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PROPERTIES AND APPLICATION OF 3 -INDEX GENERALIZED HERMITE POLYNOMIALS

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Abstract

In this article, we compute the orthogonal properties of a 3 -index generalized Hermite polynomials due to their Rodrigue's formulas in suitable regions. Also, we apply them in expressing of the Fourier series in terms of 3-index generalized Hermite polynomials. Also, we derive other related results.

Keywords and phrases: 3 -index generalized Hermite polynomials, classical Hermite polynomials, Lahiri's generalized Hermite polynomials, Rodrigue's formula, application in computing of Fourier exponential series.

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1 Introduction

Recently, Pathan et al [9] introduced a new class of generalized Hermite polynomials involving cubic Hermite polynomials to study a standard generating function to derive certain results for example, integral representation, matrix representations and differential equations not attempted by any researcher so far. Application of Rodrigues' formula and the integral representation attempted herein produce first few polynomials of cubic Hermite polynomials and some inequalities.

In the quantum physics, the classical Hermite polynomials, $H_n(x) \forall n = 0, 1, 2, 3, \ldots$; ([1], see also in [11, Section 105, p.189] and [12, Eqn. (16), p.74]) were used through Rodrigues' formula

$$H_n(x) = (-1)^n \exp\left(x^2\right) D_x^n \left\{\exp\left(-x^2\right)\right\}, D_x^n \equiv \frac{d^n}{dx^n} \tag{1.1}$$

Bell [2] presented and studied a generalization of (1.1) as a class of polynomials in form of exponential functions to use in quantum physics.

Gould and Hopper ([6, p.52], see also in [12, Eqn. (12), p. 77]) introduced operational formulas connected with generalization of (1.1).

Lahiri [7] studied the generalized Hermite polynomials $H_{n,m,v}(x), \forall x \in \mathbb{R}, n = 0, 1, 2, \ldots; m$ being a positive integer, $v > 0, x$ and t being independent variables, through generating function

$$\sum_{n=0}^{\infty} H_{n,m,v}(x)\frac{t^n}{n!} = \exp\left(vxt - t^m\right). \tag{1.2}$$

Recently, Pathan et al [9] introduced a standard exponential generating function of a class of generalized Hermite polynomials denoted by $\mathcal{H}_n^m(x)\forall x \in \mathbb{R}, n = 0, 1, 2, \ldots; m$ being a positive integer, x and t being independent variables, defined by

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} &= \exp\Bigg((-1)^{m+1}\binom{m}{0}t^m + (-1)^m\binom{m}{1}t^{m-1}x + (-1)^{m-1}\binom{m}{2}t^{m-2}x^2 \\ &\quad +(-1)^{m-2}\binom{m}{3}t^{m-3}x^3 + \cdots + \binom{m}{m-1}tx^{m-1}\Bigg) \\ &= \exp\Bigg(-\Bigg[\binom{m}{0}(-t)^m + \binom{m}{1}(-t)^{m-1}x + \binom{m}{2}(-t)^{m-2}x^2 \\ &\quad +\binom{m}{3}(-t)^{m-3}x^3 + \cdots + \binom{m}{m-1}(-t)x^{m-1} + x^m\Bigg]\Bigg)\exp\left(x^m\right)\end{aligned}$$

where,

$$\binom{m}{r} = \begin{cases} \frac{m(m-1)\ldots(m-r+1)}{r!}, 0 \le r \le m; \\ 0, r > m. \end{cases} \tag{1.3}$$

The formula (1.3) can be written as

$$\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} = \exp\left(x^m\right)\exp\left(-(x-t)^m\right) \tag{1.4}$$

Specially, setting $m = 2$ in the formula (1.4) it reduces to the standard Rodrigues' formula (1.1) given by

$$\mathcal{H}_n^2(x) = H_n(x). \tag{1.5}$$

It is noted that when $m > 2$, the generalized Hermite polynomials defined in (1.3)-(1.4) often appear in problems involving non-Gaussian statistics, higher-order differential equations, and quantum optics.

Also, it is observed that 3-index generalized Hermite polynomials do not satisfy the orthogonal properties in the interval $(-\infty, \infty)$, when $x \to -\infty$. Therefore, this becomes necessary to search the regions where the orthogonal properties exists.

Then, in this paper we investigate those regions where the 3 -index generalized Hermite polynomials satisfy the orthogonal properties and use these properties to compute the Fourier exponential series and other related results.

2 Computation of the orthogonal properties via Rodrigues' formula

In this section, we present to compute the orthogonal properties for 3 -index generalized Hermite polynomials via the Rodrigues' formula derived in [9] given by:

Lemma 2.1 ([9]). *For all $n \geq 0, m \geq 2$ the standard exponential generating function (1.4) satisfies the Rodrigues' formula*

$$\mathcal{H}_n^m(x) = (-1)^n \exp\left(x^m\right) \frac{d^n}{dx^n} \exp\left(-x^m\right). \tag{2.1}$$

Proof. Considering the standard exponential generating function (1.4) and in its right hand side on making an appeal to the Taylor's Theorem to find the Rodrigues' formula (2.1). $\square$

Lemma 2.2 ([9]). *For x and t being independent variables, $m = 3$, the Eqns. (1.3) and (1.4) and (2.1), become cubic polynomials, $\mathcal{H}_n^3(x) \forall n \geq 0$, and there exists a standard generating function in the form*

$$\sum_{n=0}^{\infty} \mathcal{H}_n^3(x) \frac{t^n}{n!} = \exp\left(t^3 - 3xt^2 + 3x^2 t\right) = \exp\left(x^3\right) \exp\left(-(x-t)^3\right), \tag{2.2}$$

and connect to Rodrigues' formula given by

$$\mathcal{H}_n^3(x) = (-1)^n e^{x^3} \frac{d^n}{dx^n}\left(e^{-x^3}\right). \tag{2.3}$$

Proof. Consider the standard exponential generating function given in the equations (1.3) and (1.4) and the Rodrigues' formula (2.1) where on setting $m = 3$, the formulae (2.2) and (2.3) are obtained. □

Theorem 2.3. *If all conditions of Lemma 2.2 are satisfied, then for all $n \geq 0, C$ is any contour of a complex plane, the cubic Hermite polynomials, $\mathcal{H}_n^3(x)$, for all $x \in \mathbb{C}, x = Re^{i\theta}$, where, $R, \theta \in \mathbb{R}, \forall \cos 3\theta > 0$, such that the ray $\theta \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right), R \to \infty$, then there exists an orthogonal formula*

$$\int_C e^{-x^3}\mathcal{H}_n^3(x)\mathcal{H}_k^3(x)dx = \begin{cases} 0, n \neq k, \\ 3^n \frac{2n!}{n!}\int_C e^{-x^3}x^n dx, n = k. \end{cases} \tag{2.4}$$

Proof. Making an appeal to the formula (2.3) in left hand side of the equation (2.4), we find

$$\begin{aligned} &\int_C e^{-x^3}\mathcal{H}_n^3(x)\mathcal{H}_k^3(x)dx = (-1)^n \int_C \frac{d^n}{dx^n}\left(e^{-x^3}\right)\mathcal{H}_k^3(x)dx \\ &= (-1)^n\left[\mathcal{H}_k^3(x)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^3}\right)\right]_{\partial C} + (-1)^{n+1}\int_C \frac{d}{dx}\mathcal{H}_k^3(x)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^3}\right)dx, \end{aligned} \tag{2.5}$$

where, ∂C is a boundary of C.

We are familiar that $\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^3}\right) = \frac{P_{n-1}(x)}{e^{x^3}}$, where, $P_{n-1}(x)$ is a polynomial of degree of $n-1$. Now for all $x \in C, x = Re^{i\theta}, \cos 3\theta > 0, \theta \in \left(\frac{4r\pi-\pi}{6}, \frac{4r\pi+\pi}{6}\right) \forall r = 0, 1, 2, R \to \infty$, here in the Theorem 2.1, taking $r = 0 \Rightarrow \theta \in \left(\frac{-\pi}{6}, \frac{\pi}{6}\right)$, to get

$$\left|(-1)^n\left[\mathcal{H}_k^3(x)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^3}\right)\right]_{\partial C}\right| = \left|\frac{\mathrm{Q}\left(Re^{i\theta}\right)}{e^{R^3e^{3i\theta}}}\right|_{R\to\infty}, \tag{2.6}$$

where, $Q(x)$ is a higher degree polynomial.

Then making an appeal to (2.6) in (2.5), we get

$\left|(-1)^n \left[\mathcal{H}_k^3(x)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^3}\right)\right]_{\partial C}\right| = \left|\frac{\mathrm{Q}(R\cos\theta)}{e^{R^3\cos 3\theta}}\right|_{R\to\infty} \to 0$, and thus applying same process $n-$ times repeatedly to get

$$\int_C e^{-x^3}\mathcal{H}_n^3(x)\mathcal{H}_k^3(x)dx = \int_C e^{-x^3}\frac{d^n}{dx^n}\mathcal{H}_k^3(x)dx. \tag{2.7}$$

By (2.7) for $n >$ degree of $\mathcal{H}_k^3(x)$, we find

$$\int_C e^{-x^3}\mathcal{H}_n^3(x)\mathcal{H}_k^3(x)dx = 0\forall n \neq k. \tag{2.8}$$

Again for $n = k$ making an appeal to (2.2) in (2.7), we write

$$\int_C e^{-x^3}\left\{\mathcal{H}_n^3(x)\right\}^2 dx = 3^n\frac{2n!}{n!}\int_C e^{-x^3}x^n dx. \tag{2.9}$$

Hence, the Theorem 2.1 is followed. □

3 Application of the Theorem 2.1 in computation of the Fourier series involving cubic Hermite polynomials

In this section, we define a Fourier series involving cubic Hermite polynomials given by

$$F(x) = \sum_{n=0}^{\infty} A_n\mathcal{H}_n^3\left(xe^{ipx}\right),$$

where, $p > 0$ and the cubic Hermite polynomials, $\mathcal{H}_n^3(x)$, are defined by Rodrigues' formula (2.3) for all $n = 0, 1, 2, 3, \ldots$.

Also, for all $n = 0, 1, 2, 3, \ldots, A_n$ are the Fourier coefficients.

We state following theorems:

Theorem 3.1. *If the polynomials $\mathcal{H}_n^3(z)$ defined by the generating function (2.2) and satisfies the Rodrigues' formula (2.3) for $\log\left(\frac{z}{x}\right) = ipx, p > 0$ in the form*

$$\mathcal{H}_n^3(z) = (-1)^n e^{z^3}\frac{d^n}{dz^n}\left(e^{-z^3}\right). \tag{3.1}$$

Also, for all $x \in \mathbb{C}, p > 0$, if the Fourier series is given by

$$F(x) = \sum_{n=0}^{\infty} A_n \mathcal{H}_n^3 \left(xe^{ipx}\right). \tag{3.2}$$

Then across the ray $\theta \in \left(\frac{-\pi}{6}, \frac{\pi}{6}\right), z = Re^{i\theta}, \cos 3\theta > 0, R \to \infty$, the Fourier coefficients are computed by

$$A_n = \frac{\int_C \exp\left(-z^3\right) \mathcal{H}_n^3(z) F\left(ze^{-\log\left(\frac{z}{x}\right)}\right) dz}{3^n (n!)^{-1} (2n)! \int_C e^{-z^3} z^n dz}, n = 0, 1, 2, 3, \ldots. \tag{3.3}$$

Proof. To prove the Theorem 3.1, in the series (3.2) we consider $xe^{ipx} = z, p > 0$, and make an appeal to the Theorem 2.1,to write the orthogonally property as

$$\int_C e^{-z^3} \mathcal{H}_n^3(z) \mathcal{H}_k^3(z) dz = \delta_{nm} \cdot (n!)^{-1} 3^n (2n)! \int_C e^{-z^3} z^n dz. \tag{3.4}$$

In the Fourier series (3.2) it is noted that xe^{ipx} acts as a transformation of the variable z, and in the orthogonally property (3.4), the $\exp\left(-z^3\right)$ acts as a weight function that ensures convergence and orthogonally across the ray

$$\theta \in \left(\frac{-\pi}{6}, \frac{\pi}{6}\right), z = Re^{i\theta}, \cos 3\theta > 0, R \to \infty.$$

Again, multiplying by $\exp\left(-z^3\right) \mathcal{H}_m^3(z)$, where $z = xe^{ipx}, p > 0$, in both sides of the series (3.2) and then integrating with respect to z, and on applying the orthogonally property (3.4), we find

$$\int_C \exp\left(-z^3\right) \mathcal{H}_n^3(z) F\left(ze^{-\log\left(\frac{z}{x}\right)}\right) dz = A_n 3^n \frac{(2n)!}{n!} \int_C e^{-z^3} z^n dz, \frac{z}{x} = e^{ipx}, p > 0. \tag{3.5}$$

Therefore, making an appeal to the equations (3.5), we obtain the Fourier coefficients given in the formula (3.3). □

Theorem 3.2. *If all conditions of the Theorem 3.1 are satisfied then the Fourier series is represented by*

$$F(x) = \sum_{n=0}^{\infty} \left[\frac{\int_C \exp\left(-z^3\right) \mathcal{H}_n^3(z) F\left(ze^{-\log\left(\frac{z}{x}\right)}\right) dz}{3^n (n!)^{-1} (2n)! \int_C e^{-z^3} z^n dz}\right] \mathcal{H}_n^3 \left(xe^{ipx}\right). \tag{3.6}$$

Proof. Making an appeal to the equations (3.2) and (3.3) of the Theorem 3.1, we derive the exponential Fourier series (3.6). □

4 Other application

In this section, some other results related to the cubic Hermite polynomials are derived on using our above generating function of cubic Hermite polynomials and concepts of the theory. We then solve a second order differential equation on applying adjointness of a differential operator.

Making an appeal to the generating function (2.2) we derive a generating function involving cubic Hermite polynomials in the form

$$\sum_{n=0}^{\infty} K_n^3(x)\frac{t^n}{n!} = \exp\left(t^3 + 3x^2 t\right),$$

where,

$$K_n^3(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (3x)^k \mathcal{H}_{n-2k}^3(x)\frac{n!}{k!(n-2k)!}. \tag{4.1}$$

Theorem 4.1. *If* $t \in \mathbb{C}, t = Re^{i\theta}$, *where,* $R, \theta \in \mathbb{R}$, *such that the ray* $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right), \cos 3\theta < 0$, *then the differential equation*

$$\left(\frac{1}{12}\left(\frac{1}{x}\frac{d}{dx}\right)^2 + 3x^2\right) Y(x) = 0, \tag{4.2}$$

has the solution

$$Y(x) = \int_C v(t)\exp\left(3x^2 t\right) dt,$$

where, C *is contour in the complex plane and*

$$v(t) = \exp\left(t^3\right). \tag{4.3}$$

Proof. We are familiar with that

$$\begin{aligned}\left(\frac{1}{12}\left(\frac{1}{x}\frac{d}{dx}\right)^2 + 3x^2\right)\exp\left(3x^2 t\right) &= \left(3t^2 + 3x^2\right)\exp\left(3x^2 t\right) \\ &= \left(3t^2 + \frac{d}{dt}\right)\exp\left(3x^2 t\right). \end{aligned} \tag{4.4}$$

By (4.4), consider that $L_x \equiv \left(\frac{1}{12}\left(\frac{1}{x}\frac{d}{dx}\right)^2 + 3x^2\right)$ and $M_t \equiv \left(3t^2 + \frac{d}{dt}\right)$, the adjoint of $M_t = {M_t}^* \equiv \left(3t^2 - \frac{d}{dt}\right)$.

And again, consider that

$$K(x,t) = \exp\left(3x^2 t\right) = u.$$

Then we have

$$\int_C \left(vM_t[u] - uM_t^*[v]\right) dt = \int_C \left\{ \frac{d}{dt}\left(v \exp\left(3x^2 t\right)\right) \right\} dt = \left[v \exp\left(3x^2 t\right)\right]_{\partial C},$$

∂C is the boundary of the contour.

Therefore, we get a solution

$$\begin{aligned}
Y(x) &= \int_C v(t) \exp\left(3x^2 t\right) dt \text{ which satisfies} \\
L_x Y(x) &= \left(\frac{1}{12}\left(\frac{1}{x}\frac{d}{dx}\right)^2 + 3x^2\right) \int_C v(t) \exp\left(3x^2 t\right) dt \\
&= \int_C v(t)\left(3t^2 + 3x^2\right) \exp\left(3x^2 t\right) dt = \int_C v(t) M_t\left[\exp\left(3x^2 t\right)\right] dt \\
&= \int_C \left\{ \exp\left(3x^2 t\right) M_t^*[v] + \frac{d}{dt}\left[v \exp\left(3x^2 t\right)\right] \right\} dt \\
&= \int_C \left\{ \exp\left(3x^2 t\right)\left(3t^2 v - \frac{dv}{dt}\right) + \frac{d}{dt}\left[v \exp\left(3x^2 t\right)\right] \right\} dt = 0,
\end{aligned}$$

which gives $v = \exp\left(t^3\right)$, also for the ray $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right), \cos 3\theta < 0$ we have

$$\left|\exp\left(t^3 + 3x^2 t\right)\right|_{\partial C} \to 0 \text{ as } R \to \infty. \tag{4.5}$$

Hence, the Theorem 4.1 is followed. $\square$

5 Conclusion

The series defined by (3.2) consisting of generalized Hermite polynomials of order three, denoted by $\mathcal{H}_n^3(x), n = 0, 1, 2, 3, \ldots$, is found in various fields of sciences involving higher-order differential equations and non-Gaussian statistics.

The cubic Hermite polynomials $\mathcal{H}_n^3(x)$ are the generalization of classical Hermite polynomials $H_n(x)$ (corresponds harmonic oscillator). On the other hand the cubic Hermite polynomials, $\mathcal{H}_n^3(x)$, can appear in the study of the systems governed by potential functions involving x^3 corresponding to the anharmonic oscillators or generalized Schrödinger equations of quantum mechanics and mathematical physics.

In the Fourier series (3.2), the function $F(x)$ is consisting of the complex term $e^{ipx}, p > 0$ that suggests applications in wave propagation and diffraction theory. Also, the expansion of $F(x)$ into these modes can represent the evolution of a wave packet in a medium with third -order dispersion. This form is also called generalized Fourier-Hermite expansion.

The orthogonal property (2.4) (implied by formula for A_n) allows for the approximation of complex functions $F(x)$ by a finite series of polynomials which is applicable in spectral methods for solving differential equations in numerical analysis.

The Theorem 4.1 is logically and mathematically proved in which it is shown that a second order differential equation has the solution in terms of the integral of the generating function 4.1 involving 3-index generalized Hermite polynomials.

By motivation of this paper and the researches done in this field due to the authors for example ([3], [4], [5], [6], [7], [8], [10] and others) may help to develop the researches in the field of special functions.

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