} = \sum_{l=1}^{k+1}\sum_{I_{k+1,l}} (A^{(m)})_{n-1,i_1}(A^{(m)})_{i_1,i_2}\cdots(A^{(m)})_{i_{l-1},0}(A^{(m)})_{0,i_{l+1}}\cdots(A^{(m)})_{i_k,i_{k+1}}$$

$$
\begin{aligned}
&= \sum_{l=2}^{k+1}\sum_{I_{k+1,p}} (A^{(m)})_{n-1,i_1}(A^{(m)})_{i_1,i_2}\cdots(A^{(m)})_{i_{l-1},0}(A^{(m)})_{0,i_{l+1}}\cdots(A^{(m)})_{i_k,i_{k+1}} + \\
&+ \sum_{I_{k+1,1}} (A^{(m)})_{n-1,0}(A^{(m)})_{0,i_2}\cdots(A^{(m)})_{i_k,0} = \\
&= \sum_{l=2}^{k+1}\sum_{I_{k+1,l}} (A^{(m)})_{n-1,i_1}(A^{(m)})_{i_1,i_2}\cdots(A^{(m)})_{i_{l-2},i_{l-1}}(A^{(m)})_{1,0} \times \\
&\times (A^{(m)})_{0,i_{l+1}}\cdots(A^{(m)})_{i_k,i_{k+1}} + (A^{(m)})_{n-1,0}\sum_{I_k}(A^{(m)})_{0,i_1}\cdots(A^{(m)})_{i_{k-1},0}
\end{aligned}
$$

(here we used that $i_{l-1} = 1$). Next, by the definition of $(A^{(m+1)})$, we consecutively find

$$
\begin{aligned}
S_{k+1}^{n,m} &= \\
&= \sum_{l=2}^{k+1}\sum_{I_{k+1,l}} (A^{(m+1)})_{n-2,i_1-1}(A^{(m+1)})_{i_1-1,i_2-1}\cdots(A^{(m+1)})_{i_{l-2}-1,0}A_{m+1,m} \times \\
&\times (A^{(m)})_{0,i_{l+1}}\cdots(A^{(m)})_{i_k,i_{k+1}} + S_1^{n,m}S_k^{1,m}.
\end{aligned}
$$

Thus, by setting $p = l-2$ and again using (15), we find that the latter can be rewritten as (13).

For $n = 1$, acting similarly as above, we get

$$
\begin{aligned}
S_{k+1}^{1,m} &= \sum_{l=2}^{k+1}\sum_{I_{k+1,l}} (A^{(m)})_{0,i_1}\cdots(A^{(m)})_{i_{l-2},i_{l-1}}A_{m+1,m}(A^{(m)})_{i_{l+1},i_{l+2}}\cdots(A^{(m)})_{i_k,0} + \\
&+ S_1^{1,m}S_k^{1,m}.
\end{aligned}
$$

Each set $I_{k+1,l}$ can be, in turn, represented as:

$$
I_{k+1,l} = \bigcup_{i=1}^{r} I_{k+1,l,i}; \quad I_{k+1,l,i} := \{(i_1,\ldots,i_{k+1}) \in I_{k+1,l},\, i_1 = i\},
$$

and again, acting as in the case $n = 2,\ldots,r$, we find

$$
\begin{aligned}
S_{k+1}^{1,m} &= \\
&= \sum_{l=2}^{k+1}\sum_{i=1}^{r}\sum_{I_{k+1,l,i}} (A^{(m)})_{0,i}(A^{(m)})_{i,i_2}\cdots(A^{(m)})_{i_{l-2},i_{l-1}}A_{m+1,m} \times \\
&\times (A^{(m)})_{i_{l+1},i_{l+2}}\cdots(A^{(m)})_{i_k,0} + S_1^{1,m}S_k^{n,m} =
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=2}^{k+1}\sum_{i=1}^{r}\sum_{I_{k+1,l,i}} (A^{(m)})_{0,i}(A^{(m+1)})_{i-1,i_2-1}\cdots(A^{(m+1)})_{i_{l-2}-1,i_{l-1}-1}A_{m+1,m} \times \\
&\times (A^{(m)})_{i_l,i_{l+1}}\cdots(A^{(m)})_{i_k,0} + S_1^{1,m}S_k^{n,m} = \\
&= \sum_{p=0}^{k-1}\sum_{i=1}^{r} A_{m,m+i}S_p^{i,m+1}A_{m+1,m}S_{k-p-1}^{1,m} + S_1^{1,m}S_k^{1,m}.
\end{aligned}
$$

□

Next, for $k \in \mathbb{Z}_+$, we introduce the following operator-valued quantities

$$\begin{aligned} D_k^1(S^m) &= \sum_{i_1+\cdots+i_l=k+1} (-1)^{l+1} S_{i_1}^{1,m} \cdots S_{i_l}^{1,m}, & (17)\\ D_k^n(S^m) &= S_{k+1}^{n,m} + \sum_{p=0}^{k-1} S_{p+1}^{n,m} \sum_{i_1+\cdots+i_l=k-p} (-1)^l S_{i_1}^{1,m} \cdots S_{i_l}^{1,m} = & (18)\\ &= S_{k+1}^{n,m} - \sum_{p=0}^{k-1} S_{p+1}^{n,m} D_{k-p-1}^1(S^m), \quad n = 2, \ldots, r, \\ & i_1, \ldots, i_l \geq 1, \qquad (D_0^n(S^m) = S_1^{n,m}); \end{aligned}$$

and the summation is over all $l$-tuples $(i_1, \ldots, i_l)$ satisfying the conditions indicated in (17)-(18). Note that using (12), one can check that

$$D_{n-2}^n(S^m) = S_{n-1}^{n,m} = A_{n-1+m,n-2+m} A_{n-2+m,n-3+m} \cdots A_{m+1,m}, \; n = 2, \ldots, r. \quad (19)$$

From their definition one easily derives the following result.

**Lemma 2.** *The operators $S_{k+1}^{1,m}, \ldots, S_{k+1}^{q,m}$ satisfy the relations:*

$$S_{k+1}^{1,m} = \sum_{p=0}^{k} S_p^{1,m} D_{k-p}^1(S^m) = \sum_{p=0}^{k} D_{k-p}^1(S^m) S_p^{1,m} = \sum_{p=0}^{k-1} S_p^{1,m} D_{k-p}^1(S^m) + S_1^{1,m} S_k^{1,m}, \quad (20)$$

$$S_{k+1}^{n,m} = \sum_{p=0}^{k} D_{k-p}^n(S^m) S_p^{1,m} = \sum_{p=0}^{k-1} D_{k-p}^n(S^m) S_p^{1,m} + S_1^{n,m} S_k^{1,m}. \quad (21)$$

Comparing (21) with (13), we obtain that the quantities $D_k^n(S^m)$ and $S_k^{n-1,m+1} A_{m+1,m}$ satisfy the same recurrent relations. As follows from (18), $D_0^n(S^m) = S_1^{n,m} = \delta_{n,2} A_{m+1,m}$. At the same time, it follows from (12), that $S_0^{i,m} = \delta_{i,1} E$. Hence for $k = 0$, $D_k^n(S^m)$ coincides with $S_k^{n-1,m+1}$, and we find that

$$S_k^{n-1,m+1} = D_k^n(S^m) A_{m+1,m}^{-1}, \quad k \in \mathbb{Z}_+, \quad n = 2, \ldots, r. \quad (22)$$

From the definition of $S^m(\mathfrak{M}, \mathcal{A})$ also follows that

$$A_{m+1,m} = S_1^{2,m} \quad \text{for} \quad r > 1, \quad (23)$$

so the above formula can be written entirely in terms of this moment sequence:

$$S_k^{n-1,m+1} = D_k^n(S^m)(S_1^{2,m})^{-1}. \quad (24)$$

Besides, from the definition of $S_k^{1,m}$ in (12) obviously follows that

$$A_{m,m} = S_1^{1,m}. \quad (25)$$

The remaining nontrivial (e. g. not necessarily zero) elements of initial row of $A^{(m)}$ can be found recurrently. Namely, similarly as above, comparing (20) with (14), we find that

$$\sum_{i=1}^{r} A_{m,m+i} S_k^{i,m+1} A_{m+1,m} = D_{k+1}^1(S^m), \quad k \in \mathbb{Z}_+; \qquad (A_{m,m+r} = E). \quad (26)$$

As follows from (12), $S_0^{i,m+1} = \delta_{1,i}E$. By setting $k = 0$ in (26) we find

$$A_{m,m+1} = D_1^1(S^m)(A_{m+1,m})^{-1}; \tag{27}$$

and, by setting $k = 1, \ldots, r-2$ we get, using (22) and (19)

$$A_{m,m+i} = (D_i^1(S^m) - \sum_{j=1}^{i-1} A_{m,m+j} D_{i-1}^{j+1}(S^m))(D_{i-1}^{i+1}(S^m))^{-1} =$$

$$= (D_i^1(S^m) - \sum_{j=1}^{i-1} A_{m,m+j} D_{i-1}^{j+1}(S^m))(S_i^{i+1,m})^{-1}, \quad i = 2, \ldots, r-1; \tag{28}$$

(note that, as follows from (19), $S_i^{i+1,m}$ are invertible operators). It remains to express the elements $S_k^{r,m+1}$ in terms of $S^m(\mathfrak{M}, \mathcal{B})$. In doing so, since $A_{m,m+i}$ are now known, we turn back to (26):

$$S_k^{r,m+1} = D_{k+1}^1(S^m)A_{m+1,m}^{-1} - \sum_{n=1}^{r-1} A_{m,m+n} S_k^{n,m+1}. \tag{29}$$

Thus we arrive at the following theorem.

**Theorem 2.** *The entries of matrix $A$ (5) for $q = 1$, $r > 1$ can be consecutively reconstructed from the moment sequence $S(\mathfrak{M}, \mathcal{A})$ via the elements $S^m(\mathfrak{M}, \mathcal{A})$ of the moment sequence corresponding to the shifted matrix $A^{(m)} := (A_{i+m,j+m})_{i,j=0}^{\infty}$ according to the following algorithm:*

*(i) Compose $D_k^n(S^m)$, $n = 1, \ldots, r$ by using (17)-(18);*

*(ii) Using (24), find the elements of $S^{m+1}(\mathfrak{M}, \mathcal{B}) : S_k^{n,m+1}$ for $k \in \mathbb{Z}_+$, $n = 1, \ldots, r-1$;*

*(iii) From (23), (25),(27)-(28), find the operators $A_{m+1,m}$, $A_{m,m+i}$, $i = 0, \ldots, r-1$, which are the elements of matrix $A^{(m)}$, but not of $A^{(m+1)}$;*

*(iv) Using (29), find the remaining elements of $S^{m+1}(\mathfrak{M}, \mathcal{A})$, namely, $S_k^{r,m+1}$.*

If $r = 1$, i. e. the matrix $A$ is tridiagonal, then this algorithm has to be slightly modified. Namely, first we calculate $D_k^n(S^m)$ according to (17). The elements $A_{m+1,m}$ in this case can be found from the relations:

$$A_{m+1,m} = S_2^{1,m} - S_1^{1,m} S_1^{1,m} = D_1^1(S^m),$$

(cf. with (23)), and $A_{m,m}$, as above, satisfy (25). Finally, for the elements $S_k^{r,m+1}$, the above formula (29) turns into

$$S_k^{1,m+1} = D_{k+1}^1(S^m)A_{m+1,m}^{-1} = D_{k+1}^1(S^m)(D_1^1(S^m))^{-1}, \ k \in \mathbb{Z}_+.$$

Such modification coincides with a previously known inverse spectral problem algorithm for the three diagonal block band operators, which correspond to our case $q = r = 1$, see [18] and references thereafter, so Theorem 2 may be regarded as its generalization to the case of arbitrary $r$.

As previously mentioned, the sparse matrices $A$, i. e. the ones which additionally satisfy (10) (in our case, for $q = 1$) are important in the application context. Turning back to the algorithm described in Theorem 2, and taking (11) into account, we obtain its version for the sparse block band operators:

**Corollary 1.** *If the condition* (10) *is fulfilled for* $q = 1$, *then the reconstruction algorithm for* $\mathcal{A}$ *is as follows:*

*(i')-(ii') Same as the steps (i) - (ii) of the Theorem 2. In doing so,we take into account* (11).

*(iii') Calculate the elements* $S_k^{r,m+1}$ :

$$S_k^{r,m+1} = D_{k+1}^1(S^m)A_{m+1,m}^{-1}, \quad k \in \mathbb{Z}_+;$$
$$\textit{if} \quad r = 1, \quad \textit{then} \quad A_{m+1,m} = D_1^1(S^m) = S_2^{1,m}.$$

## 4 Vector continued fractions with operator coefficients

First, consider the set $\mathbb{C}(\frac{1}{z}, L(H))$ of formal power series:

$$a = a(z) = \sum_{k=m}^{\infty} \frac{A_k}{z^k}, \quad m \in \mathbb{Z};\ z \in \mathbb{C},\ A_k \in L(H); \quad A_m \text{ is invertible} \tag{30}$$

Define the sum and the product of the elements of $\mathbb{C}(\frac{1}{z}, L(H))$:

$$(a+b)(z) = \sum_{k=l}^{\infty} \frac{A_k + B_k}{z^k}, \quad l = min(m, m_1); \quad (ab)(z) = \sum_{k=m+m_1}^{\infty} \sum_{l=m}^{k-m_1} \frac{A_l B_{k-l}}{z^k},$$

where $b = \sum\limits_{k=m_1}^{\infty} \frac{B_k}{z^k}$. Since $A_m$ is invertible, it can be directly checked that for each $a \in \mathbb{C}(\frac{1}{z}, L(H))$ $a \neq O$ there exist the inverse element $a^{-1}(z) = b$ such that

$$ba = E, \quad ab = E. \tag{31}$$

We also denote $a^{-1}(z)$ as $\dfrac{E}{a(z)}$. Next, for the element $a(z)$ we have

$$a(z) = (a^{-1}(z))^{-1} = \cfrac{E}{\cfrac{E}{a(z)}} = \frac{E}{a^{-1}(z)},$$

where

$$a^{-1}(z) = \sum_{k=-m}^{\infty} \frac{\tilde{A}_k}{z^k}, \quad \tilde{A}_{-m} = A_m^{-1}.$$

Suppose that $m \geq 0$ and for some $l,\ -m < l \leq 1$, the operator $\tilde{A}_l$ is invertible. Then the series $a^{-1}(z)$ admits the following representation:

$$a^{-1}(z) = \sum_{k=-m}^{l-1} \frac{\tilde{A}_k}{z^k} + \sum_{k=l}^{\infty} \frac{\tilde{A}_k \tilde{A}_l^{-1}}{z^k} \tilde{A}_l := p_0(z) + a_1(z)\tilde{A}_l = p_0(z) + \frac{E}{b_1(z)} \tilde{A}_l, \tag{32}$$

where $a_1(z), b_1(z) \in \mathbb{C}(\frac{1}{z}, L(H))$, $b_1(z) = a_1^{-1}(z)$ and $p_0(z)$ is an operator polynomial in $z : p_0(z) \in P(z, L(H))$. For convenience, denote $\tilde{A}_l$ by $B_0$ Continuing the same process with $b_1(z)$ under the same assumptions, one may eventually obtain the finite or infinite functional continued fraction:

$$a(z) \sim \cfrac{E}{p_0(z) + \cfrac{E}{p_1(z) + \cfrac{E}{\ddots} B_1} B_0}, \tag{33}$$

where $B_k$ are invertible operators and $p_k(z)$ are operator polynomials in $z$. Then, for $a_1(z), a_2(z) \in \mathbb{C}(\frac{1}{z}, L(H))$ we set [1]

$$\frac{a_2(z)}{a_1(z)} := a_2(z)(a_1(z))^{-1}. \tag{34}$$

Now we turn to the vector operator continued fractions. In doing so, we first consider the set of vectors (systems) $(a^1(z), \ldots, a^r(z))$, $r > 1$ of the elements of $\mathbb{C}(\frac{1}{z}, L(H))$ which we denote by $\mathbb{C}^r(\frac{1}{z}, L(H))$; the set $(p^1(z), \ldots, p^r(z)), p^i(z) \in P(z, L(H))$ we denote by $P^r(z, L(H))$.

As in the numeric case [12, 22, 10], for a vectors $G = G(z) = (g^1, \ldots, g^r)$ and $H = (h^1, \ldots, h^r)$, from $\mathbb{C}^r(\frac{1}{z}, L(H))$ we define their sum and product componentwise, i. e.,

$$G + H = (g^1 + h^1, \ldots, g^r + h^r), \quad F \times G = (g^1h^1, \ldots, g^rh^r).$$

Also, we define the unity $\mathbf{1} := (E, \ldots, E)_{1\times r}$ divided by a vector $G$ such that $g^r \neq O$:

$$\tilde{\mathcal{T}}G := \frac{\mathbf{1}}{G} = \left(\frac{E}{g^r}, \frac{g^1}{g^r}, \ldots, \frac{g^{r-1}}{g^r}\right), \tag{35}$$

When $g^1, \ldots, g^r$ are complex functions (or complex numbers) this operation is called the Jacobi-Perron rule [8, 10], so the above formula can be regarded as its analog for $\mathbb{C}^r(\frac{1}{z}, L(H))$. From this definition follows that if $g^1 \neq O$, then

$$G^1 := \tilde{\mathcal{T}}^{-1}G = \left(\frac{g^2}{g^1}, \frac{g^3}{g^1}, \ldots, \frac{g^r}{g^1}, \frac{E}{g^1}\right), \; G = \frac{\mathbf{1}}{\left(\frac{g^2}{g^1}, \frac{g^3}{g^1}, \frac{g^r}{g^1}, \frac{E}{g^1}\right)}, \tag{36}$$

and, if $g^n \neq O$ for all $i = 1, \ldots, r$, then $\tilde{\mathcal{T}}^{r+1}G = G$. Next, in the same manner as above, we extract the polynomial parts from the components of $G^1$ :

$$G^1 = P_0 + G^2, \; G^2 \in \mathbb{C}^r(\frac{1}{z}, L(H)), \quad P_0 = (p_0^1(z), \ldots, p_0^r(z)) \in P^r(z, L(H)).$$

Then, if the first component of $G^2$ is not zero, we can apply to $G^2$ the transformation (36). Similarly to (33), by repeating this process as long as possible (we provide more details below), one may obtain a finite or infinite vector continued fraction of $z$ with operator coefficients:

$$G(z) \sim \frac{\mathbf{1}\rfloor}{\lceil P_0} + \frac{\mathbf{1} \times \hat{B}_0\rfloor}{\lceil P_1} + \cdots + \frac{\mathbf{1} \times \hat{B}_{k-1}\rfloor}{\lceil P_k} + \ldots, \tag{37}$$
$$\hat{B}_k = (B_k^1, \ldots, B_k^r), \quad k \geq 0, B_k^i \in L(H), (B_k^i)^{-1}\text{exists}, \quad P_k \in P^r(z, L(H)).$$

In the numeric case, a similar procedure is called Jacobi-Perron algorithm [8, 22, 12], so we can conclude that the latter is extendable to the $\mathbb{C}^r(\frac{1}{z}, L(H))$ case. Thus, a natural question arises: for which systems $G$ an expansion into the infinite vector continued fraction, which has the same structure as (37) is take place? It turns out that the theory of block band operators is useful for getting an answer. To see this, consider the system

$$F = F(z) := (f^1, \ldots, f^r) = \left(\sum_{k=0}^{\infty} \frac{S_k^1}{z^{k+1}}, \ldots, \sum_{k=0}^{\infty} \frac{S_k^r}{z^{k+1}}\right), \; S_k^i \in L(H), \tag{38}$$

[1] In [20, 21], we used this notation for the left multiplication by $a_1^{-1}$, namely, $\frac{a2}{a1} = a_1^{-1}a_2$

such that the sequence $S := (S_k^1, \dots, S_k^r)_{k=0}^\infty$ satisfies the conditions of Theorem 1. Then, according to this theorem, the transposed sequence $S^T := \begin{pmatrix} S_k^1 \\ \vdots \\ S_k^r \end{pmatrix}_{k=0}^\infty$ is the moment sequence of the Weyl matrix of a certain operator $\mathcal{A}$ and, by the Theorem 2, the elements of the corresponding matrix $A$ can be reconstructed from $S = S(\mathfrak{M}, \mathcal{A})$ by applying the algorithm indicated by this theorem. By taking these facts into account, we can establish a relation between the systems $F(z)$ and the continued fractions presented in (37). Namely, the following result holds.

**Theorem 3.** *The system $F(z)$ defined in* (38) *can be expanded into the infinite vector continued fraction of the form:*

$$\begin{aligned} F(z) \sim K_r \quad &:= \frac{\mathbf{1}}{\lfloor (O, \dots, O, zE - A_{0,0})} + \frac{\mathbf{1} \times (A_{1,0}, \dots, A_{1,0})}{\lfloor (O, \dots, O, -A_{0,1}, zE - A_{1,1})} + \dots \\ &+ \frac{\mathbf{1} \times (A_{r-1,r-2}, \dots, A_{r-1,r-2})}{\lfloor (-A_{0,r-1}, \dots, -A_{r-2,r-1}, zE - A_{r-1,r-1})} + \\ &+ \frac{\mathbf{1} \times (A_{l,l-1}, \dots, A_{l,l-1})}{\lfloor (-A_{l-r+1,l}, \dots, -A_{l-1,l}, zE - A_{l,l})} + \dots, \quad l \geq r, \end{aligned} \tag{39}$$

*with the elements from $L(H)$, $A_{k+1,k}, k \in \mathbb{Z}_+$ are invertible, if and only if the coefficients $S_k^i$* (38) *satisfy the conditions of the Theorem 1. The elements of this continued fraction can be found from $S$ by using the algorithm described by the Theorem 2.*

*Proof.* According to (36),

$$F = \frac{\mathbf{1}}{\left( \frac{f^2}{f^1}, \frac{f^3}{f^1}, \dots, \frac{f^q}{f^1}, \frac{E}{f^1} \right)} \tag{40}$$

First, take the series $f^1 = f^1(z) = \sum_{k=0}^{\infty} \frac{S_k^1}{z^{k+1}}$. We have $S_0^1 = E$ and the coefficients of

$$\frac{E}{f^1} = \sum_{k=-1}^{\infty} \frac{\tilde{S}_k}{z^k}$$

satisfy the relations

$$\sum_{j=0}^{k} \tilde{S}_{k-1-j}^1 S_j^1 = \delta_{k0} E, \qquad k \in \mathbb{Z}_+,$$

from which we get for the first two coefficients, which form the polynomial (linear) part of $\frac{E}{f^1}$

$$\tilde{S}_{-1}^1 = S_0^1 = E, \quad \tilde{S}_0^1 = -S_1^1 = -D_0^1(S^0) = -A_{0,0}, \tag{41}$$

and the remaining coefficients are found consecutively:

$$\tilde{S}_k^1 = -S_{k+1}^1 + \sum_{p=1}^{k} D_{k-p}^1(S^0) S_p^1, \quad k \geq 1,$$

Comparing the latter formula with (20) and taking (41) into account we obtain that

$$\tilde{S}^1_k = -D^1_k(S^0), \quad k \geq 0.$$

and

$$\frac{E}{f^1} = zS^1_0 - S^1_1 - \sum_{k=1}^{\infty} \frac{D^1_k(S^0)}{z^k} = zE - A_{0,0} - \sum_{k=1}^{\infty} \frac{D^1_k(S^0)}{z^k}, \tag{42}$$

and for $n = 2, \ldots, r$, using (18) and (22), one gets

$$\frac{f^n}{f^1} = \sum_{k=0}^{\infty} \frac{S^n_{k+1} - \sum_{p=0}^{k-1} S^n_{p+1} D^1_{k-p-1}(S^0)}{z^{k+1}} = \sum_{k=0}^{\infty} \frac{D^n_k(S^0)}{z^{k+1}} = \sum_{k=0}^{\infty} \frac{S^{n-1,1}_k}{z^{k+1}} A_{1,0}, \tag{43}$$

For $m \in \mathbb{Z}_+$, denote by $f^{i,m}$ the series $\sum_{k=0}^{\infty} \frac{S^{i,m}_k}{z^{k+1}}$ $(f^{i,0} = f^i)$. Using (26) , we find for (42)

$$\frac{E}{f^1} = zE - A_{0,0} - \sum_{i=1}^{r} A_{0,i} f^{i,1} A_{1,0}. \tag{44}$$

Hence, (40) can be written as follows

$$\begin{aligned} F &= \frac{\mathbf{1}}{\left(f^{1,1}A_{1,0}, \ldots, f^{r-1,1}A_{1,0}, zE - A_{0,0} - \sum_{i=1}^{r} A_{0,i} f^{i,1} A_{1,0}\right)} = \\ &= \frac{\mathbf{1}}{(O, \ldots, O, zE - A_{0,0}) + \tilde{F}^1 \times (A_{1,0}, \ldots, A_{1,0})}, \end{aligned}$$

where $\tilde{F}^1 = (f^{1,1}, \ldots, f^{r-1,1}, \sum_{i=1}^{r} -A_{0,i} f^{i,1})$. Applying (36) to $G = \tilde{F}^1$ and acting similarly as above, see (42)-(44), we get

$$\tilde{F}^1 = \frac{\mathbf{1}}{(O, \ldots, O, -A_{0,1},\ zE - A_{1,1}) + \tilde{F}^2 \times (A_{2,1}, \ldots, A_{2,1})},$$

where $\tilde{F}^2 = (f^{1,2}, \ldots, f^{r-2,2}, \sum_{i=2}^{r} -A_{0,i} f^{i-1,2}, \sum_{i=1}^{r} -A_{i,i+1} f^{i,2})$. By repeating the above procedure $r-2$ times, we get the following vector continued fraction

$$\begin{aligned} F = &\frac{\mathbf{1}\ |}{|\ (O, \ldots, O, zE - A_{0,0})} + \frac{\mathbf{1} \times (A_{1,0}, \ldots, A_{1,0})\ |}{|\ (O, \ldots, O, -A_{0,1},\ zE - A_{1,1})} + \cdots + \\ &+ \frac{\mathbf{1} \times (A_{r-1,r-2}, \ldots, A_{r-1,r-2})\ |}{|\ (-A_{0,r-1}, \ldots, -A_{r-2,r-1}, zE - A_{r-1,r-1}) + \tilde{F}^r \times (A_{r,r-1}, \ldots, A_{r,r-1})}, \end{aligned}$$

where

$$\tilde{F}^r = (f^{1,r}, \sum_{i=r-1}^{r} -A_{1,i+1} f^{i-r+2,r}, \ldots, \sum_{i=1}^{r} -A_{r-1,i+r-1} f^{i,r})$$

Then, for $\tilde{F}^r$ we get

$$\tilde{F}^r = \frac{\mathbf{1}}{(-A_{1,r}, \ldots, -A_{r-1,r}, zE - A_{r,r}) + \tilde{F}^{r+1} \times (A_{r+1,r}, \ldots, A_{r+1,r})},$$

where

$$\tilde{F}^{r+k} = (f^{1,r+k}, \sum_{i=r-1}^{r} -A_{1+k,i+1+k} f^{i-r+2,r+k}, \ldots, \sum_{i=1}^{r} -A_{r-1+k,i+r-1+k} f^{i,r+k}),$$

$k \geq 1$. By repeating the same procedure with $\tilde{F}^{r+k}$, $\tilde{F}^{r+k+1}, \ldots$ infinitely many times, we get the fraction $K_r$ □

It follows from the equivalence between (11) and (10) that if the coefficients $S_k^i$ of the system $F$ (38), in addition to the conditions of Theorem 1, satisfy (11), then the sequence $S^T$ is the moment sequence of the Weyl matrix of a sparse operator $\mathcal{A}$. Then, taking into account Corollary 1 from the previous section, we get the following version of the above result for this specific case.

**Corollary 2.** *If the system $F(z)$ (38) satisfies additionally* (11)*, then it can be expanded into the following vector continued fraction*

$$F(z) \sim K_r^s := \frac{\mathbf{1}\ \big|}{\big|\ (O,\dots,O,zE)} + \frac{\mathbf{1}\times(A_{1,0},\dots,A_{1,0})\ \big|}{\big|\ (O,\dots,O,zE)} + \dots + \tag{45}$$
$$+\frac{\mathbf{1}\times(A_{r-1,r-2},\dots,A_{r-1,r-2})\ \big|}{\big|\ (O,\dots,O,zE)} + \frac{\mathbf{1}\times(A_{l,l-1},\dots,A_{l,l-1})\ \big|}{\big|\ (O,\dots,O,zE)} + \dots,\ l \geq r,$$

*The elements $A_{k+1,k}$, $k \in \mathbb{Z}_+$ can be found from the sequence $S$ by means of the algorithm of Corollary 1.*

## 5 Conclusion

In view of the above, we can conclude that we found a correspondence between the systems $F(z)$ (38), the block band operators $\mathcal{A}$ and the vector continued fractions $K_r$ and $K_r^s$ defined in (39)and (45) and showed that the algorithm of Section 3 can be regarded as a modified version of Jacobi-Perron algorithm for expansion of the system $F(z)$ into the continued fraction $K_r$ (or $K_r^s$). As follows from Theorem 1 and (8), the expansions (39),(45) can be regarded as analogs of formula (4) for the Jacobi operators.

For the $J-$ fractions, the study of their convergence on the resolvent set of the corresponding Jacobi operators is a valuable tool in the spectral theory of the latter [5]. Some similar convergence results for the operator $J-$ fractions have been obtained in [20] (where the convergence is meant in the operator norm sense). In particular, it was shown that the operator $J-$ fraction converges at a geometric rate to the Weyl function of the corresponding block Jacobi operator outside the numerical range of the latter. As noted in the Introduction, the spectral theory of the band operators (not to mention the block band ones) is much less developed than that of the Jacobi operators, so it is of certain interest to study similar issues for the vector continued fractions corresponding to the block band operators (including the ones considered in [21]). For the scalar band operators some results of this kind can be found in [9]. Also, in [20], the operator $J-$ fractions were applied to solving quadratic operator equations. The results of [4], in particular, give confidence that the continued fractions $K_r$, $K_r^s$ are applicable for solving operator equations of an arbitrary order.

All these issues can be the subject of further study.

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## References

[1] N. I. Akhiezer, The Classical Moment Problem, Oliver Boyd, Edinburgh (1965).

[2] N.I. Akhiezer, I.M. Glazman, Theory of Linear Operators in Hilbert Space, Dover Publications, New. York (1993).

[3] Aptekarev, A., Kaliaguine, V., Van Iseghem, J. The genetic sums' representation for the moments of a system of Stieltjes functions and its application, Constr. Approx., vol. 16, (2000), 487524.

[4] A. I. Aptekarev, V. A. Kalyagin, E. B. Saff, Higher-order three-term recurrences and asymptotics of multiple orthogonal polynomials, Constr. Approx., vol. 30, no. 2, (2009), 175-223.

[5] B. Beckermann, On the convergence of bounded J-fractions on the resolvent set of the corresponding second order difference operator, Journal of Approx. Theory, vol. 99 (1999), 369-408.

[6] Ju. M. Berezanskij, Expansions in Egenfunctions of Selfadjoint Operators, AMS, Providence, R.I. (1968).

[7] Yu. M. Berezanski, The integration of semi-infinite Toda chain by means of inverse spectral problem. Repts. Math. Phys., vol. 24, No 1, (1986), 21-47.

[8] L. Bernstein, The Jacobi-Perron Algorithm: Its Theory and Application, Lecture Notes in Math., vol. 207, Springer-Verlag, Berlin, 1971.

[9] J. Van Iseghem, Matrix continued fractions for the resolvent function of the band operator, Acta Math. Appl., 61 (2000), 351-365.

[10] V.A. Kalyagin, Hermite-Padé approximants and spectral analysis of nonsymmetric operators, Sb. Math., vol. 82, Issue 1, (1995), 199- 216.

[11] V. Kaliaguine, Operator moment problem, vector continued fractions and an explicit form of the Favard theorem for vector orthogonal polynomial, Journ. of Comput. Appl. Math., vol. 65 (1995), 181-193.

[12] E. M. Nikishin, V. N. Sorokin, Rational Approximations and Orthogonality, Translations of Mathematical Monographs, vol. 92, AMS, Providence, R.I., 1991.

[13] Osipov A.S., On a determinant criterion of weak perfectness of systems of functions holomorthic at infinity and their connection with some classes of higher-oder difference operators. (Russian) Fundam. Prikl. Mat. vol. 1, no 3, (1995), 711-727.

[14] A. S. Osipov, Integration of non-Ableian Langmuir type lattices by the inverse spectral problem method, Funct. Anal. Appl., 31, (1997), 67-70.

[15] A. Osipov, On some issues related to the moment problem for the band matrices with operator elements, Journ. Math. An. Appl., vol. 275, issue 2, (2002), 657-675.

[16] A. Osipov, Some resolvent set properties of band operators with matrix elements, arXiv:1412.7354, 2014.

[17] Osipov A.S., Inverse spectral problem for band operators and their sparsity criterion in terms of inverse problem data. Russ. J. Math. Phys. vol. 29, (2022), 225-237.

[18] A. Osipov, Miura type transform between non-Abelian Volterra and Toda lattices and inverse spectral problem for band operators. Russ. J. Math. Phys., 30, (2023), 382-396.

[19] A. Osipov, On the links between Miura transformations of Bogoyavlensky lattices and inverse spectral problems for band operators. Armen. J. Math. vol. 16, no. 2, (2024), 1-28.

[20] A. Osipov, On some properties and applications of operator continued $J-$fractions. Russ J. Math. Phys., 31, (2024), 335-355.

[21] A. Osipov, On one class of vector continued fractions with operator elements and the Jacobi-Perron algorithm. Russ J. Math. Phys., 32, (2025), 562-582.

[22] V.I. Parusnikov, The Jacobi-Perron algorithm and simultaneous approximation of functions, Math. USSR Sb., vol. 42, (1982), 287-296.

[23] M. Shmoish, On generalized spectral functions, the parametrization of block Hankel and block Jacobi matrices, and some root location problems, Lin. Alg. Appl., vol. 202, (1994) 91-128.

[24] M. H. Stone, Linear Transformations in Hilbert Space and Their Applications to Analysis, AMS Coll. Publ., vol. 15, NY, 1932.

[25] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, Mathematical Surveys and Monographs, Vol. 72, RI: AMS, 2000.

[26] H. Wall, Analytic Teory of Continued Fractions, Chelsea, Bronx, NY, 1973.

[27] V.A . Yurko, Inverse Spectral Problems for Linear Differential Operators and Their Applications, 1st edn, Boca Raton, FL: CRC Press, 2000.