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A NEW CRITERION THAT A CONNECTED LIE GROUP IS AMENABLE

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Abstract. As usual, we say that the uniform convergence topology for continuous irreducible unitary representations is discrete if continuous irreducible unitary representations sufficiently close with respect to the norm topology are similar. It is proved that a connected Lie group is amenable if and only if the uniform convergence topology for continuous irreducible unitary representations is discrete.

§ 1. Introduction

The following assertion was proved in [1].

Theorem 1. *Let π be a continuous unitary representation of an amenable locally compact group G. All* (*automatically continuous*) *unitary representations of G belonging to the ball of radius less than 1 with center at π are equivalent to π, which shows the discreteness of the topology of uniform convergence of continuous unitary representations on the whole group G.*

The objective of the present paper is to prove a kind of converse assertion and thus establish the following criterion.

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Theorem 2. *A connected Lie group is amenable if and only if the uniform convergence topology for continuous irreducible unitary representations is discrete.*

§ 2. Proof of Theorem 2

It suffices to prove the "if" part. We argue by contradiction. Let G be a connected Lie group which is not amenable. This means that the semisimple part of the Levi decomposition is not compact [2], and hence there is a quotient group G_1 of G which is a noncompact semisimple Lie group [3].

The dual space of G formed by the equivalence classes of irreducible continuous unitary representations of G contains the dual space of G_1 formed by the equivalence classes of irreducible continuous unitary representations of G_1 with the same topology [4], and hence we may assume that $G = G_1$.

For the case of a noncompact semisimple Lie group G, we use the Iwasawa decomposition $G = KAN$ with K a maximal compact subgroup, A Abelian, and N nilpotent (AN is solvable) and consider the subgroup MAN with M the centralizer in K of A.

Consider representations τ of MAN that are irreducible, unitary, and trivial on the subgroup N. The induced representations of these τ form the principal series. They include the so-called spherical principal series that consists of representations induced from 1-dimensional representations of MAN obtained by extending characters ν of A using the homomorphism of MAN onto A.

By [5], these representations are irreducible.

By the formula (1) in [6], the representations of the spherical principal series depend on the family of finitely many real parameters defining the character ν of A analytically. To be more precise, the factor changing from one representation of the spherical principal series to another has the base which is a continuous function, on the compact set K/M, that does not take the value zero (and thus its modulus is both bounded and bounded away from zero), and the exponent includes the linear combination of the real parameters of ν multiplied by the imaginary unit.

Thus, the family of spherical principal series is a norm continuous family of continuous irreducible unitary representations of G.

Since two representations of the spherical principal series are equivalent if and only if the corresponding characters ν are lying on the same orbit of the Weyl group [5], it follows that every neighborhood of every representation of the spherical principal series contains infinitely many representations nonequivalent to the chosen representation, and hence the norm convergence topology on the group G of the dual space to G is not discrete.

This completes the proof of the theorem. □

§ 3. Concluding remarks

Certainly, a similar assertion holds also for uniformly bounded representations [7].

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