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ON A GENERALIZED RECURRENCE FOR DEGENERATE BELL POLYNOMIALS ARISING FROM BOSON ALGEBRA

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ABSTRACT. This paper presents a novel Spivey-type formula for degenerate Bell polynomials, $\phi_{n+m,\lambda}(t)$, extending a foundational identity derived by Spivey. We derive the generalized formula for the degenerate Bell plynomials using the algebraic properties of Boson operators. This result generalizes both the Spivey-type formula for standard Bell polynomials (as $\lambda \to 0$) and the original Spivey identity (as $\lambda \to 0$ and $t = 1$), thus providing an alternative proof for an identity recently established through generating functions and other operator methods.

1. INTRODUCTION

The Bell numbers, ϕ_n, count the number of partitions of a set with n elements and hold fundamental importance in combinatorics, probability, number theory, and quantum field theory. While several recurrence relations and identities exist for these numbers, the derivation and establishment of effective new formulas remain an active area of research. A cornerstone development in this field was the remarkable identity derived by Spivey [17] using a powerful combinatorial method. This identity relates the $(n+m)$-th Bell number, ϕ_{n+m}, to a double summation that incorporates the Bell numbers of lower order, ϕ_k, and combinatorial weights derived from binomial coefficients, Stirling numbers of the second kind, and exponents. This identity is given as:

$$\phi_{n+m} = \sum_{j=0}^{m}\sum_{k=0}^{n} \left\{ {m \atop j} \right\} \binom{n}{k} j^{n-k}\phi_k, \quad (m,n \geq 0). \tag{1}$$

Since its initial publication, the identity in (1) has spurred extensive follow-up research, establishing an entire class of 'Spivey-type recurrence relation' [4,7,9,10]. Especially, for q-analogues of Spivey's Bell number formula, we let the reader refer to [2,3,13]. This resulting body of work has successfully extended the original formula to encompass various generalized combinatorial objects, including r-Whitney and degenerate Bell numbers, providing new sum formulas, combinatorial interpretations, and proofs for these generalizations.

In this paper, we prove the following Spivey-type formula for degenerate Bell polynomials using Boson operators a and $a^\dagger$ satisfying the commutation relation $[a,a^\dagger] = aa^\dagger - a^\dagger a = 1$:

$$\phi_{n+m,\lambda}(t) = \sum_{j=0}^{m}\sum_{k=0}^{n} \left\{ {m \atop j} \right\}_\lambda \binom{n}{k} (j-m\lambda)_{n-k,\lambda} t^j \phi_{k,\lambda}(t), \quad (\text{see } [6,7]). \tag{2}$$

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We note that (2) gives the Spivey-type formula for Bell polynomials as $\lambda \to 0$:

$$\phi_{n+m}(t) = \sum_{j=0}^{m} \sum_{k=0}^{n} \begin{Bmatrix} m \\ j \end{Bmatrix} \binom{n}{k} j^{n-k} t^j \phi_k(t), \quad (\text{see } [6,7]),$$

and the Spivey's formula in (1) as $\lambda \to 0$ and $t = 1$. Recently, Kim-Kim derived Spivey-type formula for degenerate Bell polynomials using other methods than the Boson operators, namely from generating functions and the operators X and D satisfying the commutation relation $DX - XD = 1$, where X is the multiplication by x operator and $D = \frac{d}{dx}$. As general references of this paper, the reader may refer to [1,14,15]. For the rest of this section, we recall what are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t) = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!}, \quad e_\lambda(t) = e_\lambda^1(t), \quad (\text{see}[7,8]),$$

where the degenerate falling sequence is given by

$$(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1),$$

and satisfies

$$(x+y)_{n,\lambda} = \sum_{l=0}^{n} \binom{n}{l} (x)_{l,\lambda} (y)_{n-l,\lambda}, \quad (n \geq 0). \tag{3}$$

The degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda (x)_k, \quad (n \geq 0), \quad (\text{see } [6,7]), \tag{4}$$

where

$$(x)_0 = 1, (x)_n = x(x-1)\cdots(x-(n-1)), \quad (n \geq 1).$$

Note that

$$\lim_{\lambda \to 0} \begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda = \begin{Bmatrix} n \\ k \end{Bmatrix},$$

where $\begin{Bmatrix} n \\ k \end{Bmatrix}$ are the Stirling numbers of the second kind given by

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k, \quad (n \geq 0), \quad (\text{see } [11,12]).$$

The degenerate Bell polynomials are defined by

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda x^k, \quad (n \geq 0), \quad (\text{see } [6,7]). \tag{5}$$

When $x = 1$, $\phi_{n,\lambda} = \phi_{n,\lambda}(1)$ are called the degenerate Bell numbers.
Note that

$$\lim_{\lambda \to 0} \phi_{n,\lambda}(x) = \phi_n(x),$$

where $\phi_n(x)$ are the Bell polynomials given by (see [6,16])

$$\phi_n(x) = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} x^k.$$

When $x = 1$, $\phi_n = \phi_n(1)$ are called the Bell numbers.
From (4) and (5), we note that

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see } [6,7]).$$

2. Spivey-type formula for degenerate Bell polynomials revisited

The boson annihilation and creation operators a and $a^\dagger$ satisfy the commutation relation given by

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1, \quad (\text{see } [5,8,11,12,14]). \tag{6}$$

The Fock space is defined by the basis $\{|n\rangle \mid n = 0,1,2,\ldots\}$ such that

$$a\,|n\rangle = \sqrt{n}\,|n-1,\rangle \quad \text{and} \quad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.$$

Thus, we get

$$a^\dagger a\,|n\rangle = n\,|n\rangle\,, \quad n = 0,1,2,\ldots, \quad (\text{see } [12]).$$

The coherent states $|z\rangle$, where z is a complex number, satisfy

$$a\,|z\rangle = z\,|z\rangle\,, \quad z\,|z\rangle = 1.$$

To show a connection to coherent states, we recall that the harmonic oscillator has Hamiltonian $\hat{n} = a^\dagger a$ (neglecting the zero point energy) and the usual eigenstates $|n\rangle$ $(n \in \mathbb{N})$ satisfying

$$\hat{n}\,|n\rangle = n\,|n\rangle \quad \text{and} \quad m\,|n\rangle = \delta_{mn},$$

where δ_{mn} is the Kronecker's symbol (see [12]). From (6), we have

$$(a^\dagger a)_{n,\lambda} = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_\lambda (a^\dagger)^k a^k, \quad (n \geq 0), \quad (\text{see } [5,8,11,12,14]). \tag{7}$$

For $k \in \mathbb{N}$, by (6), we get

$$\begin{aligned}
[a, (a^\dagger)^k] &= a(a^\dagger)^k - (a^\dagger)^k a \\
&= a(a^\dagger)^k - (a^\dagger)^{k-1} aa^\dagger + (a^\dagger)^{k-1} aa^\dagger - (a^\dagger)^k a \\
&= \left(a(a^\dagger)^{k-1} - (a^\dagger)^{k-1} a \right) a^\dagger + (a^\dagger)^{k-1}(aa^\dagger - a^\dagger a) \\
&= [a, (a^\dagger)^{k-1}] a^\dagger + (a^\dagger)^{k-1}.
\end{aligned} \tag{8}$$

From (8), we note that

$$\begin{aligned}
[a, (a^\dagger)^k] &= [a, (a^\dagger)^{k-1}] a^\dagger + (a^\dagger)^{k-1} = [a, (a^\dagger)^{k-2}](a^\dagger)^2 + 2(a^\dagger)^{k-1} \\
&= [a, (a^\dagger)^{k-3}](a^\dagger)^3 + 3(a^\dagger)^{k-1} = \cdots \\
&= [a, a^\dagger](a^\dagger)^{k-1} + (k-1)(a^\dagger)^{k-1} = k(a^\dagger)^{k-1}.
\end{aligned} \tag{9}$$

For $k \in \mathbb{N}$, by (9), we get

$$[a, (a^\dagger)^k] = k(a^\dagger)^{k-1}. \tag{10}$$

We note that

$$a|0\rangle = 0. \tag{11}$$

By (11), we get

$$\begin{aligned}[a,(a^\dagger)^k]|0\rangle &= a(a^\dagger)^k - (a^\dagger)^k a|0\rangle \\ &= a(a^\dagger)^k|0\rangle - (a^\dagger)^k a|0\rangle \\ &= a(a^\dagger)^k|0\rangle.\end{aligned} \tag{12}$$

From (10) and (12), we have

$$a(a^\dagger)^k|0\rangle = [a,(a^\dagger)^k]|0\rangle = k(a^\dagger)^{k-1}|0\rangle, \quad (k \in \mathbb{N}). \tag{13}$$

Thus, by (13), we get

$$\begin{aligned}a^k(a^\dagger)^l|0\rangle &= a^{k-1}a(a^\dagger)^l|0\rangle = la^{k-1}(a^\dagger)^{l-1}|0\rangle \\ &= l(l-1)a^{k-2}(a^\dagger)^{l-2}|0\rangle = l(l-1)(l-2)a^{k-3}(a^\dagger)^{l-3}|0\rangle = \cdots \\ &= l(l-1)\cdots(l-k+1)(a^\dagger)^{l-k}|0\rangle = \frac{l!}{(l-k)!}(a^\dagger)^{l-k}|0\rangle,\end{aligned} \tag{14}$$

where $l,k \in \mathbb{N}$ with $l \geq k$.

Remark 2.1.

$$a^k(a^\dagger)^l|0\rangle = 0, \textit{ if } k > l.$$

From (14) and Remark 2.1, we have

$$\begin{aligned}a^k e^{xa^\dagger}|0\rangle &= \sum_{l=k}^{\infty}\frac{x^l}{l!}a^k(a^\dagger)^l|0\rangle = \sum_{l=k}^{\infty}\frac{x^l}{l!}\frac{l!}{(l-k)!}(a^\dagger)^{l-k}|0\rangle \\ &= x^k\sum_{l=0}^{\infty}\frac{x^l}{l!}(a^\dagger)^l|0\rangle = x^k e^{xa^\dagger}|0\rangle.\end{aligned} \tag{15}$$

By (5) and (15), we get

$$\begin{aligned}(a^\dagger a)_{n,\lambda}e^{xa^\dagger}|0\rangle &= \sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda (a^\dagger)^k a^k e^{xa^\dagger}|0\rangle \\ &= \sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda (a^\dagger)^k x^k e^{xa^\dagger}|0\rangle \\ &= \sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda (xa^\dagger)^k e^{xa^\dagger}|0\rangle \\ &= \phi_{n,\lambda}(xa^\dagger)e^{xa^\dagger}|0\rangle, \quad (n \geq 0).\end{aligned} \tag{16}$$

Therefore, by (16), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\phi_{n,\lambda}(xa^\dagger)e^{xa^\dagger}|0\rangle = (a^\dagger a)_{n,\lambda}e^{xa^\dagger}|0\rangle. \tag{17}$$

From this, we get

$$\phi_{n,\lambda}(t)e^t|0\rangle = (a^\dagger a)_{n,\lambda}e^t|0\rangle.$$

For $k \in \mathbb{N} \cup \{0\}$, by (10), we get

$$(a^\dagger a)(a^\dagger)^k = a^\dagger(a(a^\dagger)^k) = a^\dagger\left((a^\dagger)^k a + k(a^\dagger)^{k-1}\right) = (a^\dagger)^k(a^\dagger a + k). \tag{18}$$

From (3), (7) and (18), we note that

$$\begin{aligned}(a^\dagger a)_{n+m,\lambda} &= (a^\dagger a - m\lambda)_{n,\lambda}(a^\dagger a)_{m,\lambda} = (a^\dagger a - m\lambda)_{n,\lambda}\sum_{j=0}^{m}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda (a^\dagger)^j a^j \\ &= \sum_{j=0}^{m}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda (a^\dagger a - m\lambda)_{n,\lambda}(a^\dagger)^j a^j \\ &= \sum_{j=0}^{m}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda (a^\dagger)^j (a^\dagger a + j - m\lambda)_{n,\lambda} a^j \\ &= \sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda \binom{n}{k}(j - m\lambda)_{n-k,\lambda}(a^\dagger)^j(a^\dagger a)_{k,\lambda}a^j.\end{aligned} \tag{19}$$

Therefore, by (19), we obtain the following theorem.

Theorem 2.3. *For $m,n \geq 0$, we have*

$$(a^\dagger a)_{m+n,\lambda} = \sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda \binom{n}{k}(j - m\lambda)_{n-k,\lambda}(a^\dagger)^j(a^\dagger a)_{k,\lambda}a^j. \tag{20}$$

By (15), (17) and (20), we get

$$\begin{aligned}&(a^\dagger a)_{n+m,\lambda}e^{xa^\dagger}|0\rangle \\ &= \sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda \binom{n}{k}(j - m\lambda)_{n-k,\lambda}(a^\dagger)^j(a^\dagger a)_{k,\lambda}a^j e^{xa^\dagger}|0\rangle \\ &= \sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda \binom{n}{k}(j - m\lambda)_{n-k,\lambda}(a^\dagger)^j x^j(a^\dagger a)_{k,\lambda}e^{xa^\dagger}|0\rangle \\ &= \sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda \binom{n}{k}(j - m\lambda)_{n-k,\lambda}(xa^\dagger)^j \phi_{k,\lambda}(xa^\dagger)e^{xa^\dagger}|0\rangle.\end{aligned} \tag{21}$$

On the other hand, by (17), we get

$$(a^\dagger a)_{n+m,\lambda}e^{xa^\dagger}|0\rangle = \phi_{n+m,\lambda}(xa^\dagger)e^{xa^\dagger}|0\rangle. \tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 2.4. *For $m,n \geq 0$, we have*

$$\begin{aligned}&\phi_{n+m,\lambda}(xa^\dagger)e^{xa^\dagger}|0\rangle \\ &= \sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda \binom{n}{k}(j - m\lambda)_{n-k,\lambda}(xa^\dagger)^j \phi_{k,\lambda}(xa^\dagger)e^{xa^\dagger}|0\rangle.\end{aligned} \tag{23}$$

From (23), we get

$$\phi_{n+m,\lambda}(xa^\dagger) = \sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_\lambda \binom{n}{k}(j - m\lambda)_{n-k,\lambda}(xa^\dagger)^j \phi_{k,\lambda}(xa^\dagger).$$

From this, we obtain the Spivey-type formula for degenerate Bell polynomials:

$$\phi_{n+m,\lambda}(t)=\sum_{j=0}^{m}\sum_{k=0}^{n}\left\{\begin{matrix} m \\ j \end{matrix}\right\}_{\lambda}\binom{n}{k}(j-m\lambda)_{n-k,\lambda}t^{j}\phi_{k,\lambda}(t),\quad (\text{see } [6,7]).$$

3. Conclusion

The derivation of the Spivey-type formula for degenerate Bell polynomials using Boson operators successfully demonstrates a powerful application of quantum mechanical formalism to problems in enumerative combinatorics. This approach provides a unique and effective algebraic framework for generalizing established combinatorial identities. The resulting formula encompasses key existing results for Bell polynomials and Bell numbers as special cases, reinforcing the identity's significance. This work not only complements recent derivations by other methods but also establishes the utility of Boson calculus as a versatile tool for proving and discovering new Spivey-type recurrence relations for generalized combinatorial sequences.

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