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ROUND NUMBERS ARE RARE

S SRINIVAS RAU AND BIDARAHALLI UMA

Abstract. G.H.Hardy and S.Ramanujan observed that round numbers (those with few prime factors repeated many times) are very rare. We seek to make the concepts of 'roundedness" and "rarity" precise by introducing k-round and k-Round numbers. We exploit density results for numbers with specified numbers of prime divisors. We discuss alternative approaches using i) normal orders of additive functions and ii) square-full numbers.

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1. Introduction

G.H. Hardy's *Ramanujan: Twelve Lectures* [2] is a superb book. In the 8th lecture, Hardy discusses "round numbers". These are positive integers with (relatively) few prime factors repeated a large number of times. Both Hardy and Ramanujan observed that round numbers are very rare.

Numbers such as:

$$1200 = 2^4 \cdot 3 \cdot 5^2, \quad 2187 = 3^7$$

are round numbers.

Hardy and Ramanujan's efforts led to very important results on the behavior of $d(n)$, the divisor function [3] . In particular, their common asymptotic value $\log\log x$ for the mean of $\Omega(n)$ and $\omega(n)$ (defined below) inspired Probabilistic Number Theory ([3],[1]).

In the absence of a precise definition of round numbers, we introduce criteria for *k-round* and *k-Round* numbers. We show rarity in terms of small density of k-Round numbers (Proposition 1). Two other approaches are discussed.

(1) Using the normal order of $\omega(n)$ and $\Omega(n)$.
(2) Square-full numbers as a source of round numbers.

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In [4], the means of

$$\left\{\frac{2^{\omega(n)}}{(\log x)^{\log 2}}\right\}, \quad \text{and} \quad \left\{\frac{(\log x)^{\log 2}}{d(n)}\right\}$$

were shown to be $cx(\log x)^{0.31}$ and $ax(\log x)^{0.19}$, respectively. So the normal order is closer to $2^{\omega(n)}$ than to $d(n)$ on average.

The bias is towards square-freeness (the opposite of roundness according to Hardy [2]).

$k-$*round* and k-Round Numbers

We recall certain definitions due to Hardy-Ramanujan:

(1) $\omega(n)$:=number of distinct prime divisors of n.
(2) $\Omega(n) :=$ number of prime divisors of n, counted with multiplicity.
(3) $d(n) :=$ number of divisors of n.
(4) $f(n)$ has normal order $g(n)$ if given $\epsilon > 0$:

$$\left|\frac{f(n)}{g(n)} - 1\right| < \epsilon$$

for almost all n (i.e., except for n in a negligible set).

(5) The density $d(S)$ of a set S of numbers is defined by

$$d(S) = \lim_{x\to\infty} \frac{1}{x} \sum_{s\leq x, s\epsilon S} 1$$

if the limit exists.

(6) A set S is called *negligible* if

$$d(S) = \lim_{x\to\infty} \frac{1}{x} \sum_{s\leq x, s\epsilon S} 1 = 0.$$

Intuitively, n is *round* if $\omega(n)$ is small and $d(n)$ (or $\Omega(n)$) is large ([2], p48). We seek to formalize this.

Definition 1. A number n is *k-round* if

$$(\Omega(n) - \omega(n)) \geq (k+1), \quad k = 1, 2, 3, \ldots$$

The motivation for this definition is the set identified in [1] , Example 6.16, p. 84 (extended from 2 to k); no power higher than 2 divides n:

$$S_2 = \{n \mid n = p_1^{t_1} p_2^{t_2} \ldots p_m^{t_m}, \quad t_m \leq 2 \quad \forall m\}.$$

S_k analogously has maximum prime power k in the factorization of its members ($t_m \leq k$).

Lemma 1.1. *The density of S_k denoted by $d(S_k)$ is given by:*

$$d(S_k) = \frac{1}{\zeta(k+1)}.$$

([1], p84)

Proof. Let χ be the characteristic function of S_k, then:

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{ks}}\right)$$

which simplifies to:

$$\frac{\zeta(s)}{\zeta((k+1)s)}.$$

By standard evaluation ([5],p31), the density of S_k is given by:

$$d(S_k) = \frac{1}{\zeta(k+1)} = \text{residue of } \frac{\zeta(s)}{\zeta(k+1)s}$$

(Wiener-Ikehara Theorem). □

Remark 1: A k-round number lies in the set complement of S_k. If R_k is the set of k-round numbers, then their density satisfies:

$$d(R_k) < \left(1 - \frac{1}{\zeta(k+1)}\right)$$

and R_{k+1} is contained in R_k.

Remark 2: The $d(R_k) \to 0$ as $k \to \infty$ since $\zeta(k+1) \to 1$ as $k \to \infty$.

$$\zeta(k+1) = 1 + \frac{1}{2^{k+1}} + \frac{1}{3^{k+1}} + \frac{1}{n^{k+1}} + \ldots.$$

This sequence decreases monotonically to 1 as $k \to \infty$.

Remark 3:

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} (\Omega(n) - \omega(n)) = \sum_p \frac{1}{p(p-1)} < \infty.$$

([1],p87,6.18)

Corollary 1.2. *k-round numbers are scarce as the corresponding summands in the above limit do not cause divergence.*

Definition 2: A number n is *k-Round* if

$$\Omega(n) - \omega(n) \ge k\omega(n).$$

Remark 4: If n is k-Round, then it is $(k-2)$-round since:

$$\Omega(n) - k\omega(n) \ge 0 \Rightarrow \Omega(n) - \omega(n) \ge (k-1)\omega(n) \ge k-1.$$

Proposition 1.3.

$$\frac{1}{x} \sum_{n \le x} \frac{\omega(n)}{d(n)} \le \frac{1}{x} \sum_{n \le x} \frac{\omega(n)}{\Omega(n)} \le \frac{1}{k} \left(\frac{1}{x} \sum_{n \le x} 1 \right)_{\textit{k-round}}$$

Letting $x \to \infty$,

$$\textit{Mean} \left\{ \frac{\omega(n)}{d(n)} \right\}_{\textit{k-round}} \le \textit{Mean} \left\{ \frac{\omega(n)}{\Omega(n)} \right\}_{\textit{k-round}} \le \frac{1}{k} \left(1 - \frac{1}{\zeta(k+1)}\right)$$

Proof: Using the definition of k-Round and Remark 1.

Corollary 2: k-Round numbers become rarer and rarer as $k \to \infty$ by the upper bound in Proposition 1. ("Rounder numbers are rarer.")

Corollary 3: The density of R_k tends to 0 as $k \to \infty$ since it is bounded by

$$\left(1 - \frac{1}{\zeta(k+1)}\right)$$

and as seen above this difference tends to 0.

Normal Order Approach

Hardy-Ramanujan ([2],[1], p86) proved that $\log\log x$ is the normal order of both $\omega(n)$ and $\Omega(n)$: Given $\epsilon > 0$, except for a negligible set of natural numbers, one has,

$$\left|\frac{\omega(n)}{\log\log n} - 1\right| < \epsilon, \quad \left|\frac{\Omega(n)}{\log\log n} - 1\right| < \epsilon.$$

Or equivalently one has,

$$1 - \epsilon < \frac{\omega(n)}{\log\log n} < 1 + \epsilon, \quad 1 - \epsilon < \frac{\Omega(n)}{\log\log n} < 1 + \epsilon.$$

So we have

$$\frac{1}{1+\epsilon} < \frac{\log\log n}{\Omega(n)} < \frac{1}{1-\epsilon}.$$

Given $\epsilon > 0$, except for a negligible set,

$$\left|\frac{\omega(n)}{\Omega(n)}\right| = \left|\frac{\omega(n)}{\log\log n}\frac{\log\log n}{\Omega(n)}\right| < \frac{1+\epsilon}{1-\epsilon}$$

$$= (1+\epsilon)(1+\epsilon+\epsilon^2+\dots).$$

Thus, this ratio is close to 1 except for a negligible set in which the ratio is close to 0. This latter set consists of the round numbers identified by Hardy-Ramanujan (though imprecisely).

Now we note that $\Omega(n) < d(n)$ (the prime power divisors are fewer than all divisors). Hence, the above discussion holds for $\frac{\omega(n)}{d(n)}$ as well.

III Square-full Numbers

Let

$$S = \{n/p \mid n \Rightarrow p^2 \mid n \text{ for any prime } p\}.$$

S consists of the **square-full numbers** ([?] p49, Ex 3.12).

It is clear that round numbers have a big overlap with S. Now,

$$\sum_{n \in S} \frac{1}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}, \quad (\Re s > 1/2)$$

Hence, by Kronecker's Lemma,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x, n \epsilon S,} 1 = 0,$$

i.e., the density of square-full numbers is 0. Thus, the subset of round numbers also has density 0. Perturbation of a square-full number n by multiplication by a prime p does not change $\Omega(n) - \omega(n)$, so k-round numbers 'n' give k-round numbers 'pn'.

References

[1] Jean-Marc De Koninck & Florian Luca, *Analytic Number Theory*, AMS Graduate Studies in Mathematics 134, 2012.

[2] G.H. Hardy, *Ramanujan: Twelve Lectures*, Cambridge University Press, 1940.

[3] M. Ram Murty, *Legacy of Ramanujan* (Lecture on the 125th Birth Anniversary of Ramanujan).

[4] G. Sudhaansh Mohan Reddy, S. Srinivas Rau, B. Uma, *A Remark on Hardy-Ramanujan's Approximation of Divisor Functions*, International Journal of Pure and Applied Mathematics, 118(4), 2018, pp. 797-799.

[5] G. Sudhaansh Mohan Reddy, S. Srinivas Rau, B. Uma, *Applications of Tauberian Theorems to Dirichlet Series*, Lambert Academic Publishing, 2019. ISBN 978-613-9-45507-2.

Chanakya University, Bengaluru
Email address: satyavolu.s@chanakyauniversity.edu.in

Chanakya University, Bengaluru
Email address: uma.b@chanakyauniversity.edu.in