Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 3, pp. 527 - 536

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.3.005

Reduced First Zagreb Index of Graphs with Added Edges

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5th March 2026

Abstract

Adding or deleting an edge or a vertex gives a larger or smaller graph compared to the original graph and hence the calculations on the smaller graph helps us to get the information on larger graph. In network theory, adding or deleting a member affects many things within the network including the strength of it. In this paper, we compute the change in the reduced first Zagreb index of a simple graph G when an arbitrary edge is added. In particular, some statements for the change of reduced first Zagreb indices of path, cycle, star, complete, complete bipartite and tadpole graphs are obtained.

1 Introduction

[1] [2]

Throughout this paper, we consider $G = (V, E)$ as a simple graphs which are finite and undirected graphs without any loops and multiple edges. The notation $d_G(v)$ denotes the degree of a vertex $v \in V(G)$. A vertex having degree one will be called as a pendant vertex. Here the notations $P_n,\ C_n,\ S_n,\ K_n,\ K_{r,s}$ and $T_{r,s}$ denote the path, cycle, star, complete, complete bipartite and tadpole graphs, respectively.

2 Zagreb indices

Since 1947, several degree based graph topological indices have been defined in many areas to study required properties of different objects such as atoms and molecules. The most studied

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[1]AMS 2010 Subject Classification Number: 05C07, 05C09, 05C30, 05C38

[2]Keywords: Graphs, pendant edge, vertex degrees, graph index, reduced first Zagreb index, edge addition

degree based topological indices are the family of Zagreb indices. Two of the most important indices are called the first and second Zagreb indices denoted by $M_1(G)$ and $M_2(G)$, respectively in [3]:

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v). \tag{1}$$

In a similar way, reduced first and second Zagreb indices are introduced and they are denoted by $M_1^R(G)$ and $M_2^R(G)$.

The reduced first Zagreb index is defined by

$$M_1^R(G) = \sum_{i=1}^{n} (d_i - 1)^2$$

And the reduced second Zagreb index is defined by

$$M_2^R(G) = \sum_{e=uv} (du - 1)(dv - 1).$$

Reduced first and second Zagreb indices in terms of first and second Zagreb indices are as follows:

Reduced first Zagreb index in terms of first Zagreb index is

$$\begin{aligned} M_1^R(G) &= \sum_{i=1}^{n} (d_i - 1)^2 \\ &= \sum_{i=1}^{n} d_i^2 - 2\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} 1 \\ &= M_1(G) - 4m + n \end{aligned}$$

and the reduced second Zagreb index in terms of first and second Zagreb indices is

$$\begin{aligned} M_2^R(G) &= \sum_{uv \in E} dudv - \sum_{uv \in E} (du + dv) + \sum_{uv \in E} 1 \\ &= M_2(G) - M_1(G) + m \end{aligned}$$

In this work, we find the effect of the reduced first Zagreb index when an edge e is added to a given graph. Finally we give examples to the change of reduced first Zagreb index for some well-known graphs when a new edge is added.

3 Change in the reduced first Zagreb index when a new edge is added

In this section, we will determine the amount of change in the reduced first Zagreb index when a new edge is added to a simple graph. We have

Theorem 3.1. *Let G be a simple graph and let $G+e$ be the graph obtained by adding an edge e to the graph G.*

i) *If e is a pendant edge joining the vertex $v_n \in V(G)$ of degree d_i with a new pendant vertex v_{n+1} of degree $d_{n+1}=1$, then*

$$M_1^R(G+\{e\}) = M_1(G) - 4m(G) + 2d_n + n - 1 \tag{2}$$

ii) *If the added edge e connects two vertices v_i, v_j of the graph G with degrees d_i, d_j respectively, then*

$$M_1^R(G+\{e\}) = M_1^R(G) + 2(d_G w_1 + d_G w_2 - 1). \tag{3}$$

Proof. ***i)*** Let G be the graph having the vertices $v_1, v_2, \cdots, v_n$ of degrees $d_1, d_2, \cdots, d_n$, respectively. Let us add a new pendant edge e between the vertices $v_i \in G$ of degree d_i in G with a new vertex v_{n+1} of degree $d_{n+1}=1$ which is not a vertex of G. Hence

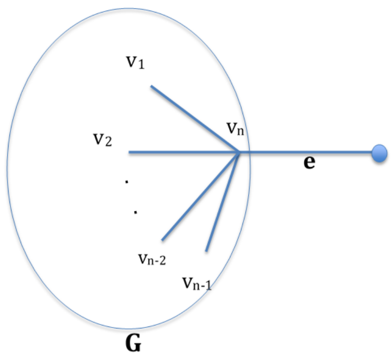

Figure 1: Adding a pendant edge e to G

$$\begin{aligned}
M_1^R(G+e) &= \sum_{i=1}^{n=1}(d_i-1)^2 + (d_{G+e}v_n - 1)^2 + (d_{G+e}v_{n+1} - 1)^2 \\
&= \sum_{i=1}^{n=1}(d_i)^2 - 2\sum_{i=1}^{n=1} d_i + (n-1)1^2 + (d_G v_n + 1 - 1)^2 \\
&= M_1(G) - (d_n)^2 - 2[2m(G) - d_n] + n - 1 + (d_n)^2 \\
&= M_1(G) - 4m(G) + 2d_n + n - 1
\end{aligned}$$

which gives the result.

ii) Consider two vertices $v_i,\ v_j \in V(G)$. Let us add a new edge $e = v_iv_j$ to the graph G. Adding an edge e increases the degrees of v_i and v_j by 1. Hence we have

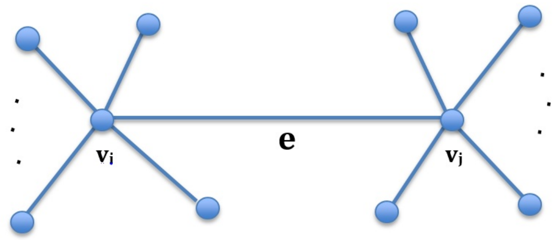

Figure 2: Adding a non-pendant edge e to G

$$\begin{aligned} M_1^R(G+e) &= \sum_{i=1}^{k}(d_{G+e}(v_i)-1)^2+\sum_{j=1}^{t}(d_{G+e}(v_j)-1)^2+d_{G+e}(w_1)-1)^2+d_{G+e}(w_2)-1)^2 \\ &= \sum_{i=1}^{k}(d_G(v_i)-1)^2+\sum_{j=1}^{t}(d_G(v_j)-1)^2+(d_Gw_1)^2+(d_Gw_2)^2 \\ &= M_1^R(G)+2(d_Gw_1+d_Gw_2-1) \end{aligned}$$

Hence we proved the theorem. $\square$

4 Effect of edge addition on M_1^R for some classes of graphs

In this section, we get the results for the graph classes $P_n,\ C_n,\ S_n,\ K_n,\ K_{r,s}$ and $T_{r,s}$ when we add a pendant or a non-pendant edge using above results.

4.1 Path graph P_n

We find that $M_1^R(P_n) = n-2$. Therefore by Theorem 3.1, we can write

$$M_1^R(P_n{+}e) = \begin{cases} n-1 & \text{if a pendant edge } e \text{ is added to one of the two end points of } P_n \text{ having degree } 1, \\ n+1 & \text{if a pendant edge } e \text{ is added to one of the midpoints of } P_n \text{ of degree } 2, \\ n+2 & \text{if a non-pendant edge } e \text{ is added at vertices } v_i,\ v_j \text{ so that } d_i=1, d_j=2 \text{ or vice versa}, \\ n+4 & \text{if a non-pendant edge } e \text{ is added at vertices } v_i,\ v_j \text{ so that } d_i=d_j=2, \\ n & \text{if a non-pendant edge } e \text{ is added at vertices } v_i,\ v_j \text{ so that } d_i=d_j=1, \end{cases}$$

So we have the following possibilities for P_n:

a) If a pendant edge e is added to one of the two end points of P_n having degree 1, then we have

$$M_1^R(P_n + e) = n - 1;$$

b) If a pendant edge e is added to one of the midponts of P_n of degree 2, then

$$M_1^R(P_n + e) = n + 1;$$

c) If a non-pendant edge e is added at vertices v_i, v_j so that $d_i = 1, d_j = 2$ or vice versa, then

$$M_1^R(P_n + e) = n + 2;$$

d) If a non-pendant edge e is added at vertices v_i, v_j so that $d_i = d_j = 2$, then

$$M_1^R(P_n + e) = n + 4.$$

e) If a non-pendant edge e is added at vertices v_i, v_j so that $d_i = d_j = 1$, then

$$M_1^R(P_n + e) = n;$$

So we proved.

Corollary 4.1. *When a new pendant or a non-pendant edge is added to a path graph P_n, its reduced first Zagreb index increases by either 1,2,3,4 or 6.*

4.2 Cycle graph C_n

We find that $M_1^R(C_n) = n$. Hence by Theorem 3.1, we can write

$$M_1^R(C_n + e) = \begin{cases} n + 3 & \text{if a pendant edge is added at } v_i, \\ n + 6 & \text{if a non-pendant edge } e \text{ is added at vertices } v_i, v_j \text{ of } C_n. \end{cases}$$

As all the vertices of C_n are of degree 2, so we have the following possibilities:

a) If a pendant edge e is added to any one of the vertices of C_n having degree 2, then

$$M_1^R(C_n + e) = n + 3;$$

b) If a non-pendant edge e is added at vertices v_i, v_j of C_n, so that, $d_i = d_j = 2$, then

$$M_1^R(C_n + e) = n + 6.$$

So we proved the following:

Corollary 4.2. *When a new pendant or a non-pendant edge is added to a cycle graph C_n, its reduced first Zagreb index increases by either 3 or 6.*

4.3 Star graph S_n

We find that for $n \geq 3$, $M_1^R(S_n) = n^2 - 4n + 4$. Hence by Theorem 3.1, we can write

$$M_1^R(S_n{+}e) = \begin{cases} n^2 - 4n + 5 & \text{if a pendant edge } e \text{ is added to a vertex } v_i \text{ of } S_n \text{ of degree 1,} \\ n^2 - 2n + 1 & \text{if a pendant edge } e \text{ is added to the vertex } v_i \text{ of } S_n \text{ of degree } n-1, \\ n^2 - 4n + 6 & \text{if a non-pendant edge } e \text{ is added at the vertices } v_i, v_j \text{ of } S_n \\ & \text{so that, } d_i = d_j = 1, \\ n^2 - 2n + 2 & \text{if a non-pendant edge } e \text{ is added at vertices } v_i, v_j \text{ of } S_n \text{ so that} \\ & d_i = 1, d_j = n-1 \text{ or vice versa.} \end{cases}$$

As a vertex of S_n is either pendant or of degree $n-1$, so we have the following possibilities:

a) If a pendant edge e is added to a vertex v_i of S_n of degree 1, then

$$M_1^R(S_n + e) = n^2 - 4n + 5;$$

b) If a pendant edge e is added to the vertex v_i of S_n of degree $n-1$, then

$$M_1^R(S_n + e) = n^2 - 2n + 1;$$

c) If a non-pendant edge e is added at the vertices v_i, v_j of S_n so that, $d_i = d_j = 1$, then

$$M_1^R(S_n + e) = n^2 - 4n + 6;$$

d) If a non-pendant edge e is added at vertices v_i, v_j of S_n so that, $d_i = 1, d_j = n-1$ or vice versa, then

$$M_1^R(S_n + e) = n^2 - 2n + 2;$$

So we proved.

Corollary 4.3. *When a new pendant or a non-pendant edge is added to a star graph S_n, its reduced first Zagreb index increases by either 1, 2, 2n-2 or 2n-3.*

4.4 Complete graph K_n

We find that $M_1^R(K_n) = n^3 - 4n^2 + 4n$. Hence by Theorem 3.1, we can write

$$M_1^R(K_n + e) = \begin{cases} n^3 - 4n^2 + 6n - 3 & \text{if a pendant edge } e \text{ is added to any one of the vertices} \\ & \text{of } K_n \text{ of degree } n-1 \\ n^3 - 4n^2 + 8n - 6 & \text{if a non-pendant edge } e \text{ is added at the vertices } v_i, v_j \\ & \text{of } K_n \text{, so that } d_i = d_j = n-1. \end{cases}$$

As all vertices of K_n are of degree $n-1$, so we have the following possibilities:

a) If a pendant edge e is added to any one of the vertices of K_n of degree $n-1$, then

$$M_1^R(K_n+e) = n^3 - 4n^2 + 6n - 3;$$

b) If a non-pendant edge e is added at the vertices v_i, v_j of K_n, so that, $d_i = d_j = n-1$, then

$$M_1^R(K_n+e) = n^3 - 4n^2 + 8n - 6;$$

So we proved.

Corollary 4.4. *When a new pendant or a non-pendant edge is added to a complete graph K_n, its reduced first Zagreb index increases by either $2n-3$ or $2(2n-3)$.*

4.5 Complete bipartite graph $K_{r,s}$

We find that $M_1^R(K_{r,s}) = r^2s + rs^2 - 4rs + r + s$. By Theorem 3.1, we can write

$$M_1^R(K_{r,s}+e) = \begin{cases} r^2s+rs^2-4rs+r+3s-1 & \text{if a pendant edge } e \text{ is added to a vertex} \\ & \text{of degree } s, \\ r^2s+rs^2-4rs+3r+s-1 & \text{if a pendant edge } e \text{ is added to a vertex} \\ & \text{of degree } r, \\ r^2s+rs^2-4rs+5r+s-2 & \text{if a non-pendant edge } e \text{ is added at} \\ & \text{the vertices } v_i, v_j \text{ of } K_{r,s}, \text{ so that } d_i = d_j = r, \\ r^2s+rs^2-4rs+r+5s-2 & \text{if a non-pendant edge } e \text{ is added at} \\ & \text{the vertices } v_i, v_j \text{ of } K_{r,s}, \text{ so that } d_i = d_j = s, \\ r^2s+rs^2-4rs+3r+3s-2 & \text{If a non-pendant edge } e \text{ is added at} \\ & \text{the vertices } v_i \text{ and } v_j \text{ of } K_{r,s}, \text{ so that } d_i = s \\ & \text{and } d_j = r. \end{cases}$$

As the degree sequence of $K_{r,s}$ is $\{r^{(s)}, s^{(r)}\}$, so we have the following possibilities:

a) If a pendant edge e is added to a vertex of degree s, then

$$M_1^R(K_{r,s}+e) = r^2s + rs^2 - 4rs + r + 3s - 1;$$

b) If a pendant edge e is added to a vertex of degree r, then

$$M_1^R(K_{r,s}+e) = r^2s + rs^2 - 4rs + 3r + s - 1;$$

c) If a non-pendant edge e is added at the vertices v_i, v_j of $K_{r,s}$, so that, $d_i = d_j = r$, then

$$M_1^R(K_{r,s}+e) = r^2s + rs^2 - 4rs + 5r + s - 2;$$

d) If a non-pendant edge e is added at the vertices v_i, v_j of $K_{r,s}$, so that, $d_i = d_j = s$, then

$$M_1^R(K_{r,s}+e) = r^2s + rs^2 - 4rs + r + 5s - 2;$$

e) If a non-pendant edge e is added at the vertices v_i and v_j of $K_{r,s}$, so that, $d_i = s$ and $d_j = r$, then

$$M_1^R(K_{r,s} + e) = r^2s + rs^2 - 4rs + 3r + 3s - 2;$$

So we proved.

Corollary 4.5. *When a new pendant or a non-pendant edge is added to a complete bipartite graph $K_{r,s}$, its reduced first Zagreb index increases by either $2r - 1$, $2s - 1$, $2(2r - 1)$, $2(2s - 1)$ or $2(r + s - 1)$.*

4.6 Tadpole graph $T_{r,s}$

We find that $M_1^R(T_{r,s}) = r + s + 2$. By Theorem 3.1, we can write

$$M_1^R(T_{r,s} + e) = \begin{cases} r + s + 3 & \text{if a pendant edge } e \text{ is added to a vertex of degree } 1, \\ r + s + 5 & \text{if a pendant edge } e \text{ is added to a vertex of degree } 2, \\ r + s + 7 & \text{if a pendant edge } e \text{ is added to a vertex of degree } 3, \\ r + s + 8 & \text{if a non-pendant edge } e \text{ is added at the vertices } v_i, v_j \\ & \text{so that } d_i = d_j = 2, \\ r + s + 8 & \text{if a non-pendant edge } e \text{ is added at the vertices } v_i, v_j \\ & \text{so that } d_i = 1, d_j = 3 \text{ or vice versa}, \\ r + s + 10 & \text{if a non-pendant edge } e \text{ is added at the vertices } v_i, v_j \\ & \text{so that } d_i = 2, d_j = 3 \text{ or vice versa}, \\ r + s + 6 & \text{if a non-pendant edge } e \text{ is added at the vertices } v_i, v_j \\ & \text{so that } d_i = 1, d_j = 2 \text{ or vice versa}. \end{cases}$$

As the degree sequence of $T_{r,s}$ is $\{1^{(1)}, 2^{(r+s-2)}, 3^{(1)}\}$, so we have the following possibilities:

a) If a pendant edge e is added to a vertex of degree 1, then

$$M_1^R(T_{r,s} + e) = r + s + 3;$$

b) If a pendant edge e is added to a vertex of degree 2, then

$$M_1^R(T_{r,s} + e) = r + s + 5;$$

c) If a pendant edge e is added to a vertex of degree 3, then

$$M_1^R(T_{r,s} + e) = r + s + 7;$$

d) If a non-pendant edge e is added at the vertices v_i, v_j, so that $d_i = d_j = 2$, then

$$M_1^R(T_{r,s} + e) = r + s + 8;$$

e) If a non-pendant edge e is added at the vertices v_i, v_j, so that $d_i = 1, d_j = 3$ or vice versa, then

$$M_1^R(T_{r,s} + e) = r + s + 8;$$

f) If a non-pendant edge e is added at the vertices v_i, v_j, so that $d_i = 2, d_j = 3$ or vice versa, then

$$M_1^R(T_{r,s} + e) = r + s + 10;$$

g) If a non-pendant edge e is added at the vertices v_i, v_j, so that $d_i = 1, d_j = 2$ or vice versa, then

$$M_1^R(T_{r,s} + e) = r + s + 6;$$

So we proved.

Corollary 4.6. *When a new pendant or a non-pendant edge is added to a tadpole graph $T_{r,s}$, its reduced first Zagreb index increases by either 1,3,4,5,6 or 8.*

5 Summary and conclusions

In this work, the effect of adding a new edge to a connected simple graph on its reduced first Zagreb index is studied. This effect differs for adding a pendant edge or a non-pendant edge. Both effects are formulized and all possible differences are determined for six most frequently used graph classes. And there will be a continuation of this paper for other indices.

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