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APPLICATIONS AND COMPUTATION OF SHEFFER-TYPE POLYNOMIALS WITH DEGENERATE PROPERTIES

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ABSTRACT. This paper has two primary contributions. First, we explore degenerate Sheffer-type polynomials, a hybrid of higher-order degenerate Bernoulli and Euler polynomials, and derive their properties. Second, assuming that the moment generating function of Y exists in a neighborhood of the origin, we introduce the degenerate Sheffer polynomials associated with Y. We then investigate their properties in general and for the specific cases of uniform and Bernoulli random variables. We also present new results for the higher-order degenerate Bernoulli and Euler polynomials.

1. INTRODUCTION

This paper explores two main areas. First, we delve into degenerate Sheffer-type polynomials, a hybrid of higher-order degenerate Bernoulli and Euler polynomials. Second, we examine degenerate Sheffer polynomials associated with a random variable Y, which are a degenerate form of the Appell polynomials associated with Y, as previously investigated in [2, 20, 22]. It's worth noting that the study of various degenerate versions of special polynomials and numbers, originated by Carlitz's work in [4, 5], has seen a renewed surge of interest among mathematicians in recent years (see [3, 7, 10, 13, 15, 16, 23, 24]). We assume that the moment generating function of Y exists in a neighborhood of the origin (see (24)).

The paper is structured as follows:

• Section 1 reviews degenerate exponentials, degenerate Bernoulli polynomials, and degenerate Euler polynomials. We also briefly remind the reader about uniform and Bernoulli random variables.

• Section 2 investigates higher-order degenerate Bernoulli and Euler polynomials, along with Sheffer-type polynomials. After presenting several properties of the higher-order degenerate Bernoulli and Euler polynomials (see Proposition 2.1, Theorems 2.3, 2.4, 2.7), we introduce the degenerate Sheffer-type polynomials $T_{n,\lambda}^{(a,b)}(x)$ (see (18)). These polynomials represent a hybrid form of the aforementioned higher-order polynomials (see Theorems 2.5, 2.6, 2.8).

• Section 3 introduces the degenerate Sheffer polynomials associated with Y, denoted as $S_{n,\lambda}^{Y}(x)$ (see (25)). We derive several properties for these polynomials:

  • In a general context (see Theorems 3.1, 3.2).
  • For a uniform random variable on $[0,1]$ (see Theorems 3.3, 3.5, 3.6).
  • For a Bernoulli random variable with the success probability $\frac{1}{2}$ (see Theorems 3.4, 3.7).

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Furthermore, Theorem 3.8 establishes an identity for $T_{n,\lambda}^{(a,b)}(x)$ by utilizing the Bernoulli random variable with the success probability of $\frac{1}{2}$. Building on the identity in Theorem 3.8 and other related identities, we deduce an identity for $T_{n,\lambda}^{(a,b)}(x)$ in Theorem 3.9, and for the higher-order degenerate Bernoulli and Euler polynomials in Theorems 3.10 and 3.11. For general references pertaining to this paper, please consult [1, 6, 14, 18, 19].

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by (see [7, 10, 13, 15, 16, 23])

$$e_\lambda^x(t) = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!}, \quad e_\lambda(t) = e_\lambda^1(t), \tag{1}$$

where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\big(x-(n-1)\lambda\big), \ (n \geq 1).$$

Note from (1) that

$$\lim_{\lambda \to 0} e_\lambda^x(t) = e^{xt}.$$

We also observe that

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda} (y)_{n-k,\lambda}. \tag{2}$$

The degenerate Bernoulli polynomials are defined by (see [13, 15])

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \tag{3}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. From (3), we note that

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda} (x)_{n-k,\lambda}, \quad (n \geq 0). \tag{4}$$

The degenerate Euler polynomials are given by (see [13, 15])

$$\frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{5}$$

When $x = 0$, $\mathscr{E}_{n,\lambda} = \mathscr{E}_{n,\lambda}(0)$, $(n \geq 0)$, are called the degenerate Euler numbers. By (5), we get

$$\mathscr{E}_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{E}_{k,\lambda} (x)_{n-k,\lambda}, \quad (n \geq 0). \tag{6}$$

Note from (3) and (5) that

$$\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x), \quad \lim_{\lambda \to 0} \mathscr{E}_{n,\lambda}(x) = E_n(x),$$

where $B_n(x)$ and $E_n(x)$ are respectively Bernoulli polynomials and Euler polynomials, respectively given by (see [6, 12, 18])

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

We recall that Y is the uniform random variable on $[a,b]$, denoted by $Y \sim U(a,b)$, if the probability density function of Y is given by (see [14, 19])

$$f_Y(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$E\big[g(Y)\big] = \int_{-\infty}^{\infty} g(y) f_Y(y) dy = \frac{1}{b-a}\int_a^b g(y) dy,$$

where g is a real-valued function.
Also, Y is the Bernoulli random variable with probability of success p, denoted by $Y \sim \mathrm{Ber}(p)$, if the probability mass function of Y is given by (see [14, 19])

$$p(0) = p\{Y=0\} = 1-p, \quad p(1) = p\{Y=1\} = p.$$

Note that

$$E\big[g(Y)\big] = (1-p)g(0) + pg(1).$$

2. Degenerate Sheffer-type polynomials

By (2) and (4), we have

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}(x)_{n-k,\lambda} = (\beta + x)_{n,\lambda}, \quad (n \geq 0), \tag{7}$$

with the understanding of replacing $(\beta)_{n,\lambda}$ by $\beta_{n,\lambda}$. The degenerate Bernoulli numbers satisfy the recurrence relation (see (3), (7)) as follows:

$$(\beta+1)_{n,\lambda} - \beta_{n,\lambda} = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{otherwise} \end{cases}, \quad \beta_{0,\lambda} = 1. \tag{8}$$

From the recurrence relation (8), we derive

$$\beta_{0,\lambda} = 1,\ \beta_{1,\lambda} = -\frac{1}{2} + \frac{1}{2}\lambda,\ \beta_{2,\lambda} = \frac{1}{6} - \frac{1}{6}\lambda^2,\ \beta_{3,\lambda} = -\frac{1}{4}\lambda + \frac{1}{4}\lambda^3,$$
$$\beta_{4,\lambda} = -\frac{1}{30} + \frac{2}{3}\lambda^2 - \frac{19}{30}\lambda^4,\ \beta_{5,\lambda} = \frac{1}{4}\lambda - \frac{5}{2}\lambda^3 + \frac{9}{4}\lambda^5, \dots.$$

From (2) and (6), we note that

$$\mathscr{E}_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{E}_{k,\lambda}(x)_{n-k,\lambda} = \left(\mathscr{E} + x\right)_{n,\lambda}, \quad (n \geq 0), \tag{9}$$

with the understanding of replacing $(\mathscr{E})_{n,\lambda}$ by $\mathscr{E}_{n,\lambda}$. The degenerate Euler numbers satisfy the recurrence relation (see (5), (9)) as follows:

$$\left(\mathscr{E}+1\right)_{n,\lambda} + \mathscr{E}_{n,\lambda} = \begin{cases} 2, & \text{if } n=0 \\ 0, & \text{if } n \neq 0 \end{cases}, \quad \mathscr{E}_{0,\lambda} = 1. \tag{10}$$

By using the recurrence relation (10), we obtain

$$\mathscr{E}_{0,\lambda} = 1,\ \mathscr{E}_{1,\lambda} = -\frac{1}{2},\ \mathscr{E}_{2,\lambda} = \frac{1}{2}\lambda,\ \mathscr{E}_{3,\lambda} = \frac{1}{4} - \lambda^2,$$
$$\mathscr{E}_{4,\lambda} = -\frac{3}{2}\lambda + 3\lambda^3,\ \mathscr{E}_{5,\lambda} = -\frac{1}{2} + \frac{35}{4}\lambda^2 - 12\lambda^4, \dots.$$

For $\alpha \in \mathbb{R}$, the higher-order degenerate Bernoulli polynomials are defined by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^\infty \beta_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{11}$$

When $x=0$, $\beta_{n,\lambda}^{(\alpha)} = \beta_{n,\lambda}^{(\alpha)}(0)$ are called the higher-order degenerate Bernoulli numbers.

The higher-order degenerate Euler polynomials are given by

$$\left(\frac{2}{e_\lambda(t)+1}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^\infty \mathscr{E}_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{12}$$

When $x=0$, $\mathscr{E}_{n,\lambda}^{(\alpha)} = \mathscr{E}_{n,\lambda}^{(\alpha)}(0)$ are called the higher-order degenerate Euler numbers. Note that

$$\lim_{\lambda\to 0} B_{n,\lambda}^{(\alpha)}(x) = B_n^{(\alpha)}(x), \quad \lim_{\lambda\to 0} \mathscr{E}_{n,\lambda}^{(\alpha)}(x) = E_n^{(\alpha)}(x),$$

where $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ are the higher-order Bernoulli and higher-order Euler polynomial, respectively given by (see [18])

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n=0}^\infty B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad \left(\frac{2}{e^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^\infty E_n^{(\alpha)}(x)\frac{t^n}{n!}.$$

From (11) and (12), we note that

$$\beta_{n,\lambda}^{(0)}(x) = (x)_{n,\lambda} \quad \text{and} \quad \mathscr{E}_{n,\lambda}^{(0)}(x) = (x)_{n,\lambda}, \quad (n \geq 0). \tag{13}$$

The following proposition is immediate from the definitions (11) and (12).

Proposition 2.1. *For $a, b \in \mathbb{R}$, we have*

$$\beta_{n,\lambda}^{(a+b)}(x+y) = \sum_{k=0}^n \binom{n}{k} \beta_{k,\lambda}^{(a)}(x)\beta_{n-k,\lambda}^{(b)}(y),$$

and

$$\mathscr{E}_{n,\lambda}^{(a+b)}(x+y) = \sum_{k=0}^n \binom{n}{k} \mathscr{E}_{k,\lambda}^{(a)}(x)\mathscr{E}_{n-k,\lambda}^{(b)}(y).$$

From Proposition 2.1 and (13), we obtain the following corollary.

Corollary 2.2. *For $a \in \mathbb{R}$, we have*

$$\beta_{n,\lambda}^{(a)}(x+y) = \sum_{k=0}^n \binom{n}{k} \beta_{k,\lambda}^{(a)}(x)(y)_{n-k,\lambda},$$

and

$$\mathscr{E}_{n,\lambda}^{(a)}(x+y) = \sum_{k=0}^n \binom{n}{k} \mathscr{E}_{k,\lambda}^{(a)}(x)(y)_{n-k,\lambda}.$$

Now, we observe that

$$\begin{aligned} \sum_{n=0}^\infty \left(\beta_{n,\lambda}^{(\alpha)}(x+1) - \beta_{n,\lambda}^{(\alpha)}(x)\right)\frac{t^n}{n!} &= t\left(\frac{t}{e_\lambda(t)-1}\right)^{\alpha-1} e_\lambda^x(t) \\ &= \sum_{n=0}^\infty (n+1)\beta_{n,\lambda}^{(\alpha-1)}(x)\frac{t^{n+1}}{(n+1)!} = \sum_{n=0}^\infty n\beta_{n-1,\lambda}^{(\alpha-1)}(x)\frac{t^n}{n!}, \end{aligned} \tag{14}$$

and

$$\sum_{n=0}^{\infty}\Big(\mathscr{E}_{n,\lambda}^{(\alpha)}(x+1)+\mathscr{E}_{n,\lambda}^{(\alpha)}(x)\Big)\frac{t^n}{n!}=2\left(\frac{2}{e_\lambda(t)+1}\right)^{\alpha-1}e_\lambda^x(t) \tag{15}$$
$$=\sum_{n=0}^{\infty}2\mathscr{E}_{n,\lambda}^{(\alpha-1)}(x)\frac{t^n}{n!}.$$

Therefore, by (14) and (15), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, and $\alpha\in\mathbb{R}$, we have*

$$\beta_{n,\lambda}^{(\alpha)}(x+1)-\beta_{n,\lambda}^{(\alpha)}(x)=n\beta_{n-1,\lambda}^{(\alpha-1)}(x),$$

and

$$\mathscr{E}_{n,\lambda}^{(\alpha)}(x+1)+\mathscr{E}_{n,\lambda}^{(\alpha)}(x)=2\mathscr{E}_{n,\lambda}^{(\alpha-1)}(x).$$

From (3) and (5), we note that

$$\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}=\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\frac{1}{2}\frac{t}{e_\lambda(t)-1}\frac{2}{e_\lambda(t)+1}\big(e_\lambda(t)+1\big)e_\lambda^x(t) \tag{16}$$
$$=\frac{1}{2}\frac{t}{e_\lambda(t)-1}e_\lambda(t)\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)+\frac{1}{2}\frac{t}{e_\lambda(t)-1}\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\Big(\beta_{k,\lambda}(1)+\beta_{k,\lambda}\Big)\mathscr{E}_{n-k,\lambda}(x)\frac{t^n}{n!}.$$

Observe that $\beta_{k,\lambda}(1)=\beta_{k,\lambda}+\delta_{k,1}$ (see (8)), where $\delta_{k,1}$ is the Kronecker's delta. By (16), we get

$$\beta_{n,\lambda}(x)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\Big(2\beta_{k,\lambda}+\delta_{k,1}\Big)\mathscr{E}_{n-k,\lambda}(x) \tag{17}$$
$$=\frac{n}{2}\mathscr{E}_{n-1,\lambda}(x)+\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}\mathscr{E}_{n-k,\lambda}(x).$$

Therefore, by (17), we obtain the following theorem.

Theorem 2.4. *For $n\geq 0$, we have*

$$\beta_{n,\lambda}(x)=\frac{n}{2}\mathscr{E}_{n-1,\lambda}(x)+\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}\mathscr{E}_{n-k,\lambda}(x).$$

Now, we consider the *degenerate Sheffer-type polynomials* given by

$$\left(\frac{t}{e_\lambda(t)-1}\right)^a\left(\frac{2}{e_\lambda(t)+1}\right)^b e_\lambda^x(t)=\sum_{n=0}^{\infty}T_{n,\lambda}^{(a,b)}(x)\frac{t^n}{n!}, \tag{18}$$

where $a,b\in\mathbb{R}$.

The following proposition follows immediately from the definition (18).

Proposition 2.5. *For $n\geq 0$, and $a,b\in\mathbb{R}$, we have*

$$T_{n,\lambda}^{(a,b)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}^{(a)}(x)\mathscr{E}_{n-k,\lambda}^{(b)}(y),$$

and

$$T_{n,\lambda}^{(a,b)}(x)=\sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}^{(a,b)}(0)(x)_{n-k,\lambda}.$$

From (18) and (3), we have

$$(19)\qquad \sum_{n=0}^{\infty} T_{n,\lambda}^{(a,b)}(x)\frac{t^n}{n!} = \left(\frac{t}{e_\lambda(t)-1}\right)^{a-1}\left(\frac{2}{e_\lambda(t)+1}\right)^{b}\left(\frac{t}{e_\lambda(t)-1}\right)e_\lambda^x(t)$$
$$= \sum_{k=0}^{\infty} T_{k,\lambda}^{(a-1,b)}(0)\frac{t^k}{k!}\sum_{l=0}^{\infty}\beta_{l,\lambda}(x)\frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}^{(a-1,b)}(0)\beta_{n-k,\lambda}(x)\frac{t^n}{n!}.$$

Similarly to (19), by (18) and (5), we get

$$(20)\qquad \sum_{n=0}^{\infty} T_{n,\lambda}^{(a,b)}(x)\frac{t^n}{n!} = \left(\frac{t}{e_\lambda(t)-1}\right)^{a}\left(\frac{2}{e_\lambda(t)+1}\right)^{b-1}\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)$$
$$= \sum_{k=0}^{\infty} T_{k,\lambda}^{(a,b-1)}(0)\frac{t^k}{k!}\sum_{l=0}^{\infty}\mathscr{E}_{l,\lambda}(x)\frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}^{(a,b-1)}(0)\mathscr{E}_{n-k,\lambda}(x)\frac{t^n}{n!}.$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, and $a,b \in \mathbb{R}$, we have*

$$T_{n,\lambda}^{(a,b)}(x) = \sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}^{(a-1,b)}(0)\beta_{n-k,\lambda}(x)$$
$$= \sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}^{(a,b-1)}(0)\mathscr{E}_{n-k,\lambda}(x).$$

Now, we observe that

$$(21)\qquad \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}\mathscr{E}_{n-k,\lambda}(x)\right)\frac{t^n}{n!} = \left(\frac{t}{e_\lambda(t)-1}\right)\left(\frac{2}{e_\lambda(t)+1}\right)e_\lambda^x(t)$$
$$= \frac{2t}{e_\lambda^2(t)-1}e_\lambda^x(t) = \frac{2t}{e_{\frac{\lambda}{2}}(2t)-1}e_{\frac{\lambda}{2}}^{\frac{x}{2}}(2t) = \sum_{n=0}^{\infty}\beta_{n,\frac{\lambda}{2}}\left(\frac{x}{2}\right)2^n\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (21),we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$2^n\beta_{n,\frac{\lambda}{2}}\left(\frac{x}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}\mathscr{E}_{n-k,\lambda}(x).$$

From (18), we note that

$$(22)\qquad \left(\frac{t}{e_\lambda(t)-1}\right)^{a}\left(\frac{t}{e_\lambda(t)-1}\right)^{b}e_\lambda^{x+1}(t) - \left(\frac{t}{e_\lambda(t)-1}\right)^{a}\left(\frac{t}{e_\lambda(t)-1}\right)^{b}e_\lambda^{x}(t)$$
$$= t\left(\frac{t}{e_\lambda(t)-1}\right)^{a-1}\left(\frac{t}{e_\lambda(t)-1}\right)^{b}e_\lambda^x(t) = \sum_{n=0}^{\infty}(n+1)T_{n,\lambda}^{(a-1,b)}(x)\frac{t^{n+1}}{(n+1)!}$$
$$= \sum_{n=0}^{\infty}nT_{n-1,\lambda}^{(a-1,b)}(x)\frac{t^n}{n!}.$$

Therefore, by (18) and (22), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$T_{n,\lambda}^{(a,b)}(x+1) - T_{n,\lambda}^{(a,b)}(x) = nT_{n-1,\lambda}^{(a-1,b)}(x). \tag{23}$$

3. Degenerate Sheffer polynomials associated with a random variable

Assume that Y is a random variable whose moment generating function exists in some neighborhood of the origin (see [2, 11, 12, 15, 16, 17, 21, 23]), that is,

$$E\left[e^{Yt}\right] < \infty, \quad (|t| < r), \quad \text{for some } r > 0. \tag{24}$$

Let $(Y_j)_{j\geq 1}$ be a sequence of mutually independent copies of the random variable Y, and let

$$Y^{(k)} = Y_1 + \cdots + Y_k,\ (k \geq 1), \quad \text{with} \quad Y^{(0)} = 0.$$

Now, we define the *degenerate Sheffer polynomials associated with* Y, $S_{n,\lambda}^{Y}(x)$, $(n \geq 0)$, by

$$\frac{1}{E\left[e_{\lambda}^{Y}(t)\right]} e_{\lambda}^{x}(t) = \sum_{n=0}^{\infty} S_{n,\lambda}^{Y}(x) \frac{t^n}{n!}. \tag{25}$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} E\left[S_{n,\lambda}^{Y}(x+Y)\right] \frac{t^n}{n!} &= E\left[\sum_{n=0}^{\infty} S_{n,\lambda}^{Y}(x+Y) \frac{t^n}{n!}\right] \\ &= E\left[\frac{1}{E\left[e_{\lambda}^{Y}(t)\right]} e_{\lambda}^{Y+x}(t)\right] = \frac{e_{\lambda}^{x}(t)}{E\left[e_{\lambda}^{Y}(t)\right]} E\left[e_{\lambda}^{Y}(t)\right] \\ &= e_{\lambda}^{x}(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}. \end{aligned} \tag{26}$$

Therefore, by comparing the coefficients on both sides of (26), we obtain the following theorem.

Theorem 3.1. *For $n \geq 0$, we have*

$$E\left[S_{n,\lambda}^{Y}(x+Y)\right] = (x)_{n,\lambda}.$$

Let Y_1 and Y_2 be independent random variables. Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} S_{n,\lambda}^{Y_1+Y_2}(x+y) \frac{t^n}{n!} &= \frac{1}{E\left[e_{\lambda}^{Y_1+Y_2}(t)\right]} e^{(x+y)t} \\ &= \frac{1}{E\left[e_{\lambda}^{Y_1}(t)\right] E\left[e_{\lambda}^{Y_2}(t)\right]} e^{xt} e^{yt} \\ &= \sum_{k=0}^{\infty} S_{k,\lambda}^{Y_1}(x) \frac{t^k}{k!} \sum_{j=0}^{\infty} S_{j,\lambda}^{Y_2}(y) \frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} S_{k,\lambda}^{Y_1}(x) S_{n-k,\lambda}^{Y_2}(y) \frac{t^n}{n!}. \end{aligned} \tag{27}$$

In particular, for $y=0$ and $Y_2=0$, we get

$$\sum_{n=0}^{\infty} S_{n,\lambda}^{Y_1}(x)\frac{t^n}{n!} = \frac{1}{E\left[e_\lambda^{Y_1}(t)\right]}e_\lambda^x(t) = \sum_{k=0}^{\infty} S_{k,\lambda}^{Y_1}(0)\frac{t^k}{k!}\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}S_{k,\lambda}^{Y_1}(0)(x)_{n-k,\lambda}\frac{t^n}{n!}. \tag{28}$$

Therefore, by (27) and (28), we obtain the following theorem.

Theorem 3.2. *For $n \geq 0$, let Y_1, Y_2 be independent random variables. Then we have*

$$S_{n,\lambda}^{Y_1+Y_2}(x+y) = \sum_{k=0}^{n}\binom{n}{k}S_{k,\lambda}^{Y_1}(x)S_{n-k,\lambda}^{Y_2}(y).$$

In particular, for $y=0$ and $Y_2=0$, we have

$$S_{1,\lambda}^{Y_1}(x) = \sum_{k=0}^{n}\binom{n}{k}S_{k,\lambda}^{Y_1}(0)(x)_{n-k,\lambda}.$$

For $Y \sim U[0,1]$, we have

$$E\left[e_\lambda^Y(t)\right] = \int_0^1 e_\lambda^y(t)dy = \frac{\lambda t}{\log(1+\lambda t)}\frac{1}{t}\big(e_\lambda(t)-1\big). \tag{29}$$

Thus, by (29), we have

$$\sum_{n=0}^{\infty} S_{n,\lambda}^{Y}(x)\frac{t^n}{n!} = \frac{1}{E\left[e_\lambda^{Y}(t)\right]}e_\lambda^x(t) = \frac{\log(1+\lambda t)}{\lambda t}\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{l=0}^{\infty}\frac{(-\lambda)^l l!}{l+1}\frac{t^l}{l!}\sum_{k=0}^{\infty}\beta_{k,\lambda}(x)\frac{t^k}{k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}(x)\frac{(-\lambda)^{n-k}(n-k)!}{n-k+1}\frac{t^n}{n!}. \tag{30}$$

Therefore, by (30), we obtain the following theorem,

Theorem 3.3. *For $Y \sim U[0,1]$, we have*

$$S_{n,\lambda}^{Y}(x) = \sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}(x)\frac{(-\lambda)^{n-k}(n-k)!}{n-k+1}, \quad (n \geq 0).$$

Let $Y \sim \mathrm{Ber}(1/2)$. Then we have

$$E\left[e_\lambda^Y(t)\right] = \frac{1}{2}+\frac{1}{2}e_\lambda(t) = \frac{e_\lambda(t)+1}{2}. \tag{31}$$

By (31), we get

$$\sum_{n=0}^{\infty} S_{n,\lambda}^{Y}(x)\frac{t^n}{n!} = \frac{1}{E\left[e_\lambda^{Y}(t)\right]}e_\lambda^x(t) = \frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 3.4. *For $Y \sim \mathrm{Ber}(1/2)$, and $n \geq 0$, we have*

$$S_{n,\lambda}^{Y}(x) = \mathscr{E}_{n,\lambda}(x).$$

For $Y \sim U[0,1]$, and $m \in \mathbb{N}$, by (29), we have

$$(33)\qquad E\left[e_{\lambda}^{Y^{(m)}}(t)\right] = E\left[e_{\lambda}^{Y_1+Y_2+\cdots+Y_m}(t)\right] = \left(e_{\lambda}(t)-1\right)^m \left(\frac{\lambda}{\log(1+\lambda t)}\right)^m = \left(\frac{e_{\lambda}(t)-1}{t}\right)^m \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^m.$$

By (33), we get

$$(34)\qquad \sum_{n=0}^{\infty} S_{n,\lambda}^{Y^{(m)}}(x)\frac{t^n}{n!} = \frac{e_{\lambda}^{x}(t)}{E\left[e_{\lambda}^{Y^{(m)}}(t)\right]} = \left(\frac{\log(1+\lambda t)}{\lambda t}\right)^m \left(\frac{t}{e_{\lambda}(t)-1}\right)^m e_{\lambda}^{x}(t)$$
$$= \frac{m!}{\lambda^m}\sum_{j=m}^{\infty} S_1(j,m)\lambda^j \frac{t^{j-m}}{j!}\sum_{k=0}^{\infty}\beta_{k,\lambda}^{(m)}(x)\frac{t^k}{k!}$$
$$= \sum_{j=0}^{\infty}\frac{S_1(j+m,m)\lambda^j}{\binom{j+m}{m}}\frac{t^j}{j!}\sum_{k=0}^{\infty}\beta_{k,\lambda}^{(m)}(x)\frac{t^k}{k!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_1(n-k+m,m)\lambda^{n-k}\frac{\binom{n}{k}}{\binom{n-k+m}{m}}\beta_{k,\lambda}^{(m)}(x)\frac{t^n}{n!},$$

where $S_1(j,m)$ are the Stirling number of the first kind defined by

$$\frac{1}{k!}\log^k(1+t) = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (k \ge 0).$$

Therefore, by (34), we obtain the following theorem.

Theorem 3.5. *For $Y \sim U[0,1]$, and $m,n \ge 0$, we have*

$$S_{n,\lambda}^{Y^{(m)}}(x) = \sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{n-k+m}{m}}\lambda^{n-k}S_1(n-k+m,m)\beta_{k,\lambda}^{(m)}(x).$$

Let m,l be integers with $m \ge l \ge 0$. Now, by (25) and (26), we observe that

$$(35)\qquad \sum_{n=0}^{\infty} E\left[S_{n,\lambda}^{Y^{(m)}}\left(x+Y^{(l)}\right)\right]\frac{t^n}{n!} = \frac{e_{\lambda}^{x}(t)}{E\left[e_{\lambda}^{Y^{(m)}}(t)\right]}E\left[e_{\lambda}^{Y^{(l)}}(t)\right] = \frac{e_{\lambda}^{x}(t)}{E\left[e_{\lambda}^{Y^{(m)}-Y^{(l)}}(t)\right]} = \sum_{n=0}^{\infty} S_{n,\lambda}^{Y^{(m)}-Y^{(l)}}(x)\frac{t^n}{n!}.$$

Thus, by (35), we get

$$(36)\qquad E\left[S_{n,\lambda}^{Y^{(m)}}\left(x+Y^{(l)}\right)\right] = S_{n,\lambda}^{Y^{(m)}-Y^{(l)}}(x), \quad (n \ge 0,\ m \ge l \ge 0).$$

Therefore, by (36) and Theorem 3.5, we obtain the following theorem.

Theorem 3.6. *For $n \ge 0$, and $Y \sim U[0,1]$, we have*

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{n-k+m}{m}}\lambda^{n-k}S_1(n-k+m,m)E\left[\beta_{n,\lambda}^{(m)}\left(x+Y^{(l)}\right)\right]$$
$$= \sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{n-k+m-l}{m-l}}\lambda^{n-k}S_1(n-k+m-l,m-l)\beta_{k,\lambda}^{(m-l)}(x),$$

where m,l are integers with $m \ge l \ge 0$.

For $Y \sim \mathrm{Ber}(1/2)$, we have

$$\sum_{n=0}^{\infty} S_{n,\lambda}^{Y^{(m)}}(x)\frac{t^n}{n!} = \frac{e_\lambda^x(t)}{E\left[e_\lambda^{Y^{(m)}}(t)\right]} = \frac{1}{E\left[e_\lambda^{Y_1+\cdots+Y_m}(t)\right]}e_\lambda^x(t) \tag{37}$$
$$= \frac{1}{E\left[e_\lambda^{Y_1}(t)\right]\cdots E\left[e_\lambda^{Y_m}(t)\right]}e_\lambda^x(t) = \left(\frac{2}{e_\lambda(t)+1}\right)^m e_\lambda^x(t)$$
$$= \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}^{(m)}(x)\frac{t^n}{n!}, \quad (m \geq 0).$$

By (37), we get

$$S_{n,\lambda}^{Y^{(m)}}(x) = \mathscr{E}_{n,\lambda}^{(m)}(x). \tag{38}$$

Therefore, by (36) and (38), we obtain the following theorem.

Theorem 3.7. *For $Y \sim \mathrm{Ber}(1/2)$, and $n \geq 0$, we have*

$$E\left[S_{n,\lambda}^{Y^{(m)}}\left(x+Y^{(l)}\right)\right] = \mathscr{E}_{n,\lambda}^{(m-l)}(x),$$

where m,l are integers with $m \geq l \geq 0$.

For $Y \sim \mathrm{Ber}(1/2)$, we have

$$\sum_{n=0}^{\infty} E\left[T_{n,\lambda}^{(a,b)}(x+Y)\right]\frac{t^n}{n!} = E\left[\sum_{n=0}^{\infty} T_{n,\lambda}^{(a,b)}(x+Y)\frac{t^n}{n!}\right] \tag{39}$$
$$= \left(\frac{t}{e_\lambda(t)-1}\right)^a\left(\frac{2}{e_\lambda(t)+1}\right)^b e_\lambda^x(t)E\left[e_\lambda^Y(t)\right]$$
$$= \left(\frac{t}{e_\lambda(t)-1}\right)^a\left(\frac{2}{e_\lambda(t)+1}\right)^b e_\lambda^x(t)\frac{e_\lambda(t)+1}{2}$$
$$= \left(\frac{t}{e_\lambda(t)-1}\right)^a\left(\frac{2}{e_\lambda(t)+1}\right)^{b-1} e_\lambda^x(t)$$
$$= \sum_{n=0}^{\infty} T_{n,\lambda}^{(a,b-1)}(x)\frac{t^n}{n!}.$$

On the other hand, by Theorem 2.5, we get

$$E\left[T_{n,\lambda}^{(a,b)}(x+Y)\right] = \sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}^{(m,l)}(0)E\left[(x+Y)_{n-k,\lambda}\right] \tag{40}$$
$$= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}^{(a,b)}(0)\left((x+1)_{n-k,\lambda}+(x)_{n-k,\lambda}\right)$$
$$= \frac{1}{2}\left(T_{n,\lambda}^{(a,b)}(x+1)+T_{n,\lambda}^{(a,b)}(x)\right).$$

From (39) and (40), we obtain the following theorem.

Theorem 3.8. *For $n \geq 0$, we have*

$$T_{n,\lambda}^{(a,b-1)}(x) = \frac{1}{2}\left(T_{n,\lambda}^{(a,b)}(x+1)+T_{n,\lambda}^{(a,b)}(x)\right), \tag{41}$$

where $a,b \in \mathbb{R}$.

From (23) and (41), we derive the following theorem.

Theorem 3.9. *For $n \geq 0$, we have*

$$T_{n,\lambda}^{(a,b)}(x) = T_{n,\lambda}^{(a,b-1)}(x) - \frac{n}{2} T_{n-1,\lambda}^{(a-1,b)}(x), \tag{42}$$

where $a, b \in \mathbb{R}$.

By Theorem 2.5 and (42), we get

(43)

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}^{(a)}(x) \mathscr{E}_{n-k,\lambda}^{(b)}(y) = T_{n,\lambda}^{(a,b)}(x+y) = T_{n,\lambda}^{(a,b-1)}(x+y) - \frac{n}{2} T_{n-1}^{(a-1,b)}(x+y) \\
&= \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}^{(a)}(x) \mathscr{E}_{n-k,\lambda}^{(b-1)}(y) - \frac{n}{2} \sum_{k=0}^{n-1} \binom{n-1}{k} \beta_{k,\lambda}^{(a-1)}(x) \mathscr{E}_{n-k-1,\lambda}^{(b)}(y) \\
&= \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}^{(a)}(x) \mathscr{E}_{n-k,\lambda}^{(b-1)}(y) - \sum_{k=0}^{n} \binom{n}{k} \frac{k}{2} \beta_{k-1,\lambda}^{(a-1)}(x) \mathscr{E}_{n-k,\lambda}^{(b)}(y).
\end{aligned}$$

From (43), we obtain the following theorem.

Theorem 3.10. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \beta_{n,\lambda}^{(a)}(x) \mathscr{E}_{n-k,\lambda}^{(b-1)}(y) = \sum_{k=0}^{n} \binom{n}{k} \left[\beta_{n,\lambda}^{(a)}(x) + \frac{k}{2} \beta_{k-1,\lambda}^{(a-1)}(x) \right] \mathscr{E}_{n-k,\lambda}^{(b)}(y),$$

where $a, b \in \mathbb{R}$.

Let $b = 1$ in Theorem 3.10. Then we have

$$\sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}^{(a)}(x) (y)_{n-k,\lambda} = \sum_{k=0}^{n} \binom{n}{k} \left[\beta_{n,\lambda}^{(a)}(x) + \frac{k}{2} \beta_{k-1,\lambda}^{(a-1)}(x) \right] \mathscr{E}_{n-k,\lambda}(y). \tag{44}$$

Thus, by (44) and Corollary 2.2, we get

$$\beta_{n,\lambda}^{(a)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \left[\beta_{k,\lambda}^{(a)}(x) + \frac{k}{2} \beta_{k-1,\lambda}^{(a-1)}(x) \right] \mathscr{E}_{n-k,\lambda}(y). \tag{45}$$

Now, we observe that

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k}{2} \beta_{k-1,\lambda}^{(a-1)}(x) \mathscr{E}_{n-k,\lambda}^{(b)}(y) = \frac{n}{2} \sum_{k=0}^{n-1} \binom{n-1}{k} \beta_{k,\lambda}^{(a-1)}(x) \mathscr{E}_{n-k-1,\lambda}^{(b)}(y). \tag{46}$$

Let $a = 1$ in (46). Then, by Corollary 2.2, we have

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \frac{k}{2} \beta_{k-1,\lambda}^{(0)}(x) \mathscr{E}_{n-k,\lambda}^{(b)}(y) &= \frac{n}{2} \sum_{k=0}^{n-1} \binom{n-1}{k} (x)_{k,\lambda} \mathscr{E}_{n-1-k,\lambda}^{(b)}(y) \\
&= \frac{n}{2} \mathscr{E}_{n-1,\lambda}^{(b)}(x+y).
\end{aligned} \tag{47}$$

By Theorem 3.10 with $a = 1$ and (47), we get

(48)

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}(x) \mathscr{E}_{n-k,\lambda}^{(b)}(y) &= \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}(x) \mathscr{E}_{n-k,\lambda}^{(b-1)}(y) - \sum_{k=0}^{n} \binom{n}{k} \frac{k}{2} \beta_{k-1,\lambda}^{(0)}(x) \mathscr{E}_{n-k,\lambda}^{(b)}(y) \\
&= \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}(x) \mathscr{E}_{n-k,\lambda}^{(b-1)}(y) - \frac{n}{2} \mathscr{E}_{n-1,\lambda}^{(b)}(x+y).
\end{aligned}$$

From (48), we note that

$$\frac{n}{2}\mathscr{E}^{(b)}_{n-1,\lambda}(x+y) = \sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}(x)\Big(\mathscr{E}^{(b-1)}_{n-k,\lambda}(y) - \mathscr{E}^{(b)}_{n-k,\lambda}(y)\Big) \tag{49}$$
$$= \sum_{k=0}^{n}\binom{n}{k}\beta_{n-k,\lambda}(x)\Big(\mathscr{E}^{(b-1)}_{k,\lambda}(y) - \mathscr{E}^{(b)}_{k,\lambda}(y)\Big),$$

where $n \geq 0$.
Thus, for $n \geq 1$, from (49) we have

$$\mathscr{E}^{(b)}_{n-1,\lambda}(x+y) = \frac{2}{n}\sum_{k=0}^{n}\binom{n}{k}\beta_{n-k,\lambda}(x)\Big(\mathscr{E}^{(b-1)}_{k,\lambda}(y) - \mathscr{E}^{(b)}_{k,\lambda}(y)\Big) \tag{50}$$
$$= \sum_{k=1}^{n}\binom{n-1}{k-1}\frac{2}{k}\beta_{n-k,\lambda}(x)\Big(\mathscr{E}^{(b-1)}_{k,\lambda}(y) - \mathscr{E}^{(b)}_{k,\lambda}(y)\Big)$$
$$= \sum_{k=0}^{n-1}\frac{2}{k+1}\binom{n-1}{k}\beta_{n-1-k,\lambda}(x)\Big(\mathscr{E}^{(b-1)}_{k+1,\lambda}(y) - \mathscr{E}^{(b)}_{k+1,\lambda}(y)\Big).$$

Therefore, by (45) and (50), we obtain the following theorem.

Theorem 3.11. *For $a, b \in \mathbb{R}$ and $n \geq 0$, we have*

$$\beta^{(a)}_{n,\lambda}(x+y) = \sum_{k=0}^{n}\binom{n}{k}\left[\beta^{(a)}_{k,\lambda}(x) + \frac{k}{2}\beta^{(a-1)}_{k-1,\lambda}(x)\right]\mathscr{E}_{n-k,\lambda}(y),$$
$$\mathscr{E}^{(b)}_{n,\lambda}(x+y) = \sum_{k=0}^{n}\binom{n}{k}\frac{2}{k+1}\beta_{n-k,\lambda}(x)\Big(\mathscr{E}^{(b-1)}_{k+1,\lambda}(y) - \mathscr{E}^{(b)}_{k+1,\lambda}(y)\Big).$$

4. Conclusion

In this paper, we studied a hybrid of higher-order degenerate Bernoulli and Euler polynomials, namely the degenerate Sheffer-type polynomials. Then, we examined a degenerate form of the Appell polynomials associated with a random variable Y, namely the degenerate Sheffer polynomials associated with Y.

We notice from (25) and (31) that, for $Y \sim \mathrm{Ber}(1/2)$, we have

$$\frac{2}{e_\lambda(t)+1}e^x_\lambda(t) = \sum_{n=0}^{\infty} S^{\mathrm{Ber}(1/2)}_{n,\lambda}(x)\frac{t^n}{n!},$$

so that $S^{\mathrm{Ber}(1/2)}_{n,\lambda}(x) = \mathscr{E}_{n,\lambda}(x)$, the degenerate Euler polynomials in (5).
While, for $Y \sim U[0,1]$, we observe from (25) and (29) that

$$\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}e^x_\lambda(t) = \sum_{n=0}^{\infty} S^{U[0,1]}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{51}$$

so that $S^{U[0,1]}_{n,\lambda}(x)$ are the 'degenerate Bernoulli polynomials arising from Volkenborn integral' (see [9]). These polynomials are different from the Carlitz degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$ in (3) and (4). However, as $\lambda \to 0$, both $\beta_{n,\lambda}(x)$ and $S^{U[0,1]}_{n,\lambda}(x)$ tend to the Bernoulli polynomial $B_n(x)$. For details on this, the interested reader may refer to [9].

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7. Data availability

No data was used in this review paper.

8. Ethics approval

This review paper does not contain any content requiring ethics approval or prior informed consent.

9. Conflict of interest

The authors declare no conflicts of interest.

References

[1] Abramowitz, M.; Stegun, I. A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* Dover, New York, 1992.

[2] Adell, J. A.; Lekuona, A. *Explicit expressions for a certain subset of Appell polynomials: a probabilistic perspective,* Integers **21** (2021), Paper No. A60, 10 pp.

[3] Aydin, M. S.; Acikgoz, M.; Araci, S. *A new construction on the degenerate Hurwitz-zeta function associated with certain applications,* Proc. Jangjeon Math. Soc. **25** (2022), no. 2, 195-203.

[4] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51-88.

[5] Carlitz, L. *A degenerate Staudt-Clausen theorem,* Arch. Math. (Basel) 7 (1956), 28-33.

[6] Comtet, L. *Advanced Combinatorics, The Art of Finite and Infinite Expansions,* Revised and enlarged edition, D. Reidel Publishing Co., Dordrecht, 1974.

[7] Kim, D.S., Kim, T. *Combinatorial Identities Related to Degenerate Stirling Numbers of the Second Kind,* Proc. Steklov Inst. Math. **330** (2025), no. 1, 176–192 (2025). https://doi.org/10.1134/S0081543825600279.

[8] Kim, T.; Kim, D. S. *Heterogeneous Stirling numbers and heterogeneous Bell polynomials,* Russ. J. Math. Phys. **32** (2025), no. 3, 498–509.

[9] Kim, T.; Kim, D. S. *Degenerate Euler–Seidel matrix method and their applications,* Math. Methods Appl. Sci. 2026, https://doi.org/10.1002/mma.70468.

[10] Kim, D. S.; Kim, T. *A note on a new type of degenerate Bernoulli numbers,* Russ. J. Math. Phys. **27** (2020), no. 2, 227-235.

[11] Kim, T.; Kim, D. S. *Explicit formulas for probabilistic multi-poly-Bernoulli polynomials and numbers,* Russ. J. Math. Phys. **31** (2024), no. 3, 450-460.

[12] Kim, T.; Kim, D. S. *Spivey-type recurrence relations for degenerate Bell and Dowling polynomials,* Russ. J. Math. Phys. **32** (2025), no. 2, 288-296.

[13] Kim, T.; Kim, D. S. *Degenerate hyperharmonic polynomials and numbers,* Filomat **40** (2026), no. 1, 1411–1422.

[14] Leon-Garcia, A. *Probability and Random Processes for Electronic Engineering,* 2nd ed., Addison-Wesley, New York, 1994.

[15] Liu, W.; Kim, T.; Kim, D. S.; Ma, Y. *Probabilisitc degenerate Bernoulli and Euler polynomials of complex variable,* Fractals (2025), 2540167, 11 pp.

[16] Liu, X.; Ma, Y.; Kim, T.; Kim, D. S.; Hei, Y. *Probabilistic multi-Stirling numbers of the second kind associated with random variables,* Appl. Math. Sci. Eng. **33** (2025), no. 1, Paper No. 2561683.

[17] Ma, Y.; Kim, T.; Kim, D. S.; Xu, R. *Recurrence for probabilistic extension of Dowling polynomials,* Open Math. **23** (2025), no. 1, Paper No. 20250150, 7 pp.
[18] Roman, S. *The umbral calculus,* Pure and Applied Mathematics 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
[19] Ross, S. M. *Introduction to Probability Models,* Twelfth edition, Academic Press, London, 2019.
[20] Salminen, P. *Optimal stopping, Apell polynomials, and Wiener-Hopf factorization,* Stochastics 83 (2011), no. 4-6, 611-622.
[21] Soni, R.; Vellaisamy, P.; Pathak, A. K. *A probabilistic generalization of the Bell polynomials,* J. Anal. **32** (2024), no. 2, 711–732.
[22] Ta, B. Q. *A note on the generalized Bernoulli and Euler Polynomials,* European J. Pure and Appl. Math. **6** (2013), no. 4, 405-412.
[23] Wang, Q.; Hei, Y.; Dolgy, D. V.; Kim, T.; Kim, D. S. *Some identities associated with degenerate harmonic and hyperharmonic numbers,* Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, Paper No.1, 2545196.
[24] Xue, P.; Ma, Y.; Kim, T.; Kim, D. S.; Zhang, W. *Probabilistic degenerate poly-Bell polynomials associated with random variables,* Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, Paper No. 2497367, 17 pp.

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