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ON THE CONSTRUCTION OF SOFT TRIOIDS

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ABSTRACT. Molodtsov's soft set theory offers a general framework for parameterized uncertainty, while trioids algebraic structures with three linked binary operations, generalizing dimonoids and semigroups arise naturally from operadic and polytopal contexts. This paper initiates the study of soft trioids, combining these two theories. We introduce the fundamental notion of a soft trioid over a trioid, establish its basic properties, and provide illustrative examples. We then investigate algebraic operations such as restricted and extended intersections, proving closure properties. The concepts of soft sub-trioids and soft homomorphisms are developed, leading to a categorical framework. A triband of soft subtrioids is defined and a corresponding decomposition theorem is presented, elucidating the internal structure of soft trioids. This work extends the theory of soft dimonoids and suggests new applications of parameterized algebraic structures in computational science and information processing.

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1. INTRODUCTION

The interplay between algebraic rigor and the need to model uncertainty or context-dependent behavior constitutes a central challenge in modern mathematics and its applications. While classical algebraic structures provide powerful tools for analyzing deterministic systems, many real-world problems—spanning decision sciences, information processing, and computational logic—involve inherent imprecision, partial data, or parameter-sensitive operations. This gap has driven the development of hybrid frameworks capable of integrating algebraic laws with flexible, adaptive parameterization. A significant advance in uncertainty modeling was achieved by Molodtsov [23] through the introduction of soft set theory. Departing from fuzzy or rough set approaches, which require auxiliary structures such as membership functions or approximation spaces, soft sets operate via parameterized families of subsets. Formally, a soft set over a universe $\mathcal{U}$ is a pair (Ω, K), where K is a set of parameters and $\Omega : K \to \mathcal{P}(\mathcal{U})$ assigns to each parameter a subset of $\mathcal{U}$. This minimalist yet versatile paradigm has since been applied successfully across diverse mathematical disciplines, including algebra [27, 9], topology [32, 12], and decision theory [18, 19]. Recent expansions include the study of soft topological transformation groups [28], soft groupoids [29], and soft BCH-algebras [12], further demonstrating the adaptability of the soft set framework to structured algebraic contexts. Broader algebraic extensions have also been explored, such as commutative hyperrings [1].

In parallel, the theory of associative algebraic structures has been extended through the study of multi-operational systems. Originating in the work of Loday on dialgebras [15] and their Gröbner–Shirshov bases [3], dimonoids $(D, \dashv, \vdash)$—equipped with two associative binary operations satisfying three coherence axioms—generalize semigroups and have deep connections to operad theory and homological algebra, with further studies on their varieties and conformal analogs [13]. This line of inquiry naturally culminates in trioids $(T, \dashv, \vdash, \perp)$, introduced by Loday and Ronco [16] in the context of operads associated with polytopes and linked to Leibniz 3-algebras [4]. A trioid is endowed with three associative operations linked by eight fundamental axioms, providing the set-theoretic foundation for trialgebras and exhibiting a rich structure theory. This framework connects to Rota-Baxter algebraic structures [8, 30] and dendriform trialgebras [25, 26]. As comprehensively documented by Zhuchok [45, 42, 44], trioids admit intricate decomposition theorems (tribands of subtrioids, semilattice decompositions)—concepts rooted in classical semigroup band theory [5, 6] and semilattice decomposition [33, 35, 31]—detailed congruence lattices building on work concerning idempotent semigroups [21] and congruences on regular and orthodox semigroups [24, 22, 20, 7],

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and sophisticated free constructions. The integration of soft set theory with such multi-operational algebras remains largely unexplored, representing a meaningful gap in the literature. Initial steps were taken by Oguz [27] with the introduction of soft dimonoids, effectively bridging soft sets with two-operation algebras. However, the more complex and structurally richer domain of trioids has not yet been examined through the lens of parameterized uncertainty. This omission is noteworthy, as the soft set framework offers a natural language for capturing how algebraic properties—encoded by trioid operations—may depend on external parameters, varying across contexts, scales, or observational modes.

The present work is motivated by the need to unify the parameterized, uncertainty-aware modeling of soft sets with the rich algebraic architecture of trioids. We introduce and systematically develop the theory of soft trioids, thereby achieving a two-fold generalization: extending soft dimonoids [27] to the three-operation setting, and enriching classical trioid theory [16, 45] with a parameterized family of substructures. This synthesis is driven by potential applications in areas requiring multi-level algebraic modeling—such as hierarchical access control, multi-criteria decision systems, modular software verification, and resource allocation under varying constraints—where operations and their properties may change depending on situational parameters. The theoretical underpinnings draw from and extend fundamental concepts in semigroup theory [17, 10], including semiretractions [34, 36], orthogonal sums [2], and algebras whose equivalences are congruences [14].

We develop the foundational theory of soft trioids and establish a comprehensive set of structural results. First, we define a soft trioid $(\mathcal{T}, \Omega, K)$ over a base trioid $\mathcal{T}$ as a soft set whose every fiber $\Omega(\alpha)$ is a sub-trioid of $\mathcal{T}$. A range of illustrative examples, including constructions from semigroups with endomorphisms [45], free trioids [45], and rectangular semigroups, demonstrates the versatility of the concept. Next, we investigate fundamental algebraic operations on soft trioids, including direct products, restricted and extended intersections, and the novel $\wedge$-intersection. We prove comprehensive closure theorems, showing that the category of soft trioids is stable under these constructions (Theorems 3.2, 3.5, 3.7). We then establish the category-theoretic foundations by introducing soft sub-trioids and soft homomorphisms, which lead to the definition of the category SoftTrioid. We prove that this category is complete and cocomplete, possesses a lattice of subobjects, and is equipped with natural forgetful and free functors (Theorems 4.10, 4.11, 4.14). A central contribution is the decomposition theory of soft trioids into simpler components. By introducing the concept of a triband of soft sub-trioids, we prove a structure theorem (Theorem 5.5) showing that every soft trioid decomposes essentially uniquely into a triband of soft simple sub-trioids. This generalizes Zhuchok's classical decomposition [45, 44] and draws on principles of semigroup bands [5]. Specializing to idempotent fibers, we obtain a soft semilattice decomposition into soft unipotent components (Theorem 5.9), extending classical semilattice decomposition theories [33, 35, 31]. Finally, we extend congruence theory to the soft setting by defining soft congruences (ρ_K, ρ_T) and soft semiretractions (τ_K, τ_T), the latter generalizing notions introduced for monoids [34], dimonoids [36], and trioids [46]. We establish a bijective correspondence between symmetric soft semiretractions and soft congruences (Theorem 6.11) and prove isomorphism theorems for quotient structures (Theorems 6.14, 6.15). By synthesizing the parameterized approach of soft sets with the ternary operadic structure of trioids, this work not only advances pure algebra—extending results on dimonoids [39, 37, 38, 40, 41, 43] and trioids [45, 42, 44]—but also provides a rigorous, category-based framework for modeling and analyzing complex systems where algebraic rules are context-dependent. The framework connects to broader mathematical landscapes, including vertex algebras [11] and the study of algebras with lattice structures [**?**]. It also aligns with emerging work on soft structures for fields and monoids [**?**]. This lays the groundwork for future research in applied algebra, computational logic, and the study of parameterized operadic structures.

2. Preliminaries

We recall the essential definitions that form the foundation of our work.

Definition 2.1. [23] *Let $\mathcal{U}$ be a universal set and E be a set of parameters. Also, $\mathcal{P}(\mathcal{U})$ denotes the power set of $\mathcal{U}$. A soft set over $\mathcal{U}$ is a pair (Ω, K) where $K \subseteq E$ and $\Omega : K \to \mathcal{P}(\mathcal{U})$ is a mapping. In adittion, the support of a soft set (Ω, K) is defined as $Supp(\Omega, K) = \{\alpha \in K : \Omega(\alpha) \neq \emptyset\}$. A soft set is called* non-null *if its support is non-empty.*

Definition 2.2. [16] *A trioid is a quadruple $(T, \dashv, \vdash, \perp)$ where T is a nonempty set equipped with three binary operations satisfying the following axioms for all $x, y, z \in T$:*

$$\text{(T1) } (x \dashv y) \dashv z = x \dashv (y \vdash z)$$
$$\text{(T2) } (x \vdash y) \dashv z = x \vdash (y \dashv z)$$
$$\text{(T3) } (x \dashv y) \vdash z = x \vdash (y \vdash z)$$
$$\text{(T4) } (x \dashv y) \dashv z = x \dashv (y \perp z)$$
$$\text{(T5) } (x \perp y) \dashv z = x \perp (y \dashv z)$$
$$\text{(T6) } (x \dashv y) \perp z = x \perp (y \vdash z)$$
$$\text{(T7) } (x \vdash y) \perp z = x \vdash (y \perp z)$$
$$\text{(T8) } (x \perp y) \vdash z = x \vdash (y \vdash z)$$

Definition 2.3. [16]] *A subset $S \subseteq T$ is called a subtrioid if it is closed under all three operations. A mapping $f : T_1 \to T_2$ between trioids is a homomorphism if it preserves all three operations.*

The richness of trioid theory is evidenced by several remarkable constructions. Of particular importance for our work are:

Example 2.4. [45] *Let $(S, \cdot)$ be a semigroup and $f : S \to S$ an idempotent endomorphism. Then $(S, \dashv, \vdash, \perp)$ with operations defined by:*

$$x \dashv y = x(yf), \quad x \vdash y = (xf)y, \quad x \perp y = (xy)f$$

forms a trioid, denoted S^f.

Example 2.5. [45] *For a set Y, the free trioid $Frt(Y)$ consists of words in $X = Y \cup \bar{Y}$ with operations:*

$$w \dashv u = w\tilde{u}, \quad w \vdash u = \tilde{w}u, \quad w \perp u = wu$$

where $\tilde{w}$ is obtained from w by replacing all $\bar{x}$ with x.

3. Soft Trioids

The fundamental concept unifying soft set theory with trioid algebra is introduced as follows:

Definition 3.1. *Let $\mathcal{T} = (T, \dashv, \vdash, \perp)$ be a trioid and let $\mathcal{P}(\mathcal{T})$ denote the power set of $\mathcal{T}$. Let K be a set of parameters. A soft trioid over $\mathcal{T}$ is a triple $(\mathcal{T}, \Omega, K)$ where (Ω, K) is a soft set over the universe T such that for every parameter $\alpha \in K$, the subset $\Omega(\alpha) \subseteq T$ is a sub-trioid of $\mathcal{T}$.*

Equivalently, a soft trioid is a soft set (Ω, K) over T with the mapping:

$$\Omega : K \to \mathcal{P}(\mathcal{T})$$

satisfying that $\Omega(\alpha)$ is a sub-trioid of $\mathcal{T}$ for all $\alpha \in K$.

Remark 3.2. *The definition ensures that for each parameter $\alpha \in K$, the algebraic structure $(\Omega(\alpha), \dashv, \vdash, \perp)$ inherits the full trioid structure from $\mathcal{T}$. This parametric family of sub-trioids captures the essence of context-dependent algebraic behavior while maintaining algebraic integrity.*

Definition 3.3. *A soft trioid $(\mathcal{T}, \Omega, K)$ is called:*

- ***commutative*** *if for each $\alpha \in K$, the sub-trioid $\Omega(\alpha)$ is commutative;*
- ***idempotent*** *if for each $\alpha \in K$, the sub-trioid $\Omega(\alpha)$ is idempotent;*
- ***non-null*** *if* $\mathrm{Supp}(\Omega, K) \neq \emptyset$.

We begin with elementary constructions that demonstrate how soft trioids naturally arise from simpler algebraic structures.

Example 3.4. *Let $(S, \cdot)$ be a semigroup and (Ω, K) be a soft semigroup over S (i.e., $\Omega(\alpha)$ is a subsemigroup of S for each $\alpha \in K$). Consider the triod $(S, \dashv, \vdash, \perp)$ where all three operations coincide with the semigroup operation:*

$$x \dashv y = x \cdot y, \quad x \vdash y = x \cdot y, \quad x \perp y = x \cdot y$$

Then for each $\alpha \in K$, $\Omega(\alpha)$ is a sub-trioid of S, and thus (Ω, K) becomes a soft trioid over S.

Example 3.5. *Let* $(D, \dashv, \vdash)$ *be a dimonoid and* (Ω, K) *be a soft dimonoid over* D*. Define a third operation* $\perp$ *on* D *to coincide with either* $\dashv$ *or* $\vdash$*. Then* $(D, \dashv, \vdash, \perp)$ *forms a trioid, and* (Ω, K) *becomes a soft trioid over* D*.*

These elementary examples demonstrate that soft trioids provide a natural generalization of both soft semigroups and soft dimonoids. We now construct more sophisticated examples that reveal the rich structure of soft trioids.

Example 3.6. *Let* $(S, \cdot)$ *be a semigroup and* $f : S \to S$ *an idempotent endomorphism. As established in* [45]*, the structure* $S^f = (S, \dashv, \vdash, \perp)$ *with operations:*

$$x \dashv y = x(yf), \quad x \vdash y = (xf)y, \quad x \perp y = (xy)f$$

forms a trioid. Now, let $K = \{low, medium, high\}$ *be a parameter set representing different "resolution levels." Define* $\Omega : K \to \mathcal{P}(S^f)$ *by:*

$$\Omega(low) = \{s \in S : sf = s_0\} \quad \textit{(fixed point set)}$$
$$\Omega(medium) = \{s \in S : s \in \operatorname{Im}(f)\}$$
$$\Omega(high) = S \quad \textit{(full semigroup)}$$

One can verify that each $\Omega(\alpha)$ *is closed under all three operations and thus forms a sub-trioid. The resulting soft trioid* (S^f, Ω, K) *models a system with varying levels of structural resolution.*

To visualize this construction, consider the following parameterized structure:

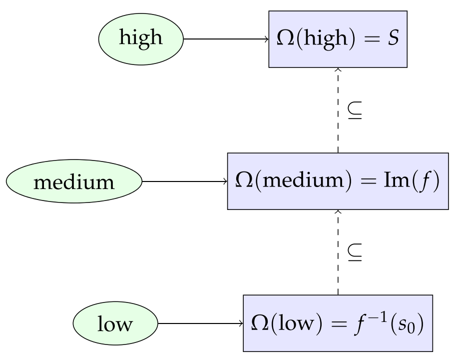

FIGURE 1. Hierarchical structure of the soft trioid (S^f, Ω, K)

Example 3.7. *Let* $\mathrm{Frt}(Y)$ *be the free trioid on a set* Y*. Consider a parameter set* K *representing different "complexity bounds." Define* $\Omega : K \to \mathcal{P}(\mathrm{Frt}(Y))$ *by:*

$$\Omega(n) = \{w \in \mathrm{Frt}(Y) : length(w) \le n\}$$

for $n \in K \subseteq \mathbb{N}$*. Each* $\Omega(n)$ *is closed under the trioid operations* $\dashv, \vdash$*, and* $\perp$*, since the length constraints are preserved. This yields a soft trioid* $(\mathrm{Frt}(Y), \Omega, K)$ *that captures words of bounded complexity.*

Example 3.8. *Let* S *be a rectangular semigroup (satisfying* $xyz = xz$*) and* M *be an arbitrary semigroup with homomorphism* $\theta : M \to S$*. As shown in* [45]*, the set* $S \times M$ *with operations:*

$$(s,t) \dashv (p,g) = (s, tg)$$
$$(s,t) \vdash (p,g) = ((t\theta)p, tg)$$
$$(s,t) \perp (p,g) = (sp, tg)$$

forms a trioid.

Now, let K *be a set of "coordinate restrictions" and define* $\Omega : K \to \mathcal{P}(S \times M)$ *by specifying allowed pairs. For instance, if* $K = \{1, 2, 3\}$*, we might define:*

$$\Omega(1) = \{(s,t) : s \in S_1 \subseteq S\}$$
$$\Omega(2) = \{(s,t) : t \in M_1 \subseteq M\}$$
$$\Omega(3) = \{(s,t) : s \in S_1, t \in M_1\}$$

where S_1 and M_1 are appropriate subsemigroups. The closure properties ensure that each $\Omega(\alpha)$ is a sub-trioid.

We now examine special classes of soft trioids that inherit additional algebraic properties. First, the characterization of commutative soft trioids.

Proposition 3.9. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid. The following are equivalent:*

(1) $(\mathcal{T}, \Omega, K)$ is commutative;
(2) For each $\alpha \in K$ and all $a, b \in \Omega(\alpha)$, the following commutation relations hold:

$$a \dashv b = b \dashv a$$
$$a \vdash b = b \vdash a$$
$$a \perp b = b \perp a$$

(3) The operations $\dashv, \vdash, \perp$ are pairwise commutative on each fiber $\Omega(\alpha)$.

Proof. The equivalence follows directly from the definition of commutativity in trioids and the fiber-wise nature of soft trioid properties. Condition (3) is the most restrictive but also the most natural from categorical perspective. □

Example 3.10. *Recall from* [45] *the trioid $T_{lr}^{\perp=(T,\dashv,\vdash,\perp)}$ where:*

$$x \dashv y = x, \quad x \vdash y = y, \quad x \perp y \text{ is an arbitrary semigroup operation}$$

If $(T, \perp)$ is an idempotent semigroup, then $T_{lr}^{\perp}$ becomes an idempotent trioid.

Now, let K be a set of "selection criteria" and define $\Omega : K \to \mathcal{P}(T)$ such that each $\Omega(\alpha)$ is a sub-semigroup of $(T, \perp)$. Then (Ω, K) forms an idempotent soft trioid over $T_{lr}^{\perp}$.

The following diagram illustrates the idempotency structure across parameters:

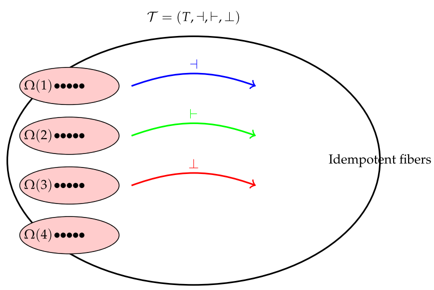

FIGURE 2. Idempotent soft trioid with idempotent elements in each fiber

Proposition 3.11. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid.*

(1) If $\mathcal{T}$ is commutative, then $(\mathcal{T}, \Omega, K)$ is commutative.
(2) If $\mathcal{T}$ is idempotent and $\Omega(\alpha) \neq \emptyset$ for all $\alpha \in K$, then $(\mathcal{T}, \Omega, K)$ is idempotent.
(3) The intersection of a family of soft trioids over the same base trioid $\mathcal{T}$ is again a soft trioid, provided it is non-null.

Proof. For (1), commutativity of $\mathcal{T}$ implies commutativity of all sub-trioids, hence all fibers $\Omega(\alpha)$. For (2), idempotency is inherited by non-empty sub-trioids. For (3), the intersection of sub-trioids is a sub-trioid when non-empty. □

4. Algebraic Operations on Soft Trioids

4.1. **The Product of Soft Trioids.** The categorical approach to soft trioids naturally leads to the consideration of product structures. We begin by defining the product of soft trioids, which generalizes the direct product construction from universal algebra to the parameterized setting.

Definition 4.1. *Let $(\mathcal{T}_1, \Omega_1, K_1)$ and $(\mathcal{T}_2, \Omega_2, K_2)$ be soft trioids over trioids $\mathcal{T}_1 = (T_1, \dashv_1, \vdash_1, \perp_1)$ and $\mathcal{T}_2 = (T_2, \dashv_2, \vdash_2, \perp_2)$, respectively. Their product is defined as the triple:*

$$(\mathcal{T}_1 \times \mathcal{T}_2, \Omega, K_1 \times K_2)$$

where $\mathcal{T}_1 \times \mathcal{T}_2 = (T_1 \times T_2, \dashv, \vdash, \perp)$ is the direct product trioid with component-wise operations:

$$(t_1, t_2) \dashv (s_1, s_2) = (t_1 \dashv_1 s_1, t_2 \dashv_2 s_2)$$
$$(t_1, t_2) \vdash (s_1, s_2) = (t_1 \vdash_1 s_1, t_2 \vdash_2 s_2)$$
$$(t_1, t_2) \perp (s_1, s_2) = (t_1 \perp_1 s_1, t_2 \perp_2 s_2)$$

and the mapping $\Omega : K_1 \times K_2 \to \mathcal{P}(T_1 \times T_2)$ is defined by:

$$\Omega(\alpha, \beta) = \Omega_1(\alpha) \times \Omega_2(\beta)$$

for all $(\alpha, \beta) \in K_1 \times K_2$.

Theorem 4.2. *The product of any two soft trioids is again a soft trioid.*

Proof. Let $(\mathcal{T}_1, \Omega_1, K_1)$ and $(\mathcal{T}_2, \Omega_2, K_2)$ be soft trioids. For any $(\alpha, \beta) \in K_1 \times K_2$, we have $\Omega(\alpha, \beta) = \Omega_1(\alpha) \times \Omega_2(\beta)$. Since $\Omega_1(\alpha)$ is a sub-trioid of $\mathcal{T}_1$ and $\Omega_2(\beta)$ is a sub-trioid of $\mathcal{T}_2$, their direct product $\Omega_1(\alpha) \times \Omega_2(\beta)$ is a sub-trioid of $\mathcal{T}_1 \times \mathcal{T}_2$. Therefore, $(\mathcal{T}_1 \times \mathcal{T}_2, \Omega, K_1 \times K_2)$ satisfies the definition of a soft trioid. □

The following diagram illustrates the product construction:

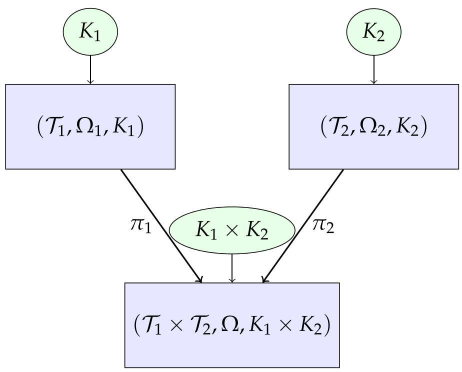

FIGURE 3. Product construction of soft trioids with projection maps

4.2. **Restricted and Extended Intersections.** The intersection operations on soft sets naturally extend to soft trioids, providing powerful tools for combining parameterized algebraic structures.

Definition 4.3. *Let $\{(\mathcal{T}, \Omega_i, K_i) \mid i \in I\}$ be a non-empty family of soft trioids over the same base trioid $\mathcal{T} = (T, \dashv, \vdash, \perp)$, with $\bigcap_{i \in I} K_i \neq \emptyset$. Their **restricted intersection** is defined as the soft set (Ω, K) where:*

- $K = \bigcap_{i \in I} K_i$
- $\Omega(\alpha) = \bigcap_{i \in I} \Omega_i(\alpha)$ *for all* $\alpha \in K$

We denote this by $(\Omega, K) = \widehat{\bigcap}_{i \in I}(\Omega_i, K_i)$.

Definition 4.4. *Let $\{(\mathcal{T}, \Omega_i, K_i) \mid i \in I\}$ be a non-empty family of soft trioids over the same base trioid $\mathcal{T}$. Their **extended intersection** is defined as the soft set (Ω, K) where:*

- $K = \bigcup_{i \in I} K_i$
- *For $\alpha \in K$, define $\mathcal{I}(\alpha) = \{i \in I \mid \alpha \in K_i\}$ and*

$$\Omega(\alpha) = \bigcap_{i \in \mathcal{I}(\alpha)} \Omega_i(\alpha)$$

We denote this by $(\Omega, K) = \widetilde{\bigcap}_{i \in I}(\Omega_i, K_i)$.

Theorem 4.5. *Let $\{(\mathcal{T}, \Omega_i, K_i) \mid i \in I\}$ be a non-empty family of soft trioids over the same base trioid $\mathcal{T}$.*

(1) The restricted intersection is a soft trioid over $\mathcal{T}$ if it is non-null.
(2) The extended intersection is a soft trioid over $\mathcal{T}$ if it is non-null.

Proof. (i) Let (Ω, K) be the restricted intersection. For any $\alpha \in \mathrm{Supp}(\Omega, K)$, we have $\Omega(\alpha) = \bigcap_{i \in I} \Omega_i(\alpha) \neq \emptyset$. Since each $\Omega_i(\alpha)$ is a sub-trioid of $\mathcal{T}$, their intersection $\Omega(\alpha)$ is also a sub-trioid (as the intersection of subalgebras is a subalgebra). Therefore, (Ω, K) is a soft trioid.

(ii) For the extended intersection, take $\alpha \in \mathrm{Supp}(\Omega, K)$. Then $\mathcal{I}(\alpha) \neq \emptyset$ and $\Omega(\alpha) = \bigcap_{i \in \mathcal{I}(\alpha)} \Omega_i(\alpha) \neq \emptyset$. Again, the intersection of sub-trioids is a sub-trioid, completing the proof. □

The following diagram illustrates the intersection constructions:

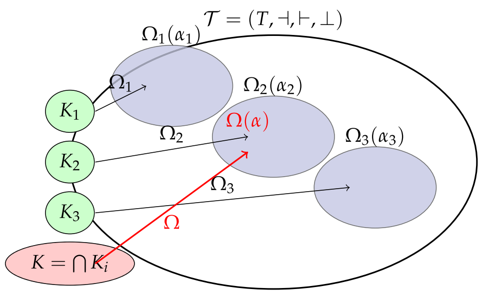

FIGURE 4. Restricted intersection of soft trioids with common parameters

4.3. **$\wedge$-Intersection of a Family of Soft Trioids.** A more sophisticated intersection operation, the $\wedge$-intersection, provides a way to combine soft trioids with different parameter sets through their Cartesian product.

Definition 4.6. *Let $\{(\mathcal{T}, \Omega_i, K_i) \mid i \in I\}$ be a non-empty family of soft trioids over the same base trioid $\mathcal{T}$. Their **$\wedge$-intersection** is defined as the soft set (Ω, K) where:*

- *$K = \prod_{i \in I} K_i$ (the Cartesian product of parameter sets)*
- *For $(\alpha_i)_{i \in I} \in K$, we define:*

$$\Omega((\alpha_i)_{i \in I}) = \bigcap_{i \in I} \Omega_i(\alpha_i)$$

We denote this by $(\Omega, K) = \widehat{\bigwedge}_{i \in I}(\Omega_i, K_i)$.

Theorem 4.7. *The $\wedge$-intersection of a non-empty family of soft trioids over the same base trioid $\mathcal{T}$ is a soft trioid over $\mathcal{T}$ if it is non-null.*

Proof. Let (Ω, K) be the $\wedge$-intersection and take $\alpha = (\alpha_i)_{i \in I} \in \mathrm{Supp}(\Omega, K)$. Then $\Omega(\alpha) = \bigcap_{i \in I} \Omega_i(\alpha_i) \neq \emptyset$. Since each $\Omega_i(\alpha_i)$ is a sub-trioid of $\mathcal{T}$, their intersection is also a sub-trioid. Therefore, (Ω, K) is a soft trioid. □

Example 4.8. *Consider a security system modeled by soft trioids over a trioid $\mathcal{T}$ representing user permissions. Let:*

- *$(\mathcal{T}, \Omega_1, K_1)$ where $K_1 = \{low, high\}$ represents security levels*
- *$(\mathcal{T}, \Omega_2, K_2)$ where $K_2 = \{day, night\}$ represents time periods*
- *$(\mathcal{T}, \Omega_3, K_3)$ where $K_3 = \{normal, emergency\}$ represents system states*

The $\wedge$-intersection (Ω, K) with $K = K_1 \times K_2 \times K_3$ models the combined effect of all three factors. For instance:

$$\Omega(high, night, emergency) = \Omega_1(high) \cap \Omega_2(night) \cap \Omega_3(emergency)$$

represents the permissions available during nighttime emergency situations with high security clearance.

The parameter space structure of the $\wedge$-intersection can be visualized as:

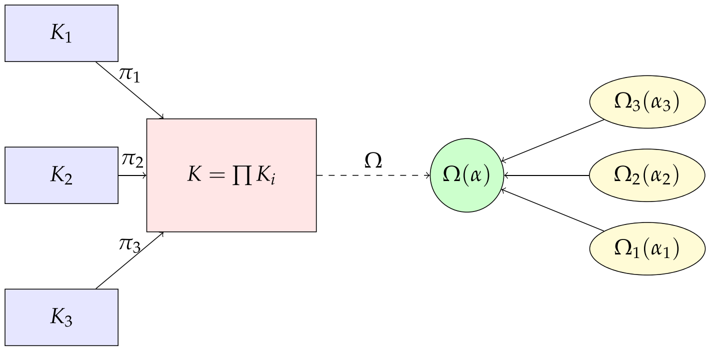

FIGURE 5. Parameter space and fiber structure of $\wedge$-intersection

4.4. **Closure Theorems.** We now establish comprehensive closure properties for the category of soft trioids.

Theorem 4.9. *The category of soft trioids over a fixed base trioid $\mathcal{T}$ is closed under:*

(1) Finite products
(2) Arbitrary restricted intersections (when non-null)
(3) Arbitrary extended intersections (when non-null)
(4) Arbitrary $\wedge$-intersections (when non-null)
(5) Sub-soft trioid formation

Proof. Let $\mathcal{C}_\mathcal{T}$ denote the category whose objects are soft trioids $(\mathcal{T}, \Omega, K)$ with underlying trioid $\mathcal{T}$ fixed, and whose morphisms are soft homomorphisms.
(1) Finite products. For a finite family $\{(\mathcal{T}, \Omega_i, K_i)\}_{i=1}^n$, define their product as

$$\prod_{i=1}^n (\mathcal{T}, \Omega_i, K_i) = \Big(\mathcal{T}, \prod_{i=1}^n \Omega_i, \prod_{i=1}^n K_i\Big),$$

where the fiber over a tuple $(\alpha_1, \ldots, \alpha_n)$ is $\bigcap_{i=1}^n \Omega_i(\alpha_i)$ (a sub-trioid of $\mathcal{T}$ by the intersection property). Projections onto each factor are soft homomorphisms, and the universal property follows directly from the component-wise definition of soft homomorphisms. Hence $\mathcal{C}_\mathcal{T}$ has finite products.
(2) Restricted intersections. Let $\{(\mathcal{T}, \Omega_i, K_i)\}_{i\in I}$ be a family with $\bigcap_{i\in I} \Omega_i \neq \varnothing$. Their restricted intersection is

$$\bigcap_R (\mathcal{T}, \Omega_i, K_i) = \Big(\mathcal{T}, \bigcap_{i\in I} \Omega_i, \bigcap_{i\in I} K_i\Big),$$

where the fiber over $\alpha \in \bigcap_i \Omega_i$ is $\bigcap_{i\in I} \Omega_i(\alpha)$, again a sub-trioid of $\mathcal{T}$. The inclusion maps into each factor are soft homomorphisms, and the intersection is the categorical limit of the diagram consisting of these inclusions. Thus $\mathcal{C}_\mathcal{T}$ is closed under arbitrary restricted intersections.
(3) Extended intersections. For the same family, the extended intersection is

$$\bigcap_\mathcal{E} (\mathcal{T}, \Omega_i, K_i) = \Big(\mathcal{T}, \bigcup_{i\in I} \Omega_i, \bigcup_{i\in I} K_i\Big),$$

with fiber over α defined as $\Omega_i(\alpha)$ if $\alpha \in \Omega_i$, and as the empty set otherwise. The universal property is verified analogously to (2), confirming closure under extended intersections.
(4) $\wedge$-intersections. The $\wedge$-intersection of the family is

$$\bigwedge (\mathcal{T}, \Omega_i, K_i) = \Big(\mathcal{T}, \prod_{i\in I} \Omega_i, \prod_{i\in I} K_i\Big),$$

where the fiber over a tuple $(\alpha_i)_{i\in I}$ is $\bigcap_{i\in I} \Omega_i(\alpha_i)$. This construction is precisely the product in $\mathcal{C}_\mathcal{T}$ when the underlying trioid is fixed, hence it is again an object of $\mathcal{C}_\mathcal{T}$.
(5) Sub-soft trioid formation. A sub-soft trioid of $(\mathcal{T}, \Omega, K)$ is a triple $(\mathcal{T}, \Omega', K')$ with $\Omega' \subseteq \Omega$ and $K' \subseteq K$ such that for every $\alpha \in K'$ the fiber $\Omega'(\alpha)$ is a sub-trioid of $\Omega(\alpha)$. The inclusion map $(\mathrm{id}_\mathcal{T}, \iota)$

with $\iota\colon K' \hookrightarrow K$ is a soft homomorphism, and the universal property of a subobject follows from the fact that any soft homomorphism whose image factors through $(\mathcal{T}, \Omega', K')$ must itself factor through the inclusion. Therefore $\mathcal{C}_{\mathcal{T}}$ is closed under all five operations. □

We now consider the lattice structure

Corollary 4.10. *The set of all soft trioids over a fixed base trioid $\mathcal{T}$ forms a complete lattice with respect to the soft inclusion relation. In this lattice:*

(1) The meet is given by the restricted intersection.
(2) The join is given by an appropriate union construction (followed by sub-trioid closure).

Proof. Let $\mathcal{S}(\mathcal{T})$ denote the collection of all soft trioids $(\mathcal{T}, \Omega, K)$ where the trioid part is fixed as $\mathcal{T}$. Recall that the **soft inclusion** $(\mathcal{T}, \Omega_1, K_1)\tilde{\subseteq}(\mathcal{T}, \Omega_2, K_2)$ is defined by $\Omega_1 \subseteq \Omega_2$ and $K_1 \subseteq K_2$. This relation is clearly reflexive, antisymmetric, and transitive; hence $\big(\mathcal{S}(\mathcal{T}), \tilde{\subseteq}\big)$ is a partially ordered set. We show that every non-empty family $\{(\mathcal{T}, \Omega_i, K_i) \mid i \in I\} \subseteq \mathcal{S}(\mathcal{T})$ has both a greatest lower bound and a least upper bound.
(1) Greatest lower bound (meet). Define

$$\bigwedge_{i\in I}(\mathcal{T}, \Omega_i, K_i) := \Big(\mathcal{T}, \bigcap_{i\in I}\Omega_i, \bigcap_{i\in I}K_i\Big),$$

provided $\bigcap_{i\in I}\Omega_i \neq \varnothing$ (otherwise we obtain the trivial empty soft trioid, which can be adjoined as a bottom element if desired). For each $j \in I$,

$$\bigcap_{i\in I}\Omega_i \subseteq \Omega_j \quad \text{and} \quad \bigcap_{i\in I}K_i \subseteq K_j,$$

so the defined soft trioid is softly contained in every $(\mathcal{T}, \Omega_j, K_j)$. If $(\mathcal{T}, \Omega', K')$ is another soft trioid with $(\mathcal{T}, \Omega', K')\tilde{\subseteq}(\mathcal{T}, \Omega_i, K_i)$ for all i, then necessarily $\Omega' \subseteq \bigcap_i \Omega_i$ and $K' \subseteq \bigcap_i K_i$; hence

$$(\mathcal{T}, \Omega', K')\tilde{\subseteq}\Big(\mathcal{T}, \bigcap_i\Omega_i, \bigcap_i K_i\Big).$$

Thus the restricted intersection yields the greatest lower bound. In the case of two soft trioids, this coincides with the restricted intersection $(\mathcal{T}, \Omega_1, K_1)\tilde{\cap}_R(\mathcal{T}, \Omega_2, K_2)$.
(2) Least upper bound (join). For the least upper bound we cannot simply take set-theoretic unions, because the union of parameter sets may not be closed under the soft-trioid structure. Define

$$\bigvee_{i\in I}(\mathcal{T}, \Omega_i, K_i) := \Big(\mathcal{T}, \bigcup_{i\in I}\Omega_i, \langle\bigcup_{i\in I}K_i\rangle_{\mathcal{T}}\Big),$$

where $\langle\cdot\rangle_{\mathcal{T}}$ denotes the sub-trioid of $\mathcal{T}$ generated by the given subset. Since each K_i is already a subset of $\mathcal{T}$, the generated sub-trioid $\langle\bigcup_i K_i\rangle_{\mathcal{T}}$ is the smallest sub-trioid of $\mathcal{T}$ containing all K_i. For every $j \in I$,

$$\Omega_j \subseteq \bigcup_i \Omega_i \quad \text{and} \quad K_j \subseteq \bigcup_i K_i \subseteq \langle\bigcup_i K_i\rangle_{\mathcal{T}},$$

so $(\mathcal{T}, \Omega_j, K_j)\tilde{\subseteq}\bigvee_i(\mathcal{T}, \Omega_i, K_i)$. If $(\mathcal{T}, \Omega'', K'')$ is any soft trioid such that $(\mathcal{T}, \Omega_i, K_i)\tilde{\subseteq}(\mathcal{T}, \Omega'', K'')$ for all i, then $\bigcup_i \Omega_i \subseteq \Omega''$ and $\bigcup_i K_i \subseteq K''$; because K'' is a sub-trioid of $\mathcal{T}$, it must contain the sub-trioid generated by $\bigcup_i K_i$. Hence

$$\Big(\mathcal{T}, \bigcup_i\Omega_i, \langle\bigcup_i K_i\rangle_{\mathcal{T}}\Big)\tilde{\subseteq}(\mathcal{T}, \Omega'', K'').$$

Consequently this construction gives the least upper bound. Since every non-empty subset of $\mathcal{S}(\mathcal{T})$ has both a meet and a join, $\big(\mathcal{S}(\mathcal{T}), \tilde{\subseteq}\big)$ is a complete lattice. The bottom element (if we admit the empty soft trioid) is $(\mathcal{T}, \varnothing, \varnothing)$; the top element is $(\mathcal{T}, \Omega_{\max}, \mathcal{T})$, where $\Omega_{\max}$ is the union of all parameter sets appearing in the family. □

Proposition 4.11. *Let $\{(\mathcal{T}, \Omega_i, K_i) \mid i \in I\}$ be a family of soft trioids over $\mathcal{T}$.*

(1) If each $(\mathcal{T}, \Omega_i, K_i)$ is commutative, then their restricted intersection, extended intersection, and $\wedge$-intersection are commutative.
(2) If each $(\mathcal{T}, \Omega_i, K_i)$ is idempotent, then their restricted intersection, extended intersection, and $\wedge$-intersection are idempotent.

(3) *The product of commutative (respectively, idempotent) soft trioids is commutative (respectively, idempotent).*

Proof. Recall that a soft trioid $(\mathcal{T}, \Omega, K)$ is called commutative if the trioid $\mathcal{T}$ is commutative, i.e., $x * y = y * x$ for all $x, y \in \mathcal{T}$, and idempotent if $\mathcal{T}$ satisfies $x * x = x$ for every $x \in \mathcal{T}$.

Let the operations on soft trioids be defined as follows:

- *Restricted intersection*: $(\mathcal{T}, \Omega_i, K_i) \tilde{\cap}_R (\mathcal{T}, \Omega_j, K_j) = (\mathcal{T}, \Omega_i \cap \Omega_j, K_i \cap K_j)$ when $\Omega_i \cap \Omega_j \neq \varnothing$;
- *Extended intersection*: $(\mathcal{T}, \Omega_i, K_i) \cap_{\mathcal{E}} (\mathcal{T}, \Omega_j, K_j) = (\mathcal{T}, \Omega_i \cup \Omega_j, K_i \cup K_j)$;
- *$\wedge$-intersection*: $(\mathcal{T}, \Omega_i, K_i) \wedge (\mathcal{T}, \Omega_j, K_j) = (\mathcal{T}, \Omega_i \times \Omega_j, K_i \times K_j)$;
- *Product*: $\prod_{i \in I} (\mathcal{T}_i, \Omega_i, K_i) = (\prod_{i \in I} \mathcal{T}_i, \prod_{i \in I} \Omega_i, \prod_{i \in I} K_i)$.

In each case the underlying trioid of the resulting soft trioid is either $\mathcal{T}$ (for intersections over the same $\mathcal{T}$) or a direct product of the $\mathcal{T}_i$ (for the product). The algebraic properties in question depend only on this underlying trioid.

(1) If every $(\mathcal{T}, \Omega_i, K_i)$ is commutative, then $\mathcal{T}$ is commutative. Since all three intersections keep $\mathcal{T}$ unchanged, their underlying trioid is again $\mathcal{T}$, hence commutative. Thus the restricted intersection, extended intersection, and $\wedge$-intersection are commutative soft trioids.

(2) Similarly, if each $(\mathcal{T}, \Omega_i, K_i)$ is idempotent, then $\mathcal{T}$ is idempotent. As the underlying trioid remains $\mathcal{T}$ in all three intersections, the resulting soft trioids are idempotent.

(3) Consider the product $\prod_{i \in I} (\mathcal{T}_i, \Omega_i, K_i)$. Its underlying trioid is $\prod_{i \in I} \mathcal{T}_i$. It is a standard algebraic fact that a direct product of commutative (resp. idempotent) trioids is again commutative (resp. idempotent): for any $(x_i), (y_i) \in \prod_i \mathcal{T}_i$,

$$(x_i) * (y_i) = (x_i * y_i) = (y_i * x_i) = (y_i) * (x_i)$$

if each $\mathcal{T}_i$ is commutative, and

$$(x_i) * (x_i) = (x_i * x_i) = (x_i)$$

if each $\mathcal{T}_i$ is idempotent. Consequently, the product soft trioid inherits the corresponding property.

All statements follow directly from the persistence of commutativity and idempotence under taking sub-trioids (here, the original $\mathcal{T}$ itself) and direct products. □

Remark 4.12. *From an applied perspective, the intersection operations correspond to logical AND operations across different parameter conditions, while the $\wedge$-intersection implements a multi-dimensional filtering mechanism. These have natural interpretations in multi-criteria decision making, access control systems, and hierarchical resource allocation problems.*

5. The Category of Soft Trioids

The notion of structure-preserving maps between soft trioids is essential for developing the categorical framework.

Definition 5.1. *Let $(\mathcal{T}_1, \Omega_1, K_1)$ and $(\mathcal{T}_2, \Omega_2, K_2)$ be soft trioids. A soft homomorphism between them is a pair (ϕ, ψ) where:*

- *$\phi : K_1 \to K_2$ is a function between parameter sets*
- *$\psi : T_1 \to T_2$ is a trioid homomorphism between base trioids*
- *For every $\alpha \in K_1$, we have $\psi(\Omega_1(\alpha)) \subseteq \Omega_2(\phi(\alpha))$ (compatibility condition)*

We denote this by $(\phi, \psi) : (\mathcal{T}_1, \Omega_1, K_1) \to (\mathcal{T}_2, \Omega_2, K_2)$.

Example 5.2. *If $(\mathcal{T}', \Omega', K')$ is a soft sub-trioid of $(\mathcal{T}, \Omega, K)$, then the inclusion maps (ι_K, ι_T) form a soft homomorphism, where $\iota_K : K' \hookrightarrow K$ and $\iota_T : T' \hookrightarrow T$ are the inclusion maps.*

Definition 5.3. *A soft homomorphism $(\phi, \psi) : (\mathcal{T}_1, \Omega_1, K_1) \to (\mathcal{T}_2, \Omega_2, K_2)$ is called a soft isomorphism if:*

- *$\phi : K_1 \to K_2$ is bijective*
- *$\psi : T_1 \to T_2$ is a trioid isomorphism*
- *For every $\alpha \in K_1$, the restriction $\psi|_{\Omega_1(\alpha)} : \Omega_1(\alpha) \to \Omega_2(\phi(\alpha))$ is a trioid isomorphism*

The following commutative diagram illustrates the structure of a soft homomorphism:

Proposition 5.4. (1) *The composition of two soft homomorphisms is a soft homomorphism.*

(2) *For any soft trioid $(\mathcal{T}, \Omega, K)$, the pair $(\mathrm{id}_K, \mathrm{id}_T)$ is a soft homomorphism, called the identity soft homomorphism.*

$$
\begin{tikzcd}
K_1 \arrow[r, "\phi"] \arrow[d, "\Omega_1"'] & K_2 \arrow[d, "\Omega_2"] \\
\mathcal{P}(T_1) \arrow[r, "\psi_*"] & \mathcal{P}(T_2) \\
\mathbf{Trioid}_1 \arrow[u, hook, "\hookrightarrow"] \arrow[r, "\psi"'] & \mathbf{Trioid}_2 \arrow[u, hook, "\hookrightarrow"']
\end{tikzcd}
$$

FIGURE 6. Morphism of soft trioids as a commutative square extended to parameter and structure spaces. The diagram illustrates how a soft trioid morphism (ϕ, ψ) induces a map ψ_* on power sets, with $\mathbf{Trioid}_i$ embedded as subsets of $\mathcal{P}(T_i)$.

(3) Soft isomorphisms are precisely the invertible soft homomorphisms.

Proof. For (1): Let $(\phi, \psi) : A \to B$ and $(\phi', \psi') : B \to C$ be soft homomorphisms. Their composition $(\phi' \circ \phi, \psi' \circ \psi)$ satisfies the compatibility condition since:

$$\psi' \circ \psi(\Omega_A(\alpha)) \subseteq \psi'(\Omega_B(\phi(\alpha))) \subseteq \Omega_C(\phi' \circ \phi(\alpha))$$

Parts (2) and (3) follow directly from the definitions. □

We now establish the categorical foundations of soft trioid theory.

Theorem 5.5. *The collection of all soft trioids with soft homomorphisms as morphisms forms a category, denoted* **SoftTrioid***, where:*

(1) Objects are soft trioids $(\mathcal{T}, \Omega, K)$
(2) Morphisms are soft homomorphisms (ϕ, ψ)
(3) Composition is given by component-wise composition
(4) Identity morphisms are pairs of identity maps

Theorem 5.6. *The category* **SoftTrioid** *has the following properties:*

(1) It has an initial object: the empty soft trioid
(2) It has all small products (as defined in Section 3)
(3) It has all equalizers
(4) It is concrete over **Set** $\times$ **Set** *via the forgetful functor* $(\mathcal{T}, \Omega, K) \mapsto (K, T)$

Proof. (1) The empty soft trioid serves as initial object since there is exactly one soft homomorphism from it to any other soft trioid.

(2) Products were constructed in Section 3 and satisfy the universal property.

(3) Equalizers can be constructed using the equalizers of the parameter maps and base trioid homomorphisms.

(4) The forgetful functor is faithful and preserves the concrete structure. □

Definition 5.7. *We define several forgetful functors that relate* **SoftTrioid** *to other categories:*

- U : **SoftTrioid** $\to$ **Set** $\times$ **Set** *with* $U(\mathcal{T}, \Omega, K) = (K, T)$
- U_{param} : **SoftTrioid** $\to$ **Set** *with* $U_{param}(\mathcal{T}, \Omega, K) = K$
- U_{trioid} : **SoftTrioid** $\to$ **Trioid** *with* $U_{trioid}(\mathcal{T}, \Omega, K) = \mathcal{T}$

Proposition 5.8. *The forgetful functor* U : **SoftTrioid** $\to$ **Set** $\times$ **Set** *has a left adjoint* F : **Set** $\times$ **Set** $\to$ **SoftTrioid** *that sends a pair* (K, X) *to the free soft trioid generated by* K *and* X.

Proof. The free soft trioid $F(K, X)$ is constructed as follows: take the free trioid $\mathrm{Frt}(X)$ on X, and for the parameter set K, define $\Omega(\alpha) = \mathrm{Frt}(X)$ for all $\alpha \in K$. This satisfies the universal property of the free object. □

The categorical relationships can be summarized in the following diagram:

$$\begin{tikzcd}
 & \mathbf{SoftTrioid} \arrow[dl, "U_{\text{param}}"'] \arrow[d, "U"] \arrow[dr, "U_{\text{trioid}}"] & \\
\mathbf{Set} & \mathbf{Set} \times \mathbf{Set} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbf{Trioid}
\end{tikzcd}$$

FIGURE 7. Categorical structure of **SoftTrioid** with forgetful functors

Theorem 5.9. *The category* **SoftTrioid** *is complete and cocomplete. That is, it has all small limits and colimits.*

Proof. Since **SoftTrioid** has products and equalizers, it has all limits. For colimits, we can construct coproducts as disjoint unions with appropriate trioid structure, and coequalizers can be built using quotient constructions. The verification follows standard categorical arguments. □

Example 5.10. *Let* $(\phi, \psi) : (\mathcal{T}_1, \Omega_1, K_1) \to (\mathcal{T}_2, \Omega_2, K_2)$ *be a soft homomorphism. The kernel of* (ϕ, ψ) *is the soft sub-trioid of* $(\mathcal{T}_1, \Omega_1, K_1)$ *defined by:*

- *Parameter set:* $\ker \phi = \{\alpha \in K_1 : \phi(\alpha) = \beta_0\}$ *for some fixed* $\beta_0 \in K_2$
- *Base trioid:* $\ker \psi$*, the kernel of the trioid homomorphism* ψ
- *Fibers:* $\Omega_{\text{ker}}(\alpha) = \Omega_1(\alpha) \cap \ker \psi$ *for* $\alpha \in \ker \phi$

This construction provides the categorical kernel in **SoftTrioid**.

Remark 5.11. *The category establishes a robust categorical foundation for advanced structural studies, including representation, homology, and deformation theory. It constitutes a mathematically coherent generalization of both classical trioid theory and soft set theory while retaining essential categorical properties. From an applied standpoint, provides a formal model for systems with multiple parameter dimensions: soft homomorphisms capture structure-preserving transformations, while the lattice of soft subtrioids describes hierarchical subsystem relationships in complex parameterized systems.*

6. SOFT SUBTRIOIDS

The hierarchical structure of soft trioids naturally leads to the concept of soft subtrioids, which formalize the notion of parameterized subsystems within a larger parameterized algebraic structure.

Definition 6.1. *Let* $(\mathcal{T}, \Omega, K)$ *and* $(\mathcal{T}', \Omega', K')$ *be two soft trioids over trioid* $\mathcal{T} = (T, \dashv, \vdash, \perp)$*. We say that* $(\mathcal{T}', \Omega', K')$ *is a soft subtrioid of* $(\mathcal{T}, \Omega, K)$*, denoted* $(\mathcal{T}', \Omega', K') \preceq (\mathcal{T}, \Omega, K)$*, if the following conditions hold:*

(1) $K' \subseteq K$ *(parameter set inclusion)*
(2) T' *is a subtrioid of* T *(base trioid inclusion)*
(3) For every $\alpha \in K'$*, we have* $\Omega'(\alpha)$ *is a subtrioid of* $\Omega(\alpha)$ *(fiber-wise inclusion)*

Example 6.2. *Let* $(\mathcal{T}, \Omega, K)$ *be a soft trioid. For any subset* $K' \subseteq K$*, we can define the restricted soft trioid* $(\mathcal{T}, \Omega|_{K'}, K')$ *where* $\Omega|_{K'}(\alpha) = \Omega(\alpha)$ *for all* $\alpha \in K'$*. This forms a soft sub-trioid. Similarly, for any sub-trioid* $\mathcal{T}'$ *of* $\mathcal{T}$*, the triple* $(\mathcal{T}', \Omega, K)$ *(with* $\Omega(\alpha)$ *restricted to* $\mathcal{T}'$*) forms a soft subtrioid when non-null.*

Proposition 6.3. *Let* $(\mathcal{T}, \Omega, K)$ *and* $(\mathcal{T}', \Omega', K')$ *be soft trioids. The following are equivalent:*

(1) $(\mathcal{T}', \Omega', K')$ *is a soft subtrioid of* $(\mathcal{T}, \Omega, K)$
(2) There exist inclusion maps $\iota_K : K' \hookrightarrow K$ *and* $\iota_T : T' \hookrightarrow T$ *such that for every* $\alpha \in K'$*, the diagram:*

$$\begin{tikzcd}
\Omega'(\alpha) \arrow[r, tail, "\iota_T"] \arrow[d, tail] & \Omega(\alpha) \arrow[d, tail] \\
T' \arrow[r, tail, "\iota_T"] & T
\end{tikzcd}$$

commutes, and ι_T *is a triod homomorphism.*
(3) $(\mathcal{T}', \Omega', K')$ *is isomorphic to a soft subtrioid of* $(\mathcal{T}, \Omega, K)$ *in the category* **SoftTrioid**.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows directly from the definitions. For (1) $\Leftrightarrow$ (3), note that if $(\mathcal{T}', \Omega', K')$ is a soft subtrioid, then the inclusion maps provide the required isomorphism with its image. Conversely, any soft trioid isomorphic to a soft subtrioid can be identified with one through the isomorphism. □

6.1. **Lattice-Theoretic Properties.** The collection of all soft subtrioids of a given soft trioid exhibits rich lattice-theoretic structure, generalizing the subgroup lattice and submodule lattice constructions from classical algebra.

Theorem 6.4. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid. The set $\mathcal{L}(\mathcal{T}, \Omega, K)$ of all soft subtrioids of $(\mathcal{T}, \Omega, K)$, ordered by the soft subtrioid relation $\preceq$, forms a complete lattice where:*

- *The meet (greatest lower bound) of a family $\{(\mathcal{T}_i, \Omega_i, K_i)\}_{i \in I}$ is given by:*

$$\bigwedge_{i \in I} (\mathcal{T}_i, \Omega_i, K_i) = \left(\bigcap_{i \in I} \mathcal{T}_i, \widehat{\bigcap}_{i \in I} \Omega_i, \bigcap_{i \in I} K_i \right)$$

- *The join (least upper bound) is given by the soft subtrioid generated by the union.*

The minimal element is the trivial soft trioid with empty parameter set, and the maximal element is $(\mathcal{T}, \Omega, K)$ itself.

Proof. For the meet construction: The intersection $\bigcap_{i \in I} \mathcal{T}_i$ is a subtrioid of $\mathcal{T}$, and for $\alpha \in \bigcap_{i \in I} K_i$, the intersection $\bigcap_{i \in I} \Omega_i(\alpha)$ is a subtrioid of $\bigcap_{i \in I} \mathcal{T}_i$. Thus the meet is well-defined and is indeed the greatest lower bound.

For the join: The soft subtrioid generated by a family is obtained by taking the subtrioid of $\mathcal{T}$ generated by the union of base sets, and for each parameter, taking the sub-trioid generated by the union of corresponding fibers. The parameter set for the join is the union of the parameter sets. □

The lattice structure can be visualized as follows:

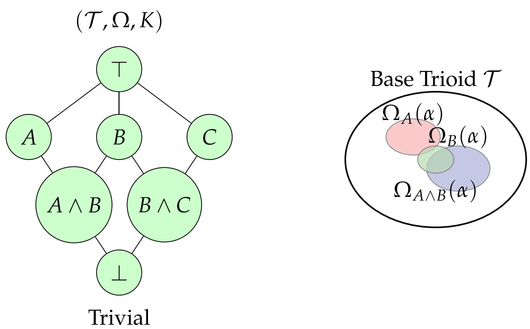

FIGURE 8. Lattice structure of soft sub-trioids with fiber inclusions

For the modularity property, we have:

Proposition 6.5. *The lattice $\mathcal{L}(\mathcal{T}, \Omega, K)$ is modular. That is, for any soft subtrioids A, B, C with $A \preceq C$, we have:*

$$A \vee (B \wedge C) = (A \vee B) \wedge C$$

Proof. This follows from the modularity of the lattice of subtrioids of a fixed trioid and the component-wise nature of the soft subtrioid operations. The verification proceeds by checking the equality on both the parameter sets and the fiber structures separately. □

7. Tribands of Soft Subtrioids and Decomposition Theorems

The decomposition of algebraic structures into simpler components represents one of the most powerful techniques in modern algebra. Building upon Zhuchok's work on tribands of trioids and the classical theory of bands in semigroup theory, we now develop the corresponding decomposition theory for soft trioids.

Definition 7.1. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid. A triband of soft subtrioids of $(\mathcal{T}, \Omega, K)$ is a family $\{(\mathcal{T}_i, \Omega_i, K_i)\}_{i \in I}$ of soft subtrioids satisfying:*

(1) $(\mathcal{T}, \Omega, K) = \bigcup_{i \in I}(\mathcal{T}_i, \Omega_i, K_i)$ (covering condition)
(2) $(\mathcal{T}_i, \Omega_i, K_i) \cap (\mathcal{T}_j, \Omega_j, K_j)$ is trivial for $i \neq j$ (disjointness condition)
(3) For every $i, j \in I$, there exist $n, m \in I$ such that:

$$(\mathcal{T}_i, \Omega_i, K_i) \dashv (\mathcal{T}_j, \Omega_j, K_j) \preceq (\mathcal{T}_n, \Omega_n, K_n)$$
$$(\mathcal{T}_i, \Omega_i, K_i) \vdash (\mathcal{T}_j, \Omega_j, K_j) \preceq (\mathcal{T}_m, \Omega_m, K_m)$$
$$(\mathcal{T}_i, \Omega_i, K_i) \perp (\mathcal{T}_j, \Omega_j, K_j) \preceq (\mathcal{T}_p, \Omega_p, K_p)$$

for some $p \in I$ (closure under operations)

We say that $(\mathcal{T}, \Omega, K)$ is decomposable into the triband $\{(\mathcal{T}_i, \Omega_i, K_i)\}_{i \in I}$.

Remark 7.2. *This definition generalizes several classical concepts:*

- *When the soft trioid has a single parameter, we recover Zhuchok's notion of a triband of trioids*
- *When all operations coincide, we obtain a band of soft semigroups*
- *The disjointness condition ensures that the decomposition is essentially unique*

The relationship between soft homomorphisms and triband decompositions reveals deep structural properties of soft trioids.

Definition 7.3. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid and $\{(\mathcal{T}_i, \Omega_i, K_i)\}_{i \in I}$ be a triband of soft subtrioids. A decomposition homomorphism is a soft homomorphism $(\phi, \psi) : (\mathcal{T}, \Omega, K) \to (J, \Omega_J, J)$ where:*

- *(J, Ω_J, J) is an idempotent soft trioid (the structure soft trioid)*
- *For each $j \in J$, the fiber $(\phi, \psi)^{-1}(j)$ is isomorphic to some $(\mathcal{T}_i, \Omega_i, K_i)$*
- *The mapping $i \mapsto j$ establishes a bijection between I and J*

Theorem 7.4. *A soft trioid $(\mathcal{T}, \Omega, K)$ is decomposable into a triband if and only if there exists a decomposition homomorphism to an idempotent soft trioid.*

Proof. ($\Rightarrow$) Suppose $(\mathcal{T}, \Omega, K)$ decomposes into $\{(\mathcal{T}_i, \Omega_i, K_i)\}_{i \in I}$. Define $J = I$ with the discrete soft trioid structure (each fiber is a singleton). The homomorphism (ϕ, ψ) sends each element to the index of its component.

($\Leftarrow$) Given a decomposition homomorphism $(\phi, \psi) : (\mathcal{T}, \Omega, K) \to (J, \Omega_J, J)$, the fibers $\{(\phi, \psi)^{-1}(j)\}_{j \in J}$ form a triband decomposition. □

The following diagram illustrates the decomposition homomorphism:
We now establish the central decomposition theorem for soft trioids, which generalizes Zhuchok's structure theorem for trioids to the soft setting.

Theorem 7.5. *Every soft trioid $(\mathcal{T}, \Omega, K)$ admits a decomposition into a triband of soft simple subtrioids. Moreover, this decomposition is essentially unique up to soft isomorphism.*

Proof. Let $(\mathcal{T}, \Omega, K)$ be a soft trioid. For each parameter $\alpha \in K$ the fiber $\Omega(\alpha)$ is a sub-trioid of $\mathcal{T}$. By Zhuchok's decomposition theorem for ordinary trioids, each $\Omega(\alpha)$ can be uniquely expressed as a triband of simple trioids; write this decomposition as $\{\Omega(\alpha)_i\}_{i \in I_\alpha}$.

Define an equivalence relation on the parameter set K by setting $\alpha \sim \beta$ if and only if the index sets coincide ($I_\alpha = I_\beta$) and the decompositions are compatible in the sense that for each $i \in I_\alpha$ the corresponding simple components are isomorphic as trioids. Denote the quotient by $K/\!\sim\, = \{[\alpha] \mid \alpha \in K\}$.

For every equivalence class $[\alpha] \in K/\!\sim$ and each index $i \in I_\alpha$, construct a soft subtrioid $(\mathcal{T}_{[\alpha],i}, \Omega_{[\alpha],i}, K_{[\alpha],i})$ as follows:

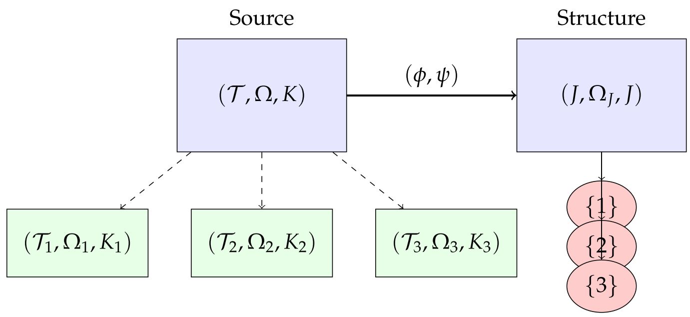

FIGURE 9. Decomposition homomorphism and fiber structure

- $K_{[\alpha],i} = \{\beta \in [\alpha] \mid \Omega(\beta)_i \text{ is defined}\}$;
- $\Omega_{[\alpha],i}(\beta) = \Omega(\beta)_i$ for all $\beta \in K_{[\alpha],i}$;
- $\mathcal{T}_{[\alpha],i}$ is the sub-trioid of $\mathcal{T}$ generated by $\bigcup_{\beta \in K_{[\alpha],i}} \Omega(\beta)_i$.

One verifies directly that each $(\mathcal{T}_{[\alpha],i}, \Omega_{[\alpha],i}, K_{[\alpha],i})$ is indeed a soft trioid. By construction the family $\{(\mathcal{T}_{[\alpha],i}, \Omega_{[\alpha],i}, K_{[\alpha],i})\}$ covers the original soft trioid, its members are pairwise disjoint on the parameter level, and the trioid operations respect the decomposition since each fiber $\Omega(\alpha)$ already decomposes as a triband. Hence the family forms a triband of soft simple sub-trioids.

For uniqueness, suppose two such decompositions are given. They induce the same equivalence relation on K (since the decompositions of the fibers are unique) and yield component soft trioids that are pairwise isomorphic. Consequently the two decompositions are isomorphic as tribands, establishing essential uniqueness. □

Definition 7.6. *A soft trioid $(\mathcal{T}, \Omega, K)$ is called soft simple if it contains no non-trivial soft subtrioids. That is, the only soft subtrioids are the trivial one and itself.*

Corollary 7.7. *Any triband decomposition of a soft trioid can be refined to the decomposition into soft simple subtrioids given by the Main Decomposition Theorem.*

Proof. Given a decomposition $\{(\mathcal{T}_i, \Omega_i, K_i)\}_{i \in I}$, apply the Main Decomposition Theorem to each component to obtain finer decompositions. The union of these refined decompositions gives the desired refinement. □

The hierarchical structure of the decomposition can be visualized as:

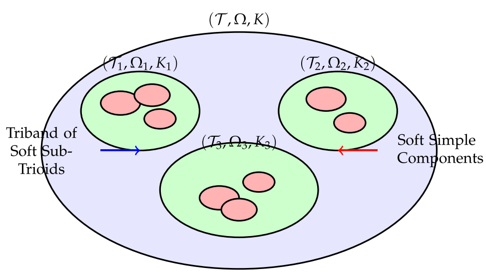

FIGURE 10. Hierarchical decomposition into soft simple components

7.1. **Semilattice Decompositions.** A particularly important special case occurs when the structure soft trioid is a semilattice, leading to semilattice decompositions of soft trioids.

Definition 7.8. *A soft semilattice decomposition of a soft trioid* $(\mathcal{T}, \Omega, K)$ *is a triband decomposition where the structure soft trioid* (J, Ω_J, J) *is a soft semilattice (i.e., each fiber* $\Omega_J(j)$ *is a semilattice and the operations are semilattice operations).*

Theorem 7.9. *Every soft trioid* $(\mathcal{T}, \Omega, K)$ *with idempotent fibers admits a soft semilattice decomposition into soft unipotent subtrioids.*

Proof. Let $(\mathcal{T}, \Omega, K)$ be a soft trioid with idempotent fibers. For each $\alpha \in K$, the trioid $\Omega(\alpha)$ admits a semilattice decomposition into unipotent trioids by Zhuchok's theorem. Let Y_α be the semilattice and $\{T_{\alpha,i}\}_{i\in Y_\alpha}$ be the decomposition.

Define an equivalence relation on K by:

$$\alpha \sim \beta \iff Y_\alpha \cong Y_\beta \text{ and the decompositions are compatible}$$

The quotient $K/\sim$ inherits a semilattice structure from the Y_α. For each $[\alpha] \in K/\sim$ and $i \in Y_{[\alpha]}$, define the soft subtrioid as in the Main Decomposition Theorem. The resulting structure soft trioid is a soft semilattice. □

Definition 7.10. *A soft trioid* $(\mathcal{T}, \Omega, K)$ *is called soft unipotent if for each* $\alpha \in K$*, the trioi*

Example 7.11. *Consider the free soft trioid* $(\mathrm{Frt}(Y), \Omega, K)$ *from Section 2, where* $\Omega(n)$ *consists of words of length at most* n*. This soft trioid admits a natural semilattice decomposition where:*

- *The structure semilattice is* $(\mathbb{N}, \min)$
- *The fiber over* $n \in \mathbb{N}$ *consists of words where certain combinatorial invariants (e.g., specific letter patterns) are bounded by* n
- *Each component is soft unipotent*

The semilattice structure can be visualized as a hierarchical tree:

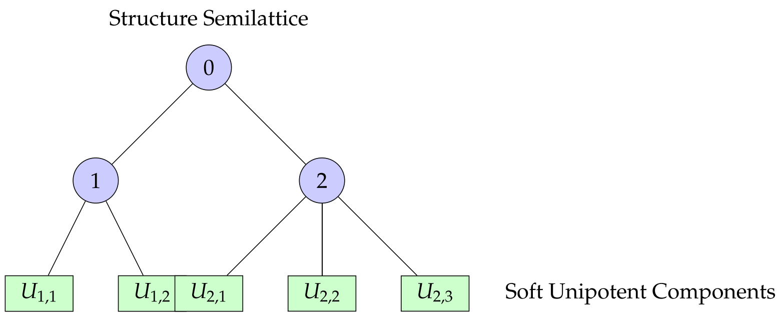

FIGURE 11. Semilattice decomposition into soft unipotent components

Theorem 7.12. *The soft semilattice decomposition of a soft trioid* $(\mathcal{T}, \Omega, K)$ *with idempotent fibers is universal: any soft homomorphism from* $(\mathcal{T}, \Omega, K)$ *to a soft semilattice factors uniquely through the decomposition homomorphism.*

Proof. Let $(\phi, \psi) : (\mathcal{T}, \Omega, K) \to (S, \Omega_S, S)$ be a soft homomorphism to a soft semilattice. The universal property follows from:

- The decomposition homomorphism $(\phi_d, \psi_d) : (\mathcal{T}, \Omega, K) \to (J, \Omega_J, J)$ to the structure semilattice
- The existence of a unique soft homomorphism $(J, \Omega_J, J) \to (S, \Omega_S, S)$ making the diagram commute

This is established using the universal property of semilattice decompositions in each fiber and the compatibility of parameter mappings. □

Corollary 7.13. *The soft semilattice decompositions of a soft trioid* $(\mathcal{T}, \Omega, K)$ *with idempotent fibers are in bijection with the soft congruences on* $(\mathcal{T}, \Omega, K)$ *whose quotient is a soft semilattice.*

Proof. Each soft semilattice decomposition corresponds to a decomposition homomorphism, whose kernel is a soft congruence with semilattice quotient. Conversely, each such soft congruence gives rise to a decomposition via its fibers. □

Remark 7.14. *The decomposition theory developed in this section provides powerful tools for analyzing the structure of soft trioids. The triband decomposition reveals the "atomic" structure, while the semilattice decomposition provides a hierarchical organization. These results have significant implications for the classification and representation theory of soft trioids, as well as for applications in parameterized system design and multi-level modeling. The connection with soft congruences established in the final corollary provides a bridge to the congruence theory that will be developed in the next section, demonstrating the deep interconnections within the theory of soft trioids.*

8. Congruences and Soft Semiretractions

The study of congruences lies at the heart of universal algebra, providing the foundation for quotient structures and homomorphism theorems. We now extend this fundamental concept to the realm of soft trioids. d $\Omega(\alpha)$ contains exactly one idempotent element.

Definition 8.1. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid. A soft congruence on $(\mathcal{T}, \Omega, K)$ is a pair (ρ_K, ρ_T) where:*

(1) ρ_K is an equivalence relation on the parameter set K
(2) ρ_T is a family of trioid congruences $\{\rho_T(\alpha)\}_{\alpha \in K}$ on the fibers $\Omega(\alpha)$
(3) For any $\alpha, \beta \in K$ with $\alpha\rho_K\beta$, and for any operations $* \in \{\dashv, \vdash, \perp\}$, we have the compatibility condition:

$$x\rho_T(\alpha)y \text{ and } z\rho_T(\beta)w \implies (x * z)\rho_T(\gamma)(y * w)$$

for some $\gamma \in K$ with $\alpha\rho_K\gamma$ and $\beta\rho_K\gamma$

Remark 8.2. *The compatibility condition ensures that the congruence relations respect both the parameter structure and the algebraic operations across different fibers. This is a significant strengthening of the classical notion of congruence.*

Example 8.3. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid and let ρ be a trioid congruence on $\mathcal{T}$ that is compatible with all fibers: for each $\alpha \in K$, if $x, y \in \Omega(\alpha)$ and $x\rho y$, then the congruence class $[x]_\rho$ is contained in some fiber $\Omega(\beta)$. Then $(\Delta_K, \{\rho|_{\Omega(\alpha)}\}_{\alpha \in K})$ forms a soft congruence, where Δ_K is the diagonal relation on K.*

Theorem 8.4. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid and (ρ_K, ρ_T) be a soft congruence. The quotient soft trioid $(\mathcal{T}, \Omega, K)/(\rho_K, \rho_T)$ is defined by:*

- *Parameter set: K/ρ_K (the quotient of K by ρ_K)*
- *For each $\bar{\alpha} \in K/\rho_K$, the fiber is:*

$$\Omega/\rho_T(\bar{\alpha}) = \bigcup_{\alpha \in \bar{\alpha}} \Omega(\alpha)/\rho_T(\alpha)$$

  with the quotient trioid structure
- *The operations are defined by:*

$$[x]_{\rho_T(\alpha)} * [y]_{\rho_T(\beta)} = [x * y]_{\rho_T(\gamma)}$$

  for some γ with $\alpha\rho_K\gamma$ and $\beta\rho_K\gamma$

This construction yields a well-defined soft trioid.

Proof. We verify the soft trioid axioms:

Well-defined operations: The compatibility condition ensures that the operations are independent of the choice of representatives and the specific γ in the equivalence class.

Soft trioid axioms: Each fiber $\Omega/\rho_T(\bar{\alpha})$ inherits the trioid structure from the quotient trioids $\Omega(\alpha)/\rho_T(\alpha)$, and the eight trioid axioms are preserved under quotient formation.

Parameter consistency: The definition ensures that for each parameter class $\bar{\alpha}$, the corresponding fiber is a well-defined trioid. □

The quotient construction can be visualized as follows:

Special classes of congruences play a crucial role in the structure theory of soft trioids, mirroring their importance in classical trioid theory.

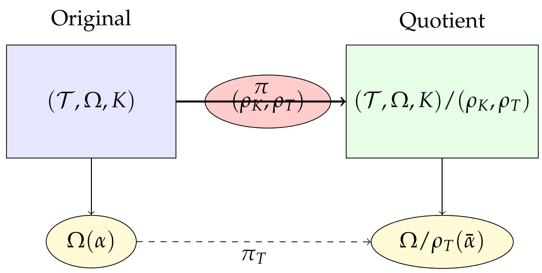

FIGURE 12. Quotient construction for soft trioids

Definition 8.5. *A soft congruence (ρ_K,ρ_T) on $(\mathcal{T},\Omega,K)$ is called idempotent if the quotient soft trioid $(\mathcal{T},\Omega,K)/(\rho_K,\rho_T)$ is idempotent. That is, for each parameter class $\bar{\alpha}\in K/\rho_K$, the fiber $\Omega/\rho_T(\bar{\alpha})$ is an idempotent trioid.*

Theorem 8.6. *Let $(\mathcal{T},\Omega,K)$ be a soft trioid with commutative $\dashv$ operation on each fiber. Define the relation $\eta=(\eta_K,\{\eta_T(\alpha)\}_{\alpha\in K})$ by:*

- *$\alpha\eta_K\beta$ if there exist parameters connecting α and β through a chain of $\dashv$-divisibility*
- *$x\eta_T(\alpha)y$ if there exist positive integers k,l such that $x\dashv y^k$ and $y\dashv x^l$ in $\Omega(\alpha)$*

Then η is the least idempotent soft congruence on $(\mathcal{T},\Omega,K)$.

Proof. The proof extends Zhuchok's construction for trioids to the soft setting:

Well-defined: The commutativity of $\dashv$ ensures that $\eta_T(\alpha)$ is a congruence on each fiber $\Omega(\alpha)$, and the parameter relation η_K is compatible with the fiber congruences.

Idempotent quotient: The quotient by η yields an idempotent soft trioid by construction.

Minimality: Any idempotent soft congruence must identify elements that are mutually $\dashv$-divisible, establishing minimality. □

Definition 8.7. *A soft congruence (ρ_K,ρ_T) is called a semilattice soft congruence if the quotient soft trioid is a soft semilattice (i.e., all operations coincide and yield semilattice operations on each fiber).*

Theorem 8.8. *Let $(\mathcal{T},\Omega,K)$ be a soft trioid with idempotent $\dashv$ operation on each fiber. Define the relation $\mathfrak{I}=(\mathfrak{I}_K,\{\mathfrak{I}_T(\alpha)\}_{\alpha\in K})$ by:*

- *$\alpha\mathfrak{I}_K\beta$ if the idempotent structures of $\Omega(\alpha)$ and $\Omega(\beta)$ are compatible*
- *$x\mathfrak{I}_T(\alpha)y$ if $x=x\dashv y\dashv x$ and $y=y\dashv x\dashv y$ in $\Omega(\alpha)$*

Then $\mathfrak{I}$ is the least semilattice soft congruence on $(\mathcal{T},\Omega,K)$.

Proof. The verification proceeds similarly to the idempotent case, using the idempotency of the $\dashv$ operation to ensure that $\mathfrak{I}_T(\alpha)$ is a congruence whose quotient is a semilattice. The minimality follows from the fact that any semilattice congruence must identify elements satisfying the given conditions. □

Semiretractions, introduced by Usenko for semigroups and extended by Zhuchok to trioids, provide a powerful tool for studying congruences through transformation theory. We now develop the soft analog of this theory.

Definition 8.9. *Let $(\mathcal{T},\Omega,K)$ be a soft trioid. A soft semiretraction is a pair (τ_K,τ_T) where:*

- *$\tau_K:K\to K$ is a transformation on parameters*
- *τ_T is a family of transformations $\{\tau_T(\alpha):\Omega(\alpha)\to\Omega(\tau_K(\alpha))\}_{\alpha\in K}$*
- *For all $\alpha\in K$, $x,y\in\Omega(\alpha)$, and $*\in\{\dashv,\vdash,\perp\}$, we have:*

$$\tau_T(\alpha)(x*y)=\tau_T(\alpha)(\tau_T(\alpha)(x)*y)=\tau_T(\alpha)(x*\tau_T(\alpha)(y))$$

If these identities hold, we call (τ_K,τ_T) a symmetric soft semiretraction.

Example 8.10. *Let $(\mathcal{T},\Omega,K)$ be a soft trioid and let $f:K\to K$ be an idempotent map. Define $\tau_K=f$ and for each $\alpha\in K$, define $\tau_T(\alpha)$ to be the identity on $\Omega(\alpha)$ when $f(\alpha)=\alpha$, and a constant map otherwise. This yields a soft semiretraction.*

Theorem 8.11. *There is a bijective correspondence between:*

(1) Symmetric soft semiretractions (τ_K, τ_T) on $(\mathcal{T}, \Omega, K)$

(2) Soft congruences (ρ_K, ρ_T) on $(\mathcal{T}, \Omega, K)$ such that each $\rho_T(\alpha)$ is the kernel congruence of $\tau_T(\alpha)$

Moreover, this correspondence preserves the quotient structures.

Proof. Given a symmetric soft semiretraction (τ_K, τ_T), define:

- $\alpha \rho_K \beta$ if $\tau_K(\alpha) = \tau_K(\beta)$
- $x \rho_T(\alpha) y$ if $\tau_T(\alpha)(x) = \tau_T(\alpha)(y)$

The semiretraction identities ensure that (ρ_K, ρ_T) is a soft congruence.

Conversely, given a soft congruence (ρ_K, ρ_T), choose representatives for each equivalence class and define $\tau_K(\alpha)$ to be the representative of $[\alpha]_{\rho_K}$, and $\tau_T(\alpha)(x)$ to be the representative of $[x]_{\rho_T(\alpha)}$. The compatibility conditions ensure this yields a symmetric soft semiretraction.

The preservation of quotient structures follows by construction. □

The following diagram illustrates the semiretraction-congruence correspondence:

$$\begin{tikzcd}
(\tau_K, \tau_T) \arrow[r, bend left, leftrightarrow] \arrow[r, bend right, leftrightarrow] \arrow[d, mapsto, "\mapsto"'] & (\rho_K, \rho_T) \arrow[d, mapsto, "\mapsto"] \\
(\mathcal{T}, \Omega, K)^\tau \arrow[r, "\cong"] & (\mathcal{T}, \Omega, K)/(\rho_K, \rho_T)
\end{tikzcd}$$

FIGURE 13. Correspondence diagram between symmetric soft semiretractions (τ_K, τ_T) and soft congruences (ρ_K, ρ_T). The top parallel arrows indicate a one-to-one correspondence. The vertical arrows show the construction of associated quotient structures, which are naturally isomorphic as shown by the bottom arrow.

Definition 8.12. *Let (τ_K, τ_T) be a symmetric soft semiretraction on $(\mathcal{T}, \Omega, K)$. The soft mutation $(\mathcal{T}, \Omega, K)^{(\tau_K, \tau_T)}$ is defined by:*

- *Parameter set: $\tau_K(K)$*
- *For each $\alpha \in \tau_K(K)$, the fiber is:*
$$\Omega^\tau(\alpha) = \{\tau_T(\beta)(x) : \beta \in \tau_K^{-1}(\alpha), x \in \Omega(\beta)\}$$
- *Operations are defined by:*
$$\tau_T(\beta)(x) *^\tau \tau_T(\gamma)(y) = \tau_T(\delta)(x * y)$$
for appropriate δ with $\tau_K(\delta) = \tau_K(\beta) = \tau_K(\gamma)$

We state the mutation theorem,

Theorem 8.13. *The soft mutation $(\mathcal{T}, \Omega, K)^{(\tau_K, \tau_T)}$ is a soft trioid, and there is a natural soft isomorphism:*
$$(\mathcal{T}, \Omega, K)^{(\tau_K, \tau_T)} \cong (\mathcal{T}, \Omega, K)/(\rho_K, \rho_T)$$
where (ρ_K, ρ_T) is the soft congruence corresponding to (τ_K, τ_T).

Proof. The soft trioid structure follows from the semiretraction identities. The isomorphism is given by mapping each element to its congruence class in the quotient. The semiretraction properties ensure this is well-defined and preserves the soft trioid structure. □

The theory of soft congruences and semiretractions provides powerful tools for constructing and analyzing quotient structures of soft trioids.

Theorem 8.14. *Let $(\phi, \psi) : (\mathcal{T}_1, \Omega_1, K_1) \to (\mathcal{T}_2, \Omega_2, K_2)$ be a soft homomorphism with kernel soft congruence $(\ker \phi, \{\ker \psi_\alpha\}_{\alpha \in K_1})$. Then there is a natural soft isomorphism:*
$$(\mathcal{T}_1, \Omega_1, K_1)/(\ker \phi, \{\ker \psi_\alpha\}) \cong \mathrm{Im}(\phi, \psi)$$

Proof. Define the isomorphism by mapping each congruence class $[x]_{\ker \psi_\alpha}$ to $\psi(x)$ in the appropriate fiber. The soft homomorphism conditions ensure this is well-defined and preserves the soft trioid structure. $\square$

Theorem 8.15. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid and (ρ_K, ρ_T) be a soft congruence. There is a lattice isomorphism between:*

- *The lattice of soft subtrioids of $(\mathcal{T}, \Omega, K)$ containing the kernel of the projection*
- *The lattice of soft subtrioids of $(\mathcal{T}, \Omega, K)/(\rho_K, \rho_T)$*

Proof. The isomorphism sends a soft subtrioid $(\mathcal{T}', \Omega', K')$ containing the kernel to its image under the projection. The inverse map takes a soft subtrioid of the quotient and takes its preimage. The soft congruence conditions ensure these operations are well-defined and preserve the lattice structure. $\square$

Example 8.16. *Consider a soft trioid modeling an access control system where:*

- *Parameters represent security contexts (e.g., "normal", "emergency", "maintenance")*
- *Fibers represent user permission sets in each context*
- *Operations represent permission combinations*

A soft congruence can model security policy equivalences: two contexts are equivalent if they grant essentially the same permissions, and two permissions are equivalent if they provide the same access capabilities. The quotient soft trioid then represents a simplified security model that captures the essential policy structure.

Theorem 8.17. *Let $(\mathcal{T}, \Omega, K)$ be a soft trioid and (ρ_K, ρ_T) be a soft congruence. For any soft homomorphism $(\phi, \psi) : (\mathcal{T}, \Omega, K) \to (\mathcal{T}', \Omega', K')$ that factors through the congruence (i.e., $x\rho_T(\alpha)y$ implies $\psi(x) = \psi(y)$ and $\alpha\rho_K\beta$ implies $\phi(\alpha) = \phi(\beta)$), there exists a unique soft homomorphism making the following diagram commute:*

$$\begin{tikzcd}
(\mathcal{T}, \Omega, K) \arrow[r, "(\phi,\psi)"] \arrow[d, "\pi"'] & (\mathcal{T}', \Omega', K') \\
(\mathcal{T}, \Omega, K)/(\rho_K, \rho_T) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Proof. The unique soft homomorphism is defined by sending each congruence class to the common image of its elements. The factorization condition ensures this is well-defined, and the soft homomorphism properties are inherited from (ϕ, ψ). $\square$

9. Conclusion

This work develops a foundational theory of soft trioids by integrating the parameterized framework of Soft Set Theory with the algebraic structure of trioids. We introduced soft trioids and established their basic properties through examples and closure results under products, intersections, and $\wedge$-intersections. From a categorical perspective in Category Theory, the category **SoftTrioid** was shown to be complete and cocomplete, with well-defined subobject lattices and natural functorial constructions. A central contribution is the decomposition theorem, which proves that every soft trioid admits an essentially unique decomposition into a triband of soft simple components, extending classical results in Universal Algebra. This was further refined for idempotent fibers via semilattice decompositions. In addition, a soft congruence theory was developed, establishing a bijective correspondence between soft congruences and symmetric soft semiretractions, along with isomorphism theorems for quotient structures. Beyond its theoretical significance, this framework opens pathways for applications such as multi-criteria decision making, while leaving computational and higher-dimensional extensions as directions for future research.

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