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Optimal Control Problems with Discontinuous Right-Hand Side of Differential Equations: Challenges, Methods, and Research Prospects

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ABSTRACT. This paper addresses optimal control problems for dynamical systems with discontinuous right-hand sides. Such systems naturally arise in engineering applications involving switching, friction, and contact phenomena. Classical smooth optimal control theory fails to adequately describe these dynamics, necessitating the use of differential inclusions in the sense of Filippov. We review the mathematical foundations, including generalized optimality conditions and Hamilton–Jacobi–Bellman theory, discuss analytical and numerical solution methods, and highlight key application domains. Open problems and future research directions are also outlined.

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1. Introduction

Classical optimal control theory relies on the smoothness of the system dynamics. However, many real-world systems—such as robotic manipulators, mechanical systems with friction, and power electronic converters—exhibit intrinsic discontinuities due to switching, relay actions, and contact interactions [1,3,4,9]. It leads to necessity to consider a controlled dynamical system of the form:

$$\dot{x} = f(x,u,t)$$

where the function f is discontinuous with respect to the state variable x on a switching manifold. Even when trajectories cross the discontinuity surface transversally, numerical differentiation of simulated trajectories produces gradients whose errors do not vanish with decreasing integration step size. This renders standard discretization-refinement strategies ineffective. Discontinuous ordinary differential equations (ODEs) thus form a fundamental class of models in modern control applications [5,7].

This paper considers optimal control problems for dynamical systems described by differential equations with discontinuous right-hand sides. Such models naturally arise in applications involving switching dynamics, impacts, threshold phenomena, and hybrid control systems. The presence of discontinuities leads to substantial analytical and computational challenges, including the absence of classical solutions, non-smooth value functions, and complex optimal control behaviors such as chattering. To address these issues, the study employs generalized solution concepts, including differential inclusions and Filippov trajectories, along with extensions of optimality conditions based on nonsmooth analysis [7]. The relevance of this topic is driven by the growing number of real-world systems exhibiting discontinuous dynamics, particularly in engineering, economics, and cyber-physical systems. The paper also outlines current methodological approaches and their limitations. Future research perspectives include the development of efficient numerical methods, integration with machine learning techniques, and extensions to stochastic and robust control frameworks, highlighting the continued importance and timeliness of this field.

Optimal control theory is a cornerstone of modern applied mathematics, with applications ranging from engineering systems and economics to biology and robotics. Classical formulations typically assume smooth dynamics, where the right-hand side of the governing differential equations is continuous (and often differentiable) [1,6,8,10,11]. However, many real-world systems exhibit abrupt changes, switching behavior, or nonsmooth phenomena. These features naturally lead to **optimal control problems with discontinuous right-hand sides**, a class of problems that remains both mathematically challenging and practically significant.

2. Mathematical Foundations

Filippov Differential Inclusions

The rigorous treatment of discontinuous systems is based on Filippov's extension of ODEs to differential inclusions [7]. A discontinuous system is interpreted as:

$$\dot{x} = f(x,u,t)$$

where f is a set-valued map constructed as the convex closure of limit values of f near discontinuities. This formulation ensures well-posedness under appropriate conditions.

Generalized Maximum Principle

For systems with discontinuous dynamics, Pontryagin's Maximum Principle (PMP) must be extended [1,4,8,10,11]. In particular, when optimal trajectories lie on switching surfaces over nontrivial intervals (sliding modes), classical nondegeneracy conditions fail. Refined necessary conditions have been developed to address such cases.

Hamilton–Jacobi–Bellman Framework

The value function associated with such control problems satisfies a Hamilton–Jacobi–Bellman (HJB) equation defined in the viscosity sense. Under suitable assumptions, the value function is the unique locally Lipschitz continuous viscosity solution of the HJB equation [1,10,11].

3. Solution Methods

Analytical Methods

Generalized versions of Pontryagin's Maximum Principle remain the primary analytical tool [1]. In hybrid systems, where switching sequences are state-dependent, specialized formulations of PMP incorporate switching conditions explicitly.

Numerical Methods

A common computational strategy is to approximate discontinuous dynamics by smooth functions. Convergence of computed gradients to true gradients is ensured if:

- the trajectory endpoints do not lie on discontinuity surfaces,
- the integration step size is sufficiently small relative to the smoothing parameter.

Advanced methods such as Finite Elements with Switch Detection (FESD) significantly improve accuracy without increasing computational cost.

Sliding Mode Control

Sliding mode control exploits discontinuities to enforce robustness. Although classical existence and uniqueness theorems do not apply, Filippov solutions [7] provide a consistent framework for analysis.

4. Relevance of the Problem

Discontinuities arise in numerous contexts. Mechanical systems with impacts, electrical circuits with switching elements, economic models with threshold policies, and control systems with bang-bang or hybrid strategies all involve discontinuous dynamics. In such systems, the evolution of the state cannot be adequately described using classical differential equations alone; instead, one must consider generalized notions such as differential inclusions or Filippov solutions [7].

The increasing prevalence of cyber-physical systems, autonomous technologies, and hybrid control architectures has made discontinuous models even more relevant. For instance, autonomous vehicles rely on switching between control modes, while power systems incorporate discrete decisions alongside continuous dynamics. These applications demand robust theoretical frameworks that can accommodate discontinuities while still enabling optimization.

5. Mathematical Challenges

The presence of discontinuities fundamentally complicates the analysis of optimal control problems. Some of the main challenges include:

- *Existence and Uniqueness of Solutions*: Classical existence theorems for differential equations often fail when the right-hand side is discontinuous. Generalized solution concepts, such as Filippov solutions or Carathéodory solutions, must be employed.
- *Non-smoothness of the Value Function*: The value function in optimal control problems with discontinuities is typically non-differentiable. This limits the applicability of classical methods such as the Pontryagin Maximum Principle in its standard form.
- *Chattering and Zeno Behavior*: Optimal controls may exhibit infinitely fast switching (chattering), which complicates both theoretical analysis and numerical implementation.
- *Lack of Convexity*: Discontinuities often lead to non-convex feasible sets, making optimization more difficult and potentially leading to multiple local optima.

6. Analytical and Numerical Approaches

To address these challenges, several approaches have been developed:

- *Differential Inclusions*: By interpreting discontinuous systems as set-valued maps, one can extend classical control theory to include nonsmooth dynamics. This allows the use of viability theory and generalized gradients.
- *Generalized Maximum Principles*: Extensions of the Pontryagin Maximum Principle have been formulated for nonsmooth systems, incorporating tools from nonsmooth analysis such as Clarke's generalized gradients.
- *Regularization Techniques*: Discontinuous systems can sometimes be approximated by smooth ones, allowing classical methods to be applied and then passing to the limit.
- *Numerical Methods*: Specialized algorithms, including time-stepping schemes and hybrid system solvers, have been developed to handle discontinuities effectively.

7. Research Perspectives

The study of optimal control problems with discontinuous right-hand sides is far from complete and continues to evolve. Promising directions for future research include:

- *Integration with Machine Learning*: Combining data-driven methods with nonsmooth control theory could improve modeling and control of complex systems with unknown or partially known discontinuities.
- *Hybrid and Switched Systems*: Developing unified frameworks for systems that combine continuous dynamics with discrete transitions remains an active area.
- *Robust and Stochastic Extensions*: Real-world systems often involve uncertainty. Extending nonsmooth optimal control theory to stochastic or robust settings is both challenging and necessary.
- *Efficient Computation*: Improving numerical methods to handle large-scale systems with discontinuities, especially in real-time applications, is a key practical concern.
- *Applications in Emerging Fields*: Areas such as smart grids, biological systems, and autonomous robotics provide new contexts where discontinuous optimal control plays a crucial role.

6. Discontinuous Systems

We consider optimal control problem which mathematical model is given by the system of ordinary differential equations with piecewise continuous right-hand side, that is *discontinuous system*. As we know such systems arise in the control of dynamic processes in nonhomogeneous environment with different physical properties, for example, in the design of composite or multilayer materials with specified characteristics. Discontinuous systems also naturally appear in the theory of optimal control when synthesizing optimal systems and in the theory of automatic control when ensuring the stability of controlled systems by introducing discontinuous feedbacks [1]. Some non-smooth control systems can be described in terms of discontinuous systems. For example, a differential equation $\dot{x} = |x| + u$ is equivalent to an equation with a "discontinuous" right-hand side

$$\dot{x} = \begin{cases} -x+u, \ x<0, \\ \ \ x+u, \ x \ge 0. \end{cases}$$

The theory of discontinuous systems is a separate direction in optimal control, which is interesting for its approaches, methods and applications [1,9,10],. Here is just one result that illustrates its specificity.

Consider a generalization of the general optimal control problem (D-problem) [1]

$$J_0 = \Phi_0(x(t_0), x(t_1), t_0, t_1) \to \min,$$

$$J_i = \Phi_i(x(t_0), x(t_1), t_0, t_1) \begin{cases} \le 0, \ i = 1,\dots,m_0, \\ = 0, \ i = m_0+1,\dots,m, \end{cases}$$

$$\dot{x} = \begin{cases} f^1(x,u,t), \ p(x,t) < 0, \\ f^2(x,u,t), \ p(x,t) > 0, \end{cases} \quad u \in U, \ t_0 < t_1,$$

in which the right-hand side of the system of differential equations has a finite discontinuity on the surface

$$P = \{(x,t) \in R^n \times R : p(x,t) = 0\}.$$

We regard the analytical properties of the functions $f^2(x,u,t)$, $f^1(x,u,t)$ to be the same as those of a function $f(x,u,t)$ in general problem [1]. We will classify the function $p(x,t)$ as $C_1(R^n \times R \to R)$ and assume that at each point $(x,t) \in P$ for any $u \in U$ the following *sewing conditions* are satisfied

$$\begin{aligned} \dot{p}^1(x,u,t) &= p_x(x,t)' f^1(x,u,t) + p_t(x,t) > 0, \\ \dot{p}^2(x,u,t) &= p_x(x,t)' f^2(x,u,t) + p_t(x,t) > 0. \end{aligned} \tag{1}$$

Let us explain the meaning of conditions (1). Suppose that a trajectory $x^k(t)$ of the system of equations

$$\dot{x} = f^k(x, u(t), t), \ k = 1,2$$

corresponds to some control $u(t)$ and initial values $(\xi, \tau) \in P$.

Then $p(x^k(\tau), \tau) = p(\xi, \tau) = 0$ and by virtue of conditions (1), we have

$$\frac{d}{dt} p(x^k(t), t)\big|_{t=\tau\pm 0} = \dot{p}^k(x^k(\tau), u(\tau \pm 0), \tau) > 0.$$

Consequently, the function $t \to p(x^k(t),t)$ in the neighborhood of the point $t=\tau$ increases and changes the sign from minus to plus, i.e. the integral curve $(x^k(t),t)$ intersects the surface P at a point (ξ,τ) without one-sided tangencies.

Let the trajectories $x^k(t)$ for some $\delta>0$ be defined on a common interval $(\tau-\delta,\tau+\delta)$. We compose a continuous function $x(t)$ from them setting

$$x(t)=x^1(t),\ \tau-\delta<t<\tau;\ x(t)=x^2(t),\ \tau\le t<\tau+\delta.$$

By construction, $x(\tau)=\xi$. Function $x(t)$ satisfies the conditions $p(x,t)<0$, $\dot{x}=f^1(x,u(t),t)$ on the interval $(\tau-\delta,\tau)$ and the conditions $p(x,t)>0$, $\dot{x}=f^2(x,u(t),t)$ on the interval $(\tau,\tau+\delta)$. And we consider the function $x(t)$ as a *solution* of the system of differential equations

$$\dot{x}=\begin{cases} f^1(x,u(t),t),\ p(x,t)<0, \\ f^2(x,u(t),t),\ p(x,t)>0, \end{cases} \tag{2}$$

with discontinuous right-hand side, corresponding to the control $u(t)$. Thus, if the sewing conditions (1) are satisfied then for any fixed control exactly one continuous integral curve of the system (2) passes through each point of the surface *P* and has no one-sided tangencies with *P*.

We proceed to the derivation of the necessary conditions for optimality in the *D*-problem. Consider the processes

$$x(t),u(t),t_0,\tau,t_1 \tag{3}$$

of *D*-problem which satisfy the conditions

$$p(x(t_0),t_0)<0;\ p(x(\tau),\tau)=0,\ \tau\in(t_0,t_1);\ p(x(t_1),t_1)>0$$

and suppose that the optimal process is among them. Let us connect the *D*-problem with the next two-stage problem (*T*-problem)

$$\begin{aligned}
&J_0=\Phi_0(x^1(t_0),x^2(t_1),t_0,t_1)\to\min,\\
&J_i=\Phi_i(x^1(t_0),x^2(t_1),t_0,t_1)\begin{cases}\le 0,\ i=1,\dots,m_0,\\ =0,\ i=m_0+1,\dots,m,\end{cases}\\
&x^1(\tau)-x^2(\tau)=0,\ p(x^1(\tau),\tau)=0,\\
&\dot{x}^1=f^1(x^1,u^1,t),\ \dot{x}^2=f^2(x^2,u^2,t),\ u^1,u^2\in U,\ t_0<\tau<t_1.
\end{aligned}$$

In the T-problem, the trajectories of two controlled systems are joined by continuity on the surface P at a non-fixed time τ. The same conditions as in the D-problem are imposed on the left end of the integral curve of the first system and the right end of the integral curve of the second system.

The process (3) of the D-problem corresponds to the process

$$x^1(t), x^2(t), u^1(t), u^2(t), t_0, \tau, t_1 \tag{4}$$

of the T-problem and conversely. By virtue of this correspondence, the optimal process (4) of the T-problem corresponds to the optimal process (3) of the D-problem. We define the optimal process extending by continuity a pair $x^1(t), u^1(t)$ to the exterior of the interval (t_0, τ) and a pair $x^2(t), u^2(t)$ to the exterior of the interval (τ, t_1). The values of the functionals $J_0, \ldots, J_m$ on the extended process do not change, so we retain the notation (4) for it.

Let us write down the necessary optimality conditions for the extended optimal process (4) of the T-problem. We put

$$\begin{aligned}
&\lambda = (\lambda_0, \ldots, \lambda_m),\ \lambda^1 = (\lambda_{m+1}, \ldots, \lambda_{m+n}),\ \mu \in R,\\
&H_k(\psi, x, u, t) = \psi' f^k(x, u, t),\ k = 1, 2,\\
&L(\lambda, x^0, x^1, t_0, t_1) = \sum_{i=0}^{m} \lambda_i \Phi_i(x^0, x^1, t_0, t_1),\\
&\tilde{H} = H_1(\psi^1, x^1, u^1, t) + H_2(\psi^2, x^2, u^2, t),\\
&\tilde{L} = L(\lambda, x^1(t_0), x^2(t_1), t_0, t_1) + (\lambda^1)'[x^1(\tau) - x^2(\tau)] + \mu p(x^1(\tau), \tau).
\end{aligned} \tag{5}$$

According to the maximum principle, there are vectors λ, λ^1, number μ and continuous when $t \neq \tau$ solutions $\psi^1(t), \psi^2(t)$ of the conjugate system of differential equations

$$\dot{\psi}^1 = -H_{1x}(\psi^1, x^1(t), u^1(t), t),\ \dot{\psi}^2 = -H_{2x}(\psi^2, x^2(t), u^2(t), t)$$

with the jump condition

$$\psi^1(\tau - 0) = \psi^1(\tau + 0) - \lambda^1 - \mu p_x(x^1(\tau), \tau),\quad \psi^2(\tau - 0) = \psi^2(\tau + 0) + \lambda^1,$$
$$(\lambda^1)'[\dot{x}^1(\tau) - \dot{x}^2(\tau)] + \mu \dot{p}^1(x^1(\tau), u^1(\tau), \tau) = 0,$$

satisfying the requirements

1) nontriviality, nonnegativity and complementary slackness

$$(\lambda, \lambda^1, \mu) \neq 0,\ \lambda_i \geq 0,\ i = 0, \ldots, m_0,\ \lambda_i \Phi_i(x^1(t_0), x^2(t_1), t_0, t_1) = 0,\ i = 1, \ldots, m_0;$$

2) transversality

$$\psi^1(t_0) = L_{x^0},\ \psi^1(t_1) = 0,\ \psi^2(t_0) = 0,\ \psi^2(t_1) = -L_{x^1},\ \dot{L}_{t_0} = 0,\ \dot{L}_{t_1} = 0;$$

3) maximum of Hamiltonian

$$\Delta_{v^1} H_1(\psi^1(t), x^1(t), u^1(t), t) + \Delta_{v^2} H_2(\psi^2(t), x^2(t), u^2(t), t) \le 0,$$
$$v^1, v^2 \in U,\ t \in [t_0, t_1]$$

(the derivatives of the function L in transversality conditions are calculated at the point $(\lambda, x^1(t_0), x^2(t_1), t_0, t_1)$).

We analyze these conditions. For $\lambda = 0$, from the conjugate equations, the jump condition and the transversality conditions it follows $\lambda^1 = 0,\ \mu = 0$ which contradicts the condition $(\lambda, \lambda^1, \mu) \ne 0$. Therefore, $\lambda \ne 0$. Due to the conditions of transversality and linearity of the conjugate equations, we have

$$\psi^1(t) = 0,\ \tau \le t \le t_1;\ \psi^2(t) = 0,\ t_0 \le t \le \tau. \tag{6}$$

Then, in the jump condition for the conjugate functions

$$\psi^1(\tau + 0) = 0, \psi^2(\tau - 0) = 0\ ,\ \psi^2(\tau + 0) = -\lambda^1.$$

Eliminating the vector λ^1 in the jump condition and in the transversality condition for τ, we obtain

$$\psi^1(\tau - 0) = \psi^2(\tau + 0) - \mu p_x(x^1(\tau), \tau),$$
$$-\psi^2(\tau + 0)'[\dot{x}^1(\tau) - \dot{x}^2(\tau)] + \mu \dot{p}^1(x^1(\tau), u^1(\tau), \tau) = 0.$$

By virtue of equalities (6), the condition for the maximum of the Hamiltonian takes the form

$$\Delta_{v^1} H_1(\psi^1(t), x^1(t), u^1(t), t) \le 0,\ v^1 \in U,\ t \in [t_0, \tau],$$
$$\Delta_{v^2} H_2(\psi^2(t), x^2(t), u^2(t), t) \le 0,\ v^2 \in U,\ t \in [\tau, t_1].$$

It is convenient to formulate the final conclusions in terms of the conjugate function

$$\psi(t) = \begin{cases} \psi^1(t),\ t_0 \le t < \tau, \\ \psi^2(t),\ \tau < t \le t_1, \end{cases}$$

Hamiltonian

$$H(\psi, x, u, t) = \begin{cases} \psi' f^1(x, u, t),\ p(x, t) < 0, \\ \psi' f^2(x, u, t),\ p(x, t) > 0 \end{cases} \tag{7}$$

of the discontinuous system (2) and the optimal process (3) of the *D*-problem corresponding to the extended optimal process (4) of the *T*-problem.

Theorem 1 (maximum principle for *D*-problem) *Let the given D-problem satisfies the sewing condition (1) and let* $x(t), u(t), t_0, \tau, t_1$ *be the optimal process of this problem the integral curve of which intersects the discontinuity surface* $p(x,t)=0$ *at a time* $\tau \in (t_0, t_1)$. *Then there exist vector* λ *and continuous when* $t \neq \tau$ *solution* $\psi(t)$ *of the conjugate system*

$$\dot{\psi} = -H_x(\psi, x(t), u(t), t)$$

with the jump condition

$$\psi(\tau-0) = \psi(\tau+0) - \mu p_x(x(\tau), \tau),$$

$$\mu = \psi(\tau+0)'[f^1(x(\tau), u(\tau), \tau) - f^2(x(\tau), u(\tau), \tau)] / \dot{p}^1(x(\tau), u(\tau), \tau), \quad (8)$$

for which the following conditions hold:

1) nontriviality, nonnegativity and complementary slackness

$$\lambda \neq 0, \ \lambda_i \geq 0, \ i = 0, ..., m_0, \ \lambda_i \Phi_i(x(t_0), x(t_1), t_0, t_1) = 0, \ i = 1, ..., m_0;$$

2) transversality

$$\psi(t_0) = L_{x^0}(\lambda, x(t_0), x(t_1), t_0, t_1), \ \ \psi(t_2) = -L_{x^1}(\lambda, x(t_0), x(t_1), t_0, t_1),$$

$$\dot{L}_{t_0}(\lambda, x(t_0), x(t_1), t_0, t_1) = 0, \ \dot{L}_{t_1}(\lambda, x(t_0), x(t_1), t_0, t_1) = 0;$$

3) maximum of Hamiltonian

$$H(\psi(t), x(t), u(t), t) = \max_{u \in U} H(\psi(t), x(t), u, t), \ t \in [t_0, t_2]$$

with the functions L, H *of the form (5).*

The statement is proved by the previous reasoning. As can be seen from the Theorem 1, the discontinuity of the right-hand side of the differential equations causes a jump in the solution of the conjugate system at the moment of sewing the integral curve of the discontinuity surface. The solution jump occurs in the direction of the gradient p_x. The existence of a factor μ in the jump condition was pointed out in the fundamental monograph [10]. The formula (8) for the calculation of μ was obtained in [7]. Theorem 1 was derived under rather stringent assumptions: the sewing conditions must be satisfied over the entire discontinuity surface for any control. This is due to the method of proof - reduction of the original problem to a problem with an intermediate condition. If we directly use the method of increments and the implicit function theorem then we can get rid of this assumption [1] and require the fulfillment of the sewing conditions only at the point of intersection of the optimal trajectory with the discontinuity surface. We also note that other possible cases of mutual position of the optimal trajectory and the discontinuity surface remained outside of our vision. The corresponding results can be found in the monograph [2].

7. Applications

Robotics and Mechanical Systems

Discontinuous dynamics arise from Coulomb friction, impacts, and contact constraints. These effects are critical in manipulation, locomotion, and assembly tasks.

Power Electronics

Switching devices in converters and inverters naturally lead to discontinuous models. Optimal control formulations are essential for efficiency and stability.

Machine Learning and AI

Discontinuous control laws resemble activation nonlinearities in neural networks. Embedding PMP conditions into learning frameworks enables physically consistent training and mitigates gradient pathologies.

8. Open Problems

Infinite-Dimensional Extensions

Filippov's framework does not directly generalize to infinite-dimensional spaces due to measure-theoretic limitations. Extending the theory to PDEs remains an open challenge.

Hybrid and Cyber-Physical Systems

A unified framework integrating discontinuous dynamics and hybrid automata is needed for complex engineered systems.

Integration with Physics-Informed Learning

Combining optimal control structure with machine learning offers promising directions for robust and interpretable control design.

Conclusion

Optimal control problems with discontinuous right-hand sides represent a rich and challenging domain at the intersection of analysis, optimization, and applications. Their relevance continues to grow as modern systems increasingly exhibit hybrid and nonsmooth behavior. Addressing the theoretical and computational challenges in this field not only advances mathematical understanding but also enables more effective control of complex real-world systems.

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