Advanced Studies in Contemporary Mathematics
36 (2026), No. 3, pp. 209 - 211

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2026.36.3.006

A NOTE ON THE LANCZOS ORTHOGONAL DERIVATIVE

J.D. BULNES, T. KIM, J. LÓPEZ-BONILLA, AND H.E. CAICEDO-ORTIZ

Abstract. Differentiation via integration can be performed using the Lanczos derivative. Here we employ a quadrature method to deduce this generalized derivative.

2000 Mathematics Subject Classification. 26A24, 26A36, 41A55.

Keywords and phrases. Orthogonal derivative, Integration by differentiation, Lanczos derivative, Legendre polynomials, Differentiation by integration, Quadrature method.

Submission date. 2th June 2026.

1. Introduction

Lanczos obtained the following remarkable quadrature formula where only are needed the values of the function and its derivatives at the end points [1]:

$$\int_a^b g(t)dt = \sum_{k=1}^{n} \frac{(2n-k)!}{(2n)!}\binom{n}{k}\Big[g^{(k-1)}(a) - (-1)^k g^{(k-1)}(b)\Big]c^k, \quad c = b-a. \tag{1}$$

In Sec. 2 we use (1) to deduce the Lanczos orthogonal derivative [1-9]:

$$f'(x_0) = \lim_{\epsilon\to 0} \frac{3}{2\epsilon^3}\int_{-\epsilon}^{\epsilon} t\, f(x_0+t)\, dt, \tag{2}$$

which represents differentiation by integration [10].

2. Lanczos derivative via a quadrature method

Now we employ (1) for the case $b = -a = \epsilon$, $c = 2\epsilon$, $g(t) = tf(x_0+t)$ and eventually we will become interested in the value of $\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_{-\epsilon}^{\epsilon} t\, f(x_0+t)\, dt$ then in (1) is only necessary to consider its first three terms, therefore:

$$\lim_{\epsilon\to 0}\frac{c}{\epsilon^3}\Big[g(a)+g(b)\Big] = 4\lim_{\epsilon\to 0}\frac{f(x_0+\epsilon)-f(x_0-\epsilon)}{2\epsilon} = 4f'(x_0),$$

$$\lim_{\epsilon\to 0}\frac{c^2}{\epsilon^3}\Big[g'(a)+g'(b)\Big] = -4\lim_{\epsilon\to 0}\Big[f'(x_0-\epsilon)+f'(x_0+\epsilon)+2\frac{f(x_0+\epsilon)-f(x_0-\epsilon)}{2\epsilon}\Big] = -16f'(x_0)$$

(3)

$$\lim_{\epsilon\to 0}\frac{c^3}{\epsilon^3}\Big[g''(a)+g''(b)\Big] = 8\lim_{\epsilon\to 0}\Big\{2\Big[f'(x_0-\epsilon)+f'(x_0+\epsilon)\Big]+\epsilon\Big[f''(x_0+\epsilon)-f''(x_0-\epsilon)\Big]\Big\} =$$

$$= 32\, f'(x_0)$$

hence from (1) and (3):

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_{-\epsilon}^{\epsilon} t\, f(x_0 + t)\, dt = \Big[2 - \frac{4(n-1)}{2n-1} + \frac{4(n-2)}{3(2n-1)}\Big] f'(x_0) = \frac{2}{3} f'(x_0), \tag{4}$$

in total agreement with (2), *q.e.d.*

Lanczos derivative for higher orders was studied by Rangarajan-Purushothaman [11-14] via Legendre polynomials [15-17], in fact:

$$f^{(n)}(x) = \lim_{\epsilon \to 0} \frac{(2n+1)!!}{2\epsilon^{n+1}} \int_{-\epsilon}^{\epsilon} P_n\Big(\frac{t}{\epsilon}\Big) f(x+t)\, dt \tag{5}$$

which reproduces (2) for $n = 1$ because $P_1\left(\frac{t}{\epsilon}\right) = \frac{t}{\epsilon}$.

References

[1] C. Lanczos, *Applied analysis*, Prentice-Hall, New Jersey (1956).
[2] N. Cioranescu, *La generalization de la premiére formule de la moyenne*, Enseign. Math. 37 (1938) 292-302.
[3] U. S. Haslam-Jones, *On a generalized derivative*, Quart. J. Math. Oxford Ser. (2) 4 (1953) 190-197.
[4] E. Diekema, T. H. Koornwinder, *Differentiation by integration using orthogonal polynomials, a survey*, J. of Approximation Theory 164 (2012) 637-667.
[5] R. Cruz-Santiago, J. López-Bonilla, R. López-Vázquez, *Lanczos derivative applied to Fourier series*, J. Sci. Res. (India) 57 (2013) 165-167.
[6] A. Hernández-Galeana, P. Laurian-Ioan, J. López-Bonilla, R. López-Vázquez, *On the Cioranescu-(Haslam-Jones)-Lanczos generalized derivative*, Global J. Adv. Res. on Classical and Modern Geom. 3, No. 1 (2014) 44-49.
[7] G. Pérez-Teruel, *A new class of generalized Lanczos derivatives*, Palestine J. of Maths. 7, No. 1 (2018) 211-221.
[8] J. D. Bulnes, J. López-Bonilla, R. Rajendra, P. S. Kota Reddy, *On the Lanczos orthogonal derivative*, South East Asian J. of Maths. and Mathematical Sci. 19, No. 3 (2023) 31-36.
[9] M. A. Pathan, H. Kumar, T. Kim, J. López-Bonilla, *Lanczos derivative via Ruffa-Toni's expression for a definite integral and solution of non-homogeneous double Fredholm integral equations*, Adv. Studies in Contemporary Maths. 35, No. 3 (2025) 203-215.
[10] A. Kempf, D. M. Jackson, A. H. Morales, *How to (path-) integrate by differentiating*, J. of Phys: Conf. Series 626 (2015) 012015.
[11] S. K. Rangarajan, S. P. Purushothaman, *Lanczos generalized derivative for higher orders*, J. Comp. Appl. Maths. 177, No. 2 (2005) 461-465.
[12] R. Cruz-Santiago, J. López-Bonilla, H. Torres-Silva, *Lanczos orthogonal derivative for higher orders*, Transactions on Maths. 3, No. 3 (2017) 12-14.
[13] A. Liptaj, *Maximal generalization of Lanczos derivative using one-dimensional integrals*, Palestine J. of Maths. 10, No. 2 (2021) 694-699.
[14] A. Liptaj, *Higher accuracy order in differentiation-by-integration*, Mathematical Modelling and Analysis 26, No. 2 (2021) 304-317.
[15] A. Sommerfeld, *Partial differential equations in Physics*, Academic Press, New York (1964).
[16] J. López-Bonilla, R. López-Vázquez, H. Torres-Silva, *On the Legendre polynomials*, Prespacetime Journal 6, No. 8 (2015) 735-739.
[17] J. D. Bulnes, J. López-Bonilla, J. Prajapati, *Certain integrals involving Legendre polynomials*, Open Access J. Sci. 6, No. 1 (2023) 1-3.

Departamento de Ciências Exatas e Tecnologia, Universidade Federal do Amapá, Rod. Juscelino Kubitschek, 68903-419, Macapá, AP, Brasil;
E-mail address: bulnes@unifap.br

Department of Mathematics, College of Natural Sciences, Kwangwoon University, Seoul 139−701, Republic of Korea;
E-mail address: tkkim@kw.ac.kr

ESIME−Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista, CP 07738, CDMX, México;
E-mail address: jlopezb@ipn.mx

Universidad Nacional Rosario Castellanos & Universidad Anáhuac México, CDMX, México.
E-mail address: hernando.caicedo@rcastellanos.cdmx.gob.mx