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EULERIAN q-INTEGRALS AND FINITE MEROMORPHIC CONTINUATIONS FOR q-APPELL AND THEIR CONFLUENT FUNCTIONS

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Abstract. The purpose of this article is to continue our studies on multiple q-hypergeometric functions by proving more Eulerian q-integrals for the q-Appell functions. By using the formula for reversion of a finite q-hypergeometric series we can prove corresponding reversion formulas for three finite q-Appell functions. These formulas are, in fact, meromorphic continuations, since q-hypergeomtric functions have an infinite number of poles and zeros. First, however, we state the corresponding Appell function reversion formulas. Also confluent forms with the Euler q-exponential function are proved.

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Contents

1. Introduction and basic definitions

The classical theory for the hypergeometric functions and basic hypergeometric functions (or q-series) are important and well-known. There are many extensions of this classical theory, for example, by H.M. Srivastava [?] and by H. Exton [?], [?].

This paper is part of a series of papers on multiple q-hypergeometric functions. The other papers were about q-difference equations, q-integral representations [?], meromorphic continuations [?] and other solutions to corresponding systems of q-difference equations [?]. The general approach by Karlsson for multiple hypergeometric functions leads to proofs with a common notation, which simplifies the understanding of the long formulas.

The paper is organized as follows: In section **??** we repeat the definition of the Kampé de Fériet function together with the necessary q-calculus from [?]. In section **??** we define the pertinent multiple q-hypergeometric functions. In section **??** we prove variants of Eulerian q-integrals for q-Appell functions. In section **??** the reversion formulas for meromorphic continuation are proved. Finally, in section **??** we shall state corresponding formulas for the confluent functions.

First, we repeat the definition of some confluent double functions and of the Kampé de Fériet functions according to Karlsson. We now repeat some notation from [**?**]. The sign $\equiv$ denotes a definition and $\cong$ denotes a formal equality. In many formulas we put $B \equiv \sum_{i=1}^{n} b_i$ etc..

Definition 1. The Pochhammer symbol is defined by

$$a_n \equiv \prod_{k=0}^{n-1} (a+k). \tag{1}$$

Definition 2. The confluent double hypergeometric functions are [**?**]:

$$\begin{aligned} &\Phi_1(a,b;c;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b)_{m_1}}{(c)_{m_1+m_2} m_1! m_2!} x_1^{m_1} x_2^{m_2}, \\ &|x_1| < 1,\ |x_2| < \infty, \end{aligned} \tag{2}$$

$$\begin{aligned} &\Psi_1(a,b;c,c';x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b)_{m_1}}{(c)_{m_1}(c')_{m_2} m_1! m_2!} x_1^{m_1} x_2^{m_2}, \\ &|x_1| < 1,\ |x_2| < \infty, \end{aligned} \tag{3}$$

$$\begin{aligned} &\Psi_2(a;c,c';x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}}{(c)_{m_1}(c')_{m_2} m_1! m_2!} x_1^{m_1} x_2^{m_2}, \\ &|x_1| < \infty,\ |x_2| < \infty. \end{aligned} \tag{4}$$

Definition 3. [**?**, (24), p.38]. Denote by

$$A, B, G, H.$$

the lengths of the vectors

$$(a), (b), (g), (h)$$

Let

$$1 + B + H - A - G \geq 0.$$

Then the generalized Kampé de Fériet function is defined by

$$\mathrm{F}^{A:G}_{B:H}\left[\begin{array}{c} (a):(g_1);\ldots;(g_n) \\ (b):(h_1);\ldots;(h_n) \end{array}\middle|\vec{x}\right] \equiv \sum_{\vec{m}} \frac{((a))_m \prod_{j=1}^{n} ((g_j))_{m_j} x_j^{m_j}}{((b))_m \prod_{j=1}^{n} ((h_j))_{m_j} m_j!}. \tag{5}$$

The notation will be explained in the corresponding q-series definition.

Definition 4. Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane. In the whole paper, $|q| < 1$.

The power function is defined by

$$q^a \equiv e^{a \log(q)}. \tag{6}$$

The following notation, similar to EXP for e^x, is often used when we have long exponents, since otherwise the right hand side would be difficult to read.

Definition 5.
$$\mathrm{QE}(x) \equiv q^x. \tag{7}$$

Definition 6. [**?**, p.19] The q-analogues of a complex number a, a natural number n and the factorial are defined as follows:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{0,1\}, \tag{8}$$

$$\{n\}_q \equiv \sum_{k=1}^{n} q^{k-1},\ \{0\}_q = 0,\ q \in \mathbb{C}\backslash\{0,1\}, \tag{9}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}\backslash\{0,1\}. \tag{10}$$

Definition 7. The q-shifted factorial [**?**] is defined by

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m}). \tag{11}$$

Sometimes we also use

$$(a;q)_n \equiv \prod_{m=0}^{n-1}(1-aq^m). \tag{12}$$

There are three other types of q-shifted factorials [**?**]: in the equations (**??**) to (**??**) we assume that $(m,l)=1$.

Definition 8. In the following, $\dfrac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod$\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R}\times e^{2\pi i\theta},\ \theta \in \mathbb{R}$. The operator

$$\widetilde{\ }: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by the 2-torsion

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{13}$$

By (**??**) it follows that

$$\langle \widetilde{a};q\rangle_n = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{14}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$. Furthermore we define

$$\widetilde{\langle a;q\rangle}_n \equiv \langle \widetilde{a};q\rangle_n. \tag{15}$$

The generalized tilde operator

$$\widetilde{\frac{m}{l}}: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l \log q}. \tag{16}$$

We also need another generalization of the tilde operator.

$$\widetilde{{}_k\langle a;q\rangle_n} \equiv \prod_{m=0}^{n-1}(\sum_{i=0}^{k-1} q^{i(a+m)}). \tag{17}$$

Formula (**??**) is used in (**??**).

The following, simple congruence rules [**?**] are easily proved. The first formula follows because of the linearity in (**??**)). The second equation is a consequence of the fact that we work mod$\frac{2\pi i}{\log q}$. The third formula follows since the left hand side is multiplicative and $\widetilde{a}$ is a multiplication with $\frac{1}{2}$. The fourth formula follows by q-exponentiation of (**??**)).

Theorem 1.

$$\widetilde{\tfrac{m}{l}} a \pm b \equiv \widetilde{\tfrac{m}{l}(a \pm b)} \pmod{\frac{2\pi i}{\log q}}, \tag{18}$$

$$\sum_{k=1}^{n} \widetilde{\tfrac{1}{n} \pm a_k} \equiv \sum_{k=1}^{n} \pm a_k \pmod{\frac{2\pi i}{\log q}}, \tag{19}$$

$$\frac{m}{l} \times \widetilde{a} \equiv \widetilde{\tfrac{m}{2l} \frac{am}{l}} \pmod{\frac{2\pi i}{\log q}}, \tag{20}$$

$$\mathrm{QE}(\widetilde{\tfrac{m}{l} a}) = \mathrm{QE}(a) e^{\frac{2\pi i m}{l}}, \tag{21}$$

Definition 9.

$$\langle\lambda;q\rangle_{kn} \equiv \langle\triangle(q;k;\lambda);q\rangle_n \equiv \prod_{m=0}^{k-1}\langle\frac{\lambda+m}{k};q\rangle_n \times_k \langle\widetilde{\frac{\lambda+m}{k}};q\rangle_n. \tag{22}$$

We also use the notation $\triangle(q;k;\lambda)$ as a parameter in q-hypergeometric functions.

If λ is a vector, we mean the corresponding product of vector elements. If λ is replaced by a sequence of numbers, separated by commas, we mean the corresponding product, as in the case of q-factorials.

The last factor in (**??**) corresponds to k^{nk}.

With this notation, q-hypergeometric function- and hypergeometric function equations become very similar.

Definition 10. The q-derivative is defined by

$$(\mathrm{D}_q\varphi)(x) \equiv \begin{cases} \dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{when } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \dfrac{d\varphi}{dx}(x), & \text{when } q = 1; \\ \dfrac{d\varphi}{dx}(0), & \text{when } x = 0. \end{cases} \tag{23}$$

Definition 11. The q-integral is the inverse of the q-derivative.

$$\int_a^b f(t,q)\, d_q(t) \equiv \int_0^b f(t,q)\, d_q(t) - \int_0^a f(t,q)\, d_q(t),\ a,b \in \mathbb{R}, \tag{24}$$

where

$$\int_0^a f(t,q)\, d_q(t) \equiv a(1-q)\sum_{n=0}^{\infty} f(aq^n,q)q^n,\ 0<|q|<1,\ a\in\mathbb{R}. \tag{25}$$

Definition 12. We shall define a q-hypergeometric series by

$$\begin{aligned} &{}_{p+p'}\phi_{r+r'}\left[\begin{matrix} \hat{a}_1,\ldots,\hat{a}_p \\ \hat{b}_1,\ldots,\hat{b}_r \end{matrix}\ |q;z||\ \begin{matrix} \prod_i f_i(k) \\ \prod_j g_j(k) \end{matrix}\right] \equiv \\ &\sum_{k=0}^{\infty} \frac{\langle \hat{a}_1;q\rangle_k \ldots \langle \hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k \ldots \langle \hat{b}_r;q\rangle_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'} z^k \frac{\prod_i f_i(k)}{\prod_j g_j(k)}, \end{aligned} \tag{26}$$

where

$$\widehat{a} \equiv a \vee \widetilde{a} \vee \widetilde{\tfrac{m}{l}a} \vee_k \widetilde{a} \vee \triangle(q;l;\lambda). \tag{27}$$

In case of $\triangle(q;l;\lambda)$ the index is adjusted accordingly. It is assumed that the denominator contains no zero factors, i.e. $\widehat{b_k} \neq -l + \frac{2m\pi i}{logq}$, $k = 1,\ldots,r, l, m \in \mathbb{N}$. We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle \widehat{a(k)};q\rangle_k$ or $(s(k);q)_k$ respectively. In a few cases the parameter $\widehat{a}$ in (**??**) will be the real plus infinity
$(0<|q|<1)$. They correspond to multiplication by 1.

Theorem 2. *(A q-analogue of* [**?**, p. 42(3)] *and of a corrected form of Slater* [**?**, p. 48 (2.2.3.2)]*). Reversal of a finite q-hypergeometric series. Assume that* (α) *and* (β) *are vectors of size A. Then*

$$\begin{aligned} &{}_{A+1}\phi_A\left[\begin{matrix} (\alpha),-m \\ (\beta) \end{matrix}\middle| q;z\right] \\ &= \frac{\langle(\alpha);q\rangle_m}{\langle(\beta);q\rangle_m} q^{\frac{-m^2-m}{2}} (-z)^m {}_{A+1}\phi_A\left[\begin{matrix} 1-(\beta)-m,-m \\ 1-(\alpha)-m \end{matrix}\middle| q;q^{1+m+\sum_i \beta_i-\alpha_i} z^{-1}\right]. \end{aligned} \tag{28}$$

2. Definitions of multiple q-hypergeometric functions

The following definition, like in the one-varible case, allows easy limits for parameters to $\pm\infty$.

Definition 13. [**?**, p.367 f]. Let the vectors $\vec{x}$ and $\vec{q}$ have lengths n. To simplify notation for the q-exponential, let m denote $m \vee k$. In these cases, m does not equal $\sum m_i$. Let the vectors

$$(a),(b),(g_i),(h_i),(a'),(b'),(g_i'),(h_i')$$

have lengths

$$A,B,G_i,H_i,A',B',G_i',H_i'.$$

Let

$$1+B+B'+H_i+H_i'-A-A'-G_i-G_i' \geq 0, i=1,\ldots,n.$$

Then the generalized q-Kampé de Fériet function is defined by

$$\Phi^{A+A':G_1+G_1';\ldots;G_n+G_n'}_{B+B':H_1+H_1';\ldots;H_n+H_n'}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g_1');\ldots;(g_n')\\(b'):(h_1');\ldots;(h_n')\end{array}\right]\equiv$$
$$\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,k)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g_j')(q_j,k_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,k)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h_j')(q_j,k_j)\langle 1;q_j\rangle_{m_j})}\times$$
$$(-1)^{\sum_{j=1}^n m_j(1+H_j+H_j'-G_j-G_j'+B+B'-A-A')}\times$$
$$\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)$$
$$\times\prod_{j=1}^n\mathrm{QE}\left((1+H_j+H_j'-G_j-G_j')\binom{m_j}{2},q_j\right), \tag{29}$$

where

$$\widehat{a}\equiv a\vee\widetilde{a}\vee\widetilde{\tfrac{m}{l}a}\vee_k\widetilde{a}\vee\triangle(q;l;\lambda). \tag{30}$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0,k),(g_j')(q_j,k_j),(b')(q_0,k),(h_j')(q_j,k_j)$ contain factors of the form $\langle a\hat{(k)};q\rangle_k$, $(s;q)_k$, $(s(k);q)_k$ or $\mathrm{QE}\,(f(\vec{m}))$.

The numbers in front of the colon denote the number of q-shifted factorials with index $m+k$ in numerator and denominator. The numbers after the colon denote the number of q-shifted factorials with index m_i in numerator and denominator. Equally, the numbers after the semicolon denote the number of q-shifted factorials with index k_i in numerator and denominator. We can skip G_2 if it is equal to G_1 for two variables etc. Every ∞ corresponds to multiplication with 1.

Definition 14. The q-Appell functions are defined by [?]

$$\Phi_1(a;b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \quad \max(|x_1|,|x_2|)<1. \tag{31}$$

$$\Phi_2(a;b,b';c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |x_1|\oplus_q|x_2|<1. \tag{32}$$

$$\Phi_3(a,a';b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \quad \max(|x_1|,|x_2|)<1. \tag{33}$$

$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |\sqrt{x_1}|\oplus_q|\sqrt{x_2}|<1. \tag{34}$$

Definition 15. Three confluent double q-hypergeometric functions [?], [?] are defined by

$$\Upsilon_1(a;b;c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \quad |x_1|<1,\ |(1-q)x_2|<\infty, \tag{35}$$

$$\Psi_1(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |x_1|<1,\ |(1-q)x_2|<\infty, \tag{36}$$

$$\Psi_2(a;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |(1-q)x_1|<\infty,\ |(1-q)x_2|<\infty. \tag{37}$$

3. Variants of Eulerian q-integrals for q-Appell functions

In this section we shall prove variants of Eulerian q-integrals for q-Appell functions. Proofs are given by using the following formula for multiplicative substitution in q-integrals

Lemma 3. [**?**, (6.64)] *Linear substitution in a q-integral.*

$$\int_0^x f(t,q)\ d_q(t) = a\int_0^{\frac{x}{a}} f(at,q)\ d_q(t). \tag{38}$$

Definition 16. The Γ_q function [**?**] is defined by

$$\Gamma_q(z) \equiv \begin{cases} \frac{\langle 1;q\rangle_\infty}{\langle z;q\rangle_\infty}(1-q)^{1-z} & \text{if } 0<|q|<1 \\ \frac{\langle 1;q^{-1}\rangle_\infty}{\langle z;q^{-1}\rangle_\infty}(\text{q}-1)^{1-z}q^{\binom{z}{2}}, & \text{if } |q|>1. \end{cases} \tag{39}$$

Definition 17. [**?**, (7.54)] The q-beta function is defined by

$$\text{B}_q(x,y) \equiv \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{40}$$

Theorem 4. [**?**, (7.55)] *Let $\Re(x) > 0$. Then the q-Euler beta integral is given by*

$$\text{B}_q(x,y) \cong \int_0^1 t^{x-1}(qt;q)_{y-1}\, d_q(t),\ \ y \neq 0,-1,-2,\ldots. \tag{41}$$

Theorem 5. *A new q-Eulerian integral transformation for the first q-Appell function and a q-analogue of* [**?**, (A.1.2.34)].

$$\begin{aligned} &\text{B}_q(s,b-s)\Phi_1(a,s,b';c|q;x,y) \\ &\cong \int_0^1 t^{s-1}(qt;q)_{b-s-1}\Phi_1(a,b,b';c|q;xt,y)\ d_q(t). \end{aligned} \tag{42}$$

Proof. Use formula (**??**). □

Theorem 6. *A q-analogue of* [**?**, (3.13)]*:*

$$\begin{aligned} &\text{B}_q(s,b-s)w^{b-1}\Phi_1(a;s,b';c|q;uw,v) \\ &\cong \int_0^w t^{s-1}w^{b-s-1}(qtw^{-1};q)_{b-s-1}\Phi_1(a;b,b';c|q;ut,v)\, d_q(t). \end{aligned} \tag{43}$$

Proof. By using the above formulas we have

$$\begin{aligned} &\text{B}_q(s,b-s)w^{b-1}\Phi_1(a;s,b';c|q;uw,v) \\ &\overset{\text{by(??)}}{\cong} w^{b-1}\int_0^1 t^{s-1}(qt;q)_{b-s-1}\Phi_1(a,b,b';c|q;uwt,v)\ d_q(t) \\ &\overset{\text{by(??)}}{\cong}\ . \end{aligned} \tag{44}$$

This equals the left hand side. □

Theorem 7. *A q-analogue of* [**?**, A.1.2.61].

$$\begin{aligned} &\int_0^1 t^{f-1}(tq;q)_{g-1}\Phi_1(a;b,b';c|q;tx,ty)d_q(t) \\ &\cong \Gamma_q\left[\begin{array}{c} f,g \\ f+g \end{array}\right]\Phi^{2:1}_{2:0}\left[\begin{array}{c} f,a:b;b' \\ f+g,c:\cdot;\cdot \end{array}\middle|q;x,y\right]. \end{aligned} \tag{45}$$

Proof.

$$\begin{aligned} &\int_0^1 t^{f-1}(tq;q)_{g-1}\sum_{m,n=0}^{\infty}\frac{\langle a;q\rangle_{m+n}\langle b;q\rangle_m\langle b';q\rangle_n}{\langle c;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1;q\rangle_n}x^m y^n t^{m+n}d_q(t) \\ &\overset{\text{by}[?,(7.55)]}{=} \sum_{m,n=0}^{\infty}\frac{\langle a;q\rangle_{m+n}\langle b;q\rangle_m\langle b';q\rangle_n}{\langle c;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1;q\rangle_n}x^m y^n\Gamma_q\left[\begin{array}{c} f+n+m,g \\ f+g+n+m \end{array}\right] \\ &\overset{\text{by}[?,(1.46)]}{=} . \end{aligned} \tag{46}$$

This equals the right hand side. □

Theorem 8. *A q-analogue of* [**?**, (3.14)]*:*

$$\begin{aligned} &\mathrm{B}_q(s,a-s)w^{a-1}\Phi_1(s;b,b';c|q;uw,vw) \\ &\cong \int_0^w t^{s-1}w^{a-s-1}(qtw^{-1};q)_{a-s-1}\Phi_1(a;b,b';c|q;ut,vt)\,d_q(t). \end{aligned} \tag{47}$$

Proof. Use formulas (**??**) and (**??**). □

Theorem 9. *A q-analogue of Feldheim* [**?**, (14)] *and Exton* [**?**, A.1.2.60].

$$\begin{aligned} &\int_0^1 t^{f-1}(tq;q)_{g-1}\Phi_1(a;b,b';f|q;tx,ty)d_q(t) \\ &\cong \Gamma_q\left[\begin{array}{c} f,g \\ f+g \end{array}\right]\Phi_1(a;b,b';f+g|q;x,y). \end{aligned} \tag{48}$$

Theorem 10. *A q-analogue of* [**?**, (3.15)]*:*

$$\begin{aligned} &\mathrm{B}_q(c,s-c)w^{s-1}\Phi_1(a;b,b';s|q;uw,vw) \\ &\cong \int_0^w t^{c-1}w^{s-c-1}(qtw^{-1};q)_{s-c-1}\Phi_1(a;b,b';c|q;ut,v)\,d_q(t). \end{aligned} \tag{49}$$

Proof. Use formulas (**??**) and (**??**). □

4. Finite reversions of three q-Appell functions

Earlier in [**?**], we proved meromorphic continuations of several non-terminating q-hypergeometric functions. As usual in q-calculus, the finite function formulas are different. Therefore, we shall prove three q-Appell function finite reversal formulas, one of which is a q-analogue of a formula by Brychkov and Nasser. First we present the two finite reversions of Appell functions F_1 and F_2. Their proofs are relegated to their respective q-analogues.

Theorem 11. *Finite reversion of F_1.*

$$\mathrm{F}_1(-n;b,b';c;x,y)=\frac{(b)_n}{(c)_n}(-x)^n\mathrm{F}_1\left(-n;1-c-n,b';1-b-n;\frac{1}{x},\frac{y}{x}\right). \tag{50}$$

Finite reversion of F_2.

$$\mathrm{F}_2(-n;b,b';c,c';x,y)=\frac{(b)_n}{(c)_n}(-x)^n\mathrm{F}^{2:0;1}_{1:0;1}\left[\begin{array}{c}-n,1-c-n:\cdot;b'\\ 1-b-n:\cdot;c'\end{array}\middle|\frac{1}{x},-\frac{y}{x}\right]. \tag{51}$$

We need the following lemmas for the proofs

Lemma 12.

$$\begin{aligned}\frac{\langle b;q\rangle_{n-k}}{\langle 1-b-n+k;q\rangle_l}&=\frac{\langle b;q\rangle_n}{\langle 1-b-n;q\rangle_{k+l}}(-1)^k\\ &\times\mathrm{QE}\left(\binom{k}{2}+k((1-b-n)\right),\ n\geq k.\end{aligned} \tag{52}$$

Lemma 13.

$$\frac{\langle b+n;q\rangle_k\langle 1-b-n;q\rangle_l}{\langle 1-b-n-k;q\rangle_{k+l}}=(-1)^k\mathrm{QE}\left(\binom{k}{2}+k((b+n)\right). \tag{53}$$

Proof. Use formula [**?**, (6.14)]. □

We start with the first q-Appell function.

Theorem 14. *The reversion of the finite first q-Appell function and a q-analogue of (**??**) is given by*

$$\begin{aligned}\Phi_1(-n;b,b';c|q;x,y)&=\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}}\\ &\times\Phi_1\left(-n,1-c-n,b';1-b-n\middle|q;q^{1+c+n-b}\frac{1}{x},q^{1-b}\frac{y}{x}\right).\end{aligned} \tag{54}$$

Proof. The left hand side equals

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\langle -n,b';q\rangle_k}{\langle 1,c;q\rangle_k}y^k{}_2\phi_1\left[\begin{array}{c}b,-n+k\\ c+k\end{array}\middle|q;x\right]\\ &\overset{\text{by}(??)}{=}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{\langle -n,b';q\rangle_k\langle b;q\rangle_{n-k}\langle -n+k,1-c-n;q\rangle_l}{\langle 1,c;q\rangle_k\langle c+k;q\rangle_{n-k}\langle 1,1-b-n+k;q\rangle_l}\\ &\times y^k x^{-l}(-x)^{n-k}\mathrm{QE}\left(l(1+c+n-b)-\frac{(n-k)^2+(n-k)}{2}\right)\\ &\overset{\text{by}(??)}{=}\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}}\sum_{k=0}^{n}\sum_{l=0}^{n-k}y^k x^{-k-l}\\ &\times\frac{\langle -n;q\rangle_{k+l}\langle b';q\rangle_k\langle 1-c-n;q\rangle_l}{\langle 1;q\rangle_k\langle 1;q\rangle_l\langle 1-b-n;q\rangle_{k+l}}\mathrm{QE}\left(l(1+c+n-b)+k(1-b)\right).\end{aligned} \tag{55}$$

This equals the right hand side. □

Theorem 15. *Reversion of the finite second q-Appell function and a q-analogue of (**??**).*

$$\Phi_2(-n;b,b';c,c'|q;x,y)=\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}} \\ \times\Phi^{2:0;2}_{1:0;1}\left[\begin{array}{c}-n,1-c-n:\cdot;b',\infty\\ 1-b-n:\cdot;c'\end{array}\middle|q;-q^{1+c-b+n}\frac{1}{x},-q^{c-b+n}\frac{y}{x}\right]. \tag{56}$$

Proof. The left hand side equals

$$\sum_{k=0}^{n}\frac{\langle -n,b';q\rangle_k}{\langle 1,c';q\rangle_k}y^k{}_2\phi_1\left[\begin{array}{c}b,-n+k\\ c\end{array}\middle|q;x\right] \\ \overset{\text{by}(??)}{=}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{\langle -n,b';q\rangle_k\langle b;q\rangle_{n-k}\langle -n+k,1-c-n+k;q\rangle_l}{\langle 1,c';q\rangle_k\langle c;q\rangle_{n-k}\langle 1,1-b-n+k;q\rangle_l} \\ \times y^k x^{-l}(-x)^{n-k}\mathrm{QE}\left(l(1+c+n-k-b)-\frac{n^2+k^2-2nk+n-k}{2}\right) \\ \overset{\text{by}2\times(??)}{=}\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}}\sum_{k=0}^{n}\sum_{l=0}^{n-k}y^k x^{-k-l}(-1)^k \\ \times\frac{\langle -n;q\rangle_{k+l}\langle b';q\rangle_k\langle 1-c-n;q\rangle_{k+l}}{\langle 1,c';q\rangle_k\langle 1;q\rangle_l\langle 1-b-n;q\rangle_{k+l}}\mathrm{QE}\left(l(1+c+n-b)+k(c-b+n)\right) \\ \times\mathrm{QE}\left(-\binom{k+l}{2}+\binom{l}{2}\right). \tag{57}$$

This equals the right hand side. □

Theorem 16. *Reversion of the finite fourth q-Appell function and a q-analogue of* [**?**, (3.8)]*:*

$$\Phi_4(-n;b;c,c'|q;x,y)=\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}} \\ \times\Phi^{2:0;2}_{0:1;1}\left[\begin{array}{ccc}-n,1-c-n: & \cdot; & 2\infty\\ \cdot: & 1-b-n; & c'\end{array}\middle|q;q^{1+c-b+n}\frac{1}{x},q^{c+2n-1}\frac{y}{x}\right]. \tag{58}$$

Proof. The left hand side equals

$$\sum_{k=0}^{n}\frac{\langle -n,b;q\rangle_k}{\langle 1,c';q\rangle_k}y^k{}_2\phi_1\left[\begin{matrix} b+k,-n+k \\ c\end{matrix}\middle|q;x\right]$$

$$\overset{\text{by(??)}}{=}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{\langle -n,b;q\rangle_k\langle b+k;q\rangle_{n-k}\langle -n+k,1-c-n+k;q\rangle_l}{\langle 1,c';q\rangle_k\langle c;q\rangle_{n-k}\langle 1,1-b-n;q\rangle_l}$$

$$\times\, y^k x^{-l}(-x)^{n-k}\mathrm{QE}\left(l(1+c+n-b-2k)-\frac{(n-k)^2+(n-k)}{2}\right)$$

$$\overset{\text{by}2\times\text{(??)}}{=}(-x)^n\sum_{k=0}^{n}\sum_{l=0}^{n-k}y^k x^{-k-l}(-1)^k\frac{\langle b;q\rangle_{n+k}\langle -n,1-c-n;q\rangle_{k+l}}{\langle c;q\rangle_n\langle 1-b-n-k;q\rangle_{k+l}\langle 1;q\rangle_l\langle 1,c';q\rangle_k}$$

$$\times\,\mathrm{QE}\left(l(1+c+n-b-2k)-\frac{(n-k)^2+(n-k)}{2}+k(-b-k+c)\right)$$

$$\overset{\text{by(??)}}{=}\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{y^k x^{-k-l}\langle -n,1-c-n;q\rangle_{k+l}}{\langle 1,1-b-n;q\rangle_l\langle 1,c';q\rangle_k}$$

$$\times\,\mathrm{QE}\left(l(1+c+n-b)+k(-1+2n+c)+2\left(-\binom{k+l}{2}+\binom{l}{2}\right)\right). \tag{59}$$

Again, this equals the right hand side. ☐

5. Confluent formulas

We can now obtain corresponding formulas for the confluent (q-)functions by a simple limiting process. For the q-functions this means letting some parameters $\to\infty$. We start with the formulas for Υ_1.

Theorem 17. *Four q-integral formulas for Υ_1 are given by*

$$\begin{aligned}&\mathrm{B}_q(s,b-s)\Upsilon_1(a,s;c|q;x,y)\\ &\cong\int_0^1 t^{s-1}(qt;q)_{b-s-1}\Upsilon_1(a,b;c|q;xt,y)\ d_q(t),\end{aligned}\tag{60}$$

$$\begin{aligned}&\mathrm{B}_q(s,b-s)w^{b-1}\Upsilon_1(a,s;c|q;uw,v)\\ &\cong\int_0^w t^{s-1}w^{b-s-1}(qtw^{-1};q)_{b-s-1}\Upsilon_1(a,b;c|q;ut,v)\,d_q(t),\end{aligned}\tag{61}$$

$$\begin{aligned}&\mathrm{B}_q(s,a-s)w^{a-1}\Upsilon_1(s,b;c|q;uw,vw)\\ &\cong\int_0^w t^{s-1}w^{a-s-1}(qtw^{-1};q)_{a-s-1}\Upsilon_1(a,b;c|q;ut,vt)\,d_q(t),\end{aligned}\tag{62}$$

A q-analogue of [**?**, (3.17)] is given by

$$\begin{aligned}&\mathrm{B}_q(c,s-c)w^{s-1}\Upsilon_1(a,b;s|q;uw,vw)\\&\cong\int_0^w t^{c-1}w^{s-c-1}(qtw^{-1};q)_{s-c-1}\Upsilon_1(a,b;c|q;ut,v)\,d_q(t).\end{aligned}\tag{63}$$

Proof. Let $b'\to\infty$ in (**??**). □

Theorem 18. *Finite reversion of the hypergeometric function Φ_1 is given by*

$$\Phi_1(-n,b;c;x,y)=\frac{(b)_n}{(c)_n}(-x)^n\Phi_1\left(-n,1-c-n;1-b-n;\frac{1}{x},\frac{y}{x}\right).\tag{64}$$

Theorem 19. *The finite reversion of Υ_1 is given by*

$$\begin{aligned}&\Upsilon_1(-n,b;c|q;x,y)=\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}}\\&\times\Upsilon_1\left(-n,1-c-n;1-b-n\middle|q;q^{1+c+n-b}\frac{1}{x},q^{1-b}\frac{y}{x}\right).\end{aligned}\tag{65}$$

We continue with the formulas for Ψ_1.

Theorem 20. *Finite reversion of the hypergeometric function Ψ_1 is given by*

$$\Psi_1(-n,b;c,c';x,y)=\frac{(b)_n}{(c)_n}(-x)^n\mathrm{F}^{2:0;0}_{1:0;1}\left[\begin{array}{ccc}-n,1-c-n: & \cdot; & \cdot\\ 1-b-n: & \cdot; & c'\end{array}\middle|\frac{1}{x},-\frac{y}{x}\right].\tag{66}$$

Theorem 21. *The finite reversion of Ψ_1 is given by*

$$\begin{aligned}&\Psi_1(-n,b;c,c'|q;x,y)=\frac{\langle b;q\rangle_n}{\langle c;q\rangle_n}(-x)^n q^{\frac{-n^2-n}{2}}\\&\times\Phi^{2:0;2}_{1:0;1}\left[\begin{array}{c}-n,1-c-n:\cdot;2\infty\\ 1-b-n:\cdot;c'\end{array}\middle|q;-q^{1+c-b+n}\frac{1}{x},-q^{c-b+n}\frac{y}{x}\right].\end{aligned}\tag{67}$$

Finally, we state the formulas for Ψ_2, which is obtained by letting $b\to\infty$ in Φ_4.

Theorem 22. *The finite reversion of the hypergeometric function Ψ_2 is given by*

$$\Psi_2(-n;c,c';x,y)=\frac{(-x)^n}{(c)_n}\mathrm{F}^{2:0;0}_{0:0;1}\left[\begin{array}{ccc}-n,1-c-n: & \cdot; & \cdot\\ \cdot: & \cdot; & c'\end{array}\middle|\frac{1}{x},\frac{y}{x}\right].\tag{68}$$

Theorem 23. *The finite reversion of Ψ_2 is given by*

$$\begin{aligned}&\Psi_2(-n;c,c'|q;x,y)=\frac{(-x)^n}{\langle c;q\rangle_n}q^{\frac{-n^2-n}{2}}\\&\times\Phi^{2:0;2}_{0:0;1}\left[\begin{array}{ccc}-n,1-c-n: & \cdot; & 2\infty\\ \cdot: & \cdot; & c'\end{array}\middle|q;q^{c+2n}\frac{1}{x},q^{c+2n-1}\frac{y}{x}\right].\end{aligned}\tag{69}$$

6. Conclusion

We managed to present our formulas in a systematic way, which has not been done earlier. It may be pointed out that all three reversions for q-Appell functions begin with the same factor. The third Appell function was not discussed because of its lack of symmetry. The reversion formulas for n large do not converge for large values of the variable x, which indicates analytic or meromorphic continuations. This was not pointed out in [**?**].

7. Discussion

There is a double Euler integral representation of the second (q-)Appell function, which is proved by manipulation with (q-)Γ functions. However, the limit (in the q-case) b, b' and $c' \to \infty$ does not work out well. This formula is therefore not adaptable to these confluences. Our formulas complement the single q-integral formulas from [**?**]. Observe that the more symmetric function, like Φ_1, the more confluent function formulas. We could have included many formulas for q-derivatives, these were published in our paper [**?**].

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