

Exact Solutions for Two New $(2 + 1)$ and $(3 + 1)$ -Dimensional Clannish Random Walker's Parabolic Equations: $\left(\frac{1}{G'''} \bigg/ \frac{G'}{G''}\right)$ -Expansion Method

Mohammed Salem Ahmed AL-Amry¹, Eman Fadhl Abdullah AL-Abdali^{2,*}

^{1,2,*} Department of Mathematics, Faculty of Education-Aden, University of Aden, Yemen

¹ salehm251251@gmail.com

¹ ORCID:0009-0005-3305-2020

^{2,*} emanfadhel328@gmail.com

^{2,*} ORCID:0009-0000-6883-6128

Abstract

In this paper, we presented two new $(2 + 1)$ and $(3 + 1)$ -dimensional Clannish Random Walker's Parabolic (CRWP) equations of nonlinear partial differential equations (NPDEs). We will use a new method, namely, the $\left(\frac{1}{G'''} \bigg/ \frac{G'}{G''}\right)$ -expansion method to conduct this analysis. Exact travelling wave solutions are obtained and expressed in terms of hyperbolic functions, trigonometric functions, rational functions.

Key Words: $\left(\frac{1}{G'''} \bigg/ \frac{G'}{G''}\right)$ - expansion method, Clannish Random Walker's Parabolic equation, solitary wave solutions.

1 Introduction

Nonlinear partial differential equations (NPDEs) are used to simulate wide range of complicated processes arising in real life and all fields of science. Exact solutions of NPDEs can be implemented to produce significant predictions for many applications related to nature and living. Modern theories of nonlinear science have been highly developed over the last half century. Particularly, integrable nonlinear systems have attracted great interest among a number of mathematicians and physicists. It is well known that the study of soliton generating nonlinear partial differential equations are of great interest in recent years not only in $(1 + 1)$ -dimensional systems, but also in $(2 + 1)$ -dimensional systems under certain conditions [1, 2]. Generally speaking, analyzing coherent type soliton structures of $(2 + 1)$ -dimensional systems are more complicated

than $(1+1)$ -dimensional systems. The study of higher-dimensional integrable systems is one of the main themes in integrability theory. Several models in the context of $(2+1)$ -dimensional equations, which are integrable have been developed by Toda and Yu [3]. So field of integrable equations is an active multidisciplinary area of research due to the fact that integrable equations describe the real features and reveal the mysterious nature of the nonlinearity in science and engineering applications [4]-[10].

In recent years, the study of integrable equations has become a hot research area in nonlinear mathematical physics and wave propagations. This originates from the fact that these equations give qualitative and quantitative features of several scientific aspects of many unrelated phenomena such as plasmas, fluids, lattice vibrations of a crystal at low temperature, propagation of waves in shallow water area, traffic flow, soliton propagation in nonlinear transmission lines, etc [11]-[17].

The $(1+1)$ -dimensional CRWP equation reads [18]-[20]

$$u_t - u_x + u_{xx} + \alpha uu_x = 0. \quad (1)$$

In this work, we aim to presentation two new $(2+1)$ -dimensional and $(3+1)$ -dimensional CRWP equations, with constant coefficients. We first introduce a $(2+1)$ -dimensional CRWP equation the form

$$u_t - u_x + u_{xx} + \alpha (u \partial_y^{-1} u_x)_x = 0,$$

or equivalently

$$\begin{aligned} u_t - u_x + u_{xx} + \alpha (uv)_x &= 0, \\ v_y &= u_x. \end{aligned} \quad (2)$$

We introduce a $(3+1)$ -dimensional CRWP equation with constant coefficients takes the form

$$u_t - u_x + \alpha uu_y + \beta u_x \partial_y^{-1} u_x + \gamma u_z \partial_y^{-1} u_z + \varepsilon (\alpha u_{yy} + \beta u_{xx} + \gamma u_{zz}) = 0,$$

or equivalently

$$\begin{aligned} u_t - u_x + \alpha uu_y + \beta v u_x + \gamma w u_z + \varepsilon (\alpha u_{yy} + \beta u_{xx} + \gamma u_{zz}) &= 0, \\ v_y = u_x, w_y &= u_z. \end{aligned} \quad (3)$$

Where α , β , and γ are constants.

2 New $\left(\frac{1}{G''} \Big/ \frac{G'}{G''}\right)$ -Expansion Method

In this section, we describe our new method, namely, the $\left(\frac{1}{G''} \Big/ \frac{G'}{G''}\right)$ expansion method for finding travelling wave solutions of nonlinear evolution equations. As preparations, consider the third order ordinary linear differential equation:

$$G''' + \lambda G' + \mu = 0, \quad (4)$$

and defining for the simplicity

$$\phi(\xi) = \frac{1}{G''}, \psi(\xi) = \frac{G'}{G''}, \quad (5)$$

and

$$\begin{aligned} \phi' &= \mu\phi^2 + \lambda\phi\psi, \\ \psi' &= \lambda\psi^2 + \mu\phi\psi + 1. \end{aligned} \quad (6)$$

Using the general solutions of the Eq. (4), we have the following solutions:

Case 1. when $\lambda < 0$, the general solutions of the Eq. (4) is

$$G(\xi) = C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h)) - \frac{\mu}{\lambda}\xi, \quad (7)$$

and we have

$$\phi^2 = \frac{-\lambda}{\lambda^3 v + \mu^2} (\lambda\psi^2 + 2\mu\phi\psi + 1), \quad (8)$$

where C_1 and C_2 are two arbitrary constants and $v = C_2^2 - C_1^2$.

Case 2. when $\lambda > 0$, the general solutions of the Eq. (4) is

$$G(\xi) = C_1 \cos(\sqrt{\lambda}(\xi + h)) + C_2 \sin(\sqrt{\lambda}(\xi + h)) - \frac{\mu}{\lambda}\xi, \quad (9)$$

and we have

$$\phi^2 = \frac{\lambda}{\lambda^3 v - \mu^2} (\lambda\psi^2 + 2\mu\phi\psi + 1), \quad (10)$$

where C_1 and C_2 are two arbitrary constants and $v = C_2^2 + C_1^2$.

Case 3. when $\lambda = 0$, the general solutions of the Eq. (4) is

$$G(\xi) = -\frac{\mu}{6}\xi^3 + \frac{1}{2}C_1\xi^2 + C_2\xi + C_3, \quad (11)$$

and we have

$$\phi^2 = \frac{1}{C_1^2 + 2\mu C_2} (2\mu\phi\psi + 1), \quad (12)$$

where C_1, C_2 and C_3 are arbitrary constants.

Now consider the general nonlinear partial differential equations (NLPDEs), say, in two variables,

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \quad (13)$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u(x, t)$ and the subscripts stand for the partial derivatives.

We suppose that the combination of real variables x and t by a complex variable ξ

$$u(x, t) = u(\xi), \quad \xi = ax - ct \quad (14)$$

where a is the wave number and c is the speed of the traveling wave. Now using Eq. (14), Eq. (13) is converted into an ordinary differential equation for $u = u(\xi)$:

$$F(u, -cu', au', c^2u'', -cau'', a^2u'', \dots) = 0, \quad \equiv \frac{\partial}{\partial \xi}. \quad (15)$$

Suppose that the traveling wave solution of Eq. (15) can be expressed by a polynomial in ϕ and ψ as:

$$u(\xi) = \sum_{i=0}^N a_i \psi^i + \sum_{i=1}^N b_i \psi^{i-1} \phi, \quad (16)$$

where $G = G(\xi)$ satisfies the third order LODE (4), $a_i (i = 0, 1, 2, \dots, N)$, $b_i (i = 1, 2, \dots, N)$, a, c, λ and μ are arbitrary constants, and the positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (15).

Substituting Eq. (16) into Eq. (15), using Eq. (6) and Eq. (8) (or using Eq. (6), Eq. (10) and Eq. (6), Eq. (12)) the left-hand side of Eq. (15) can be converted into a polynomial in ψ and ϕ , in which the degree of ϕ is not larger than 1. Compare the like powers of ψ^N and $\psi^N \phi$ equal to zero, yields a set of algebraic equations for $a_i (i = 0, 1, 2, \dots, N)$, $b_i (i = 1, 2, \dots, N)$, a, c, λ, μ, C_1 and C_2 . Solve the algebraic solutions with the aid of Maple. Substituting the values of $a_i (i = 0, 1, 2, \dots, N)$, $b_i (i = 1, 2, \dots, N)$, a, c, λ, μ, C_1 and C_2 obtained into Eq. (16), one can obtain the travelling wave solutions expressed by the hyperbolic functions of Eq. (15) (or expressed by trigonometric functions and rational functions).

3 (2 + 1)-Dimensional CRWP Equations

In this section, we aim first to apply the our new method for solve Eq. (2).

Substituting $u(x, y, t) = u(\xi)$, $\xi = ax + by - ct$ in Eq. (2) and integrating once yields

$$-(c + a)u(\xi) + \frac{a^2\alpha}{b}(u(\xi))^2 + a^2u'(\xi) + k = 0. \quad (17)$$

Balancing Eq. (17) give:

$$2N = N + 1 \Rightarrow N = 1.$$

By the use of Eq. (16), we present the solution of Eq. (17) as:

$$u(\xi) = a_0 + a_1 \psi(\xi) + b_1 \phi(\xi). \quad (18)$$

Case 1. When $\lambda < 0$ (hyperbolic function solutions)

Substituting Eq. (18) into Eq. (17), using Eq. (6) and Eq. (8) the left-hand side of Eq. (17) becomes a polynomial in ψ and ϕ . Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \mu$ and λ . After solving the algebraic equations by Maple we get,

set 1.

$$a_0 = a_0, a_1 = \frac{-b\lambda}{\alpha}, b_1 = \frac{-b\mu}{\alpha}, c = \frac{a(2a\alpha a_0 - b)}{b},$$

set 2.

$$a_0 = a_0, a_1 = \frac{-b\lambda}{2\alpha}, b_1 = \frac{b(-\mu \pm \lambda\sqrt{-(c_2^2 - c_1^2)\lambda})}{2\alpha}, c = \frac{a(2a\alpha a_0 - b)}{b}.$$

Putting the set (1) into Eq. (18), we get

$$u_1 = a_0 + \frac{b\sqrt{-\lambda}(C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + C_2 \cosh(\sqrt{-\lambda}(\xi + h)))}{\alpha(C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h)))},$$

$$v_1 = \frac{aa_0}{b} + \frac{a\sqrt{-\lambda}(C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + C_2 \cosh(\sqrt{-\lambda}(\xi + h)))}{\alpha(C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h)))}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_2 = a_0 + \frac{b\sqrt{-\lambda}}{\alpha} \tanh(\sqrt{-\lambda}(\xi + h)),$$

$$v_2 = \frac{aa_0}{b} + \frac{a\sqrt{-\lambda}}{\alpha} \tanh(\sqrt{-\lambda}(\xi + h)).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_3 = a_0 + \frac{b\sqrt{-\lambda}}{\alpha} \coth(\sqrt{-\lambda}(\xi + h)),$$

$$v_3 = \frac{aa_0}{b} + \frac{a\sqrt{-\lambda}}{\alpha} \coth(\sqrt{-\lambda}(\xi + h)).$$

Putting the set (2) into Eq. (18), we get

$$u_{4,5} = a_0 + \frac{b(C_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}(\xi + h)) + C_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{-(c_2^2 - c_1^2)\lambda})}{2\alpha(C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h)))},$$

$$v_{4,5} = \frac{aa_0}{b} + \frac{a \left(C_1 \sqrt{-\lambda} \sinh(\sqrt{-\lambda}(\xi + h)) + C_2 \sqrt{-\lambda} \cosh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{-(c_2^2 - c_1^2)\lambda} \right)}{2\alpha (C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h)))}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_{6,7} = a_0 + \frac{b\sqrt{-\lambda}}{2\alpha} \tanh(\sqrt{-\lambda}(\xi + h)) \mp \frac{a\sqrt{\lambda}}{2\alpha} \operatorname{sech}(\sqrt{-\lambda}(\xi + h)),$$

$$v_{6,7} = \frac{aa_0}{b} + \frac{a\sqrt{-\lambda}}{2\alpha} \tanh(\sqrt{-\lambda}(\xi + h)) \mp \frac{a\sqrt{\lambda}}{2\alpha} \operatorname{sech}(\sqrt{-\lambda}(\xi + h)).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_{8,9} = a_0 + \frac{b\sqrt{-\lambda}}{2\alpha} \left(\coth(\sqrt{-\lambda}(\xi + h)) \mp \operatorname{csch}(\sqrt{-\lambda}(\xi + h)) \right)$$

$$v_{8,9} = \frac{aa_0}{b} + \frac{a\sqrt{-\lambda}}{2\alpha} \left(\coth(\sqrt{-\lambda}(\xi + h)) \mp \operatorname{csch}(\sqrt{-\lambda}(\xi + h)) \right)$$

Case 2. When $\lambda > 0$ (trigonometric function solutions)

Substituting Eq. (18) into Eq. (17), using Eq. (6) and Eq. (10) the left-hand side of Eq. (17) becomes a polynomial in ψ and ϕ . Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \mu$ and λ . After solving the algebraic equations by Maple we get,

set 1.

$$a_0 = a_0, a_1 = \frac{-b\lambda}{\alpha}, b_1 = \frac{-b\mu}{\alpha}, c = \frac{a(2\alpha a_0 - b)}{b}.$$

set 2.

$$a_0 = a_0, a_1 = \frac{-b\lambda}{2\alpha}, b_1 = \frac{b(-\mu \pm \lambda\sqrt{(c_2^2 + c_1^2)\lambda})}{2\alpha}, c = \frac{a(2\alpha a_0 - b)}{b}.$$

Putting the set (1) into Eq. (18), we get

$$u_1 = a_0 + \frac{b\sqrt{\lambda} \left(-C_1 \sin(\sqrt{\lambda}(\xi + h)) + C_2 \cos(\sqrt{\lambda}(\xi + h)) \right)}{\alpha \left(C_1 \cos(\sqrt{\lambda}(\xi + h)) + C_2 \sin(\sqrt{\lambda}(\xi + h)) \right)},$$

$$v_1 = \frac{aa_0}{b} + \frac{a\sqrt{\lambda} \left(-C_1 \sin(\sqrt{\lambda}(\xi + h)) + C_2 \cos(\sqrt{\lambda}(\xi + h)) \right)}{\alpha \left(C_1 \cos(\sqrt{\lambda}(\xi + h)) + C_2 \sin(\sqrt{\lambda}(\xi + h)) \right)}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_2 = a_0 - \frac{b\sqrt{\lambda}}{\alpha} \tan\left(\sqrt{\lambda}(\xi + h)\right),$$

$$v_2 = \frac{aa_0}{b} - \frac{a\sqrt{\lambda}}{\alpha} \tan\left(\sqrt{\lambda}(\xi + h)\right).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_3 = a_0 + \frac{b\sqrt{\lambda}}{\alpha} \cot\left(\sqrt{\lambda}(\xi + h)\right),$$

$$v_3 = \frac{aa_0}{b} + \frac{a\sqrt{\lambda}}{\alpha} \cot\left(\sqrt{\lambda}(\xi + h)\right).$$

Putting the set (2) into Eq. (18), we get

$$u_{4,5} = a_0 + \frac{b\left(-C_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}(\xi + h)\right) + C_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}(\xi + h)\right) \mp \sqrt{(c_2^2 + c_1^2)\lambda}\right)}{2\alpha\left(C_1\cos\left(\sqrt{\lambda}(\xi + h)\right) + C_2\sin\left(\sqrt{\lambda}(\xi + h)\right)\right)},$$

$$v_{4,5} = \frac{aa_0}{b} + \frac{a\left(-C_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}(\xi + h)\right) + C_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}(\xi + h)\right) \mp \sqrt{(c_2^2 + c_1^2)\lambda}\right)}{2\alpha\left(C_1\cos\left(\sqrt{\lambda}(\xi + h)\right) + C_2\sin\left(\sqrt{\lambda}(\xi + h)\right)\right)}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_{6,7} = a_0 - \frac{b\sqrt{\lambda}}{2\alpha} \left(\tan\left(\sqrt{\lambda}(\xi + h)\right) \mp \sec\left(\sqrt{\lambda}(\xi + h)\right)\right),$$

$$v_{6,7} = \frac{aa_0}{b} - \frac{a\sqrt{\lambda}}{2\alpha} \left(\tan\left(\sqrt{\lambda}(\xi + h)\right) \mp \sec\left(\sqrt{\lambda}(\xi + h)\right)\right).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_{8,9} = a_0 + \frac{b\sqrt{\lambda}}{2\alpha} \left(\cot\left(\sqrt{\lambda}(\xi + h)\right) \mp \csc\left(\sqrt{\lambda}(\xi + h)\right)\right),$$

$$v_{8,9} = \frac{aa_0}{b} + \frac{a\sqrt{\lambda}}{2\alpha} \left(\cot\left(\sqrt{\lambda}(\xi + h)\right) \mp \csc\left(\sqrt{\lambda}(\xi + h)\right)\right).$$

where $\xi = ax + by - \left(\frac{a(2a\alpha a_0 - b)}{b}\right)t$.

Case 3. When $\lambda = 0$ (rational function solutions)

Substituting Eq. (18) into Eq. (17), using Eq. (6) and Eq. (12) the left-hand side of Eq. (17) becomes a polynomial in ψ and ϕ . Setting each coefficient of this

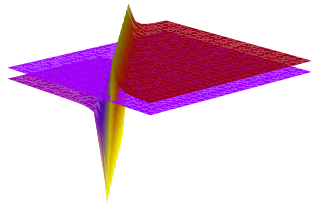
polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \mu$ and λ . After solving the algebraic equations by Maple we get,

$$a_0 = 0, a_1 = 0, b_1 = \frac{-b\mu}{\alpha}, c = -a.$$

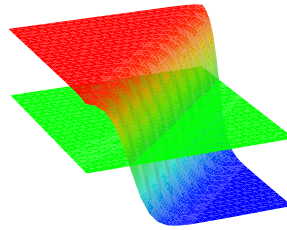
Putting the values of the constants into Eq. (18), we get

$$u_1 = \frac{-b\mu}{\alpha(-\mu(ax + by + at) + C_1)},$$

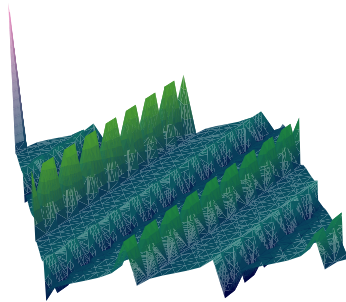
$$v_1 = \frac{-a\mu}{\alpha(-\mu(ax + by + at) + C_1)}.$$



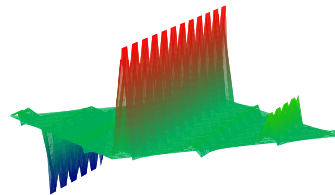
$u_1(x, y, t)$ if $\lambda < 0$



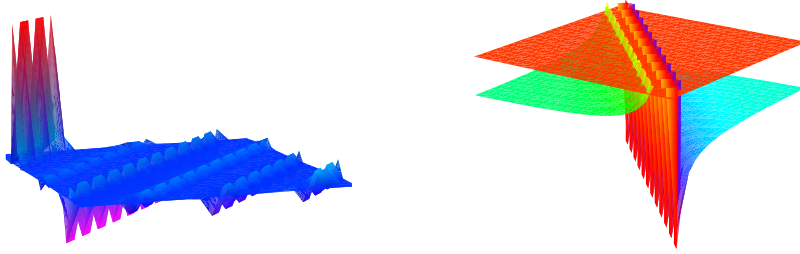
$u_2(x, y, t)$ if $\lambda < 0$



$u_1(x, y, t)$ if $\lambda > 0$



$u_2(x, y, t)$ if $\lambda > 0$



$$u_8(x, y, t) \text{ if } \lambda > 0$$

$$u_1(x, y, t) \text{ if } \lambda = 0$$

Fig. 3D-plot for $u_1(x, y, t)$, $u_2(x, y, t)$ when $\lambda < 0$, $u_1(x, y, t)$, $u_2(x, y, t)$ and $u_8(x, y, t)$, when $\lambda > 0$ and $\alpha = a = b = a_0 = 1, c_1 = 2, c_2 = 3, y = 1.5, h = 0.5$. $u_1(x, y, t)$ when $\lambda = 0, \alpha = a = b = \mu = 1, c_1 = 2$ and $y = 2$.

4 $(3 + 1)$ -Dimensional CRWP Equations

In this section, we apply the our new method for solve Eq. (3).

Substituting $u(x, y, z, t) = u(\xi), \xi = x + ay + bz - ct$ in Eq. (3) and integrating once yields

$$-(c + 1)u(\xi) + \frac{1}{2} \left(\alpha a + \frac{\beta}{a} + \frac{\gamma b^2}{a} \right) (u(\xi))^2 + \varepsilon (\alpha a^2 + \beta + \gamma b^2) u'(\xi) + k = 0. \quad (19)$$

Balancing Eq. (19) give:

$$2N = N + 1 \Rightarrow N = 1.$$

By the use of Eq. (16), we present the solution of Eq. (19) as:

$$u(\xi) = a_0 + a_1 \psi(\xi) + b_1 \phi(\xi). \quad (20)$$

Case 1. When $\lambda < 0$ (hyperbolic function solutions)

Substituting Eq. (20) into Eq. (19), using Eq. (6) and Eq. (8) the left-hand side of Eq. (19) becomes a polynomial in ψ and ϕ . Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \beta, \gamma, \mu$ and λ . After solving the algebraic equations by Maple we get,

set 1.

$$a_0 = a_0, a_1 = -2a\lambda\varepsilon, b_1 = -2a\mu\varepsilon, c = \frac{a^2\alpha a_0 + b^2\gamma a_0 + \beta a_0 - a}{a},$$

set 2.

$$\begin{aligned} a_0 &= a_0, a_1 = -a\lambda\varepsilon, b_1 = a\varepsilon \left(-\mu \pm \lambda\sqrt{-(c_2^2 - c_1^2)\lambda} \right), \\ c &= \frac{a^2\alpha a_0 + b^2\gamma a_0 + \beta a_0 - a}{a}. \end{aligned}$$

Putting the set (1) into Eq. (20), we get

$$\begin{aligned} u_1 &= a_0 + \frac{2a\varepsilon\sqrt{-\lambda} (C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + C_2 \cosh(\sqrt{-\lambda}(\xi + h)))}{C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h))}, \\ v_1 &= \frac{a_0}{a} + \frac{2\varepsilon\sqrt{-\lambda} (C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + C_2 \cosh(\sqrt{-\lambda}(\xi + h)))}{C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h))}, \\ w_1 &= \frac{ba_0}{a} + \frac{2\varepsilon b\sqrt{-\lambda} (C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + C_2 \cosh(\sqrt{-\lambda}(\xi + h)))}{C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h))}. \end{aligned}$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$\begin{aligned} u_2 &= a_0 + 2a\varepsilon\sqrt{-\lambda} \tanh(\sqrt{-\lambda}(\xi + h)), \\ v_2 &= \frac{a_0}{a} + 2\varepsilon\sqrt{-\lambda} \tanh(\sqrt{-\lambda}(\xi + h)), \\ w_2 &= \frac{ba_0}{a} + 2\varepsilon b\sqrt{-\lambda} \tanh(\sqrt{-\lambda}(\xi + h)). \end{aligned}$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$\begin{aligned} u_3 &= a_0 + 2a\varepsilon\sqrt{-\lambda} \coth(\sqrt{-\lambda}(\xi + h)), \\ v_3 &= \frac{a_0}{a} + 2\varepsilon\sqrt{-\lambda} \coth(\sqrt{-\lambda}(\xi + h)), \\ w_3 &= \frac{ba_0}{a} + 2\varepsilon b\sqrt{-\lambda} \coth(\sqrt{-\lambda}(\xi + h)). \end{aligned}$$

Putting the set (2) into Eq. (20), we get

$$\begin{aligned} u_{4,5} &= a_0 + \frac{a\varepsilon \left(\sqrt{-\lambda}C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + \sqrt{-\lambda}C_2 \cosh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{-(c_2^2 - c_1^2)\lambda} \right)}{C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h))}, \\ v_{4,5} &= \frac{a_0}{a} + \frac{\varepsilon \left(\sqrt{-\lambda}C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + \sqrt{-\lambda}C_2 \cosh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{-(c_2^2 - c_1^2)\lambda} \right)}{C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h))}, \end{aligned}$$

$$w_{4,5} = \frac{ba_0}{a} + \frac{\varepsilon b \left(\sqrt{-\lambda} C_1 \sinh(\sqrt{-\lambda}(\xi + h)) + \sqrt{-\lambda} C_2 \cosh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{-(c_2^2 - c_1^2)\lambda} \right)}{C_1 \cosh(\sqrt{-\lambda}(\xi + h)) + C_2 \sinh(\sqrt{-\lambda}(\xi + h))}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_{6,7} = a_0 + a\varepsilon \left(\sqrt{-\lambda} \tanh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{\lambda} \operatorname{sech}(\sqrt{-\lambda}(\xi + h)) \right)$$

$$v_{6,7} = \frac{a_0}{a} + \varepsilon \left(\sqrt{-\lambda} \tanh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{\lambda} \operatorname{sech}(\sqrt{-\lambda}(\xi + h)) \right)$$

$$w_{6,7} = \frac{ba_0}{a} + b\varepsilon \left(\sqrt{-\lambda} \tanh(\sqrt{-\lambda}(\xi + h)) \mp \sqrt{\lambda} \operatorname{sech}(\sqrt{-\lambda}(\xi + h)) \right)$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_{8,9} = a_0 + a\varepsilon \sqrt{-\lambda} \left(\coth(\sqrt{-\lambda}(\xi + h)) \mp \operatorname{csch}(\sqrt{-\lambda}(\xi + h)) \right)$$

$$v_{8,9} = \frac{a_0}{a} + \varepsilon \sqrt{-\lambda} \left(\coth(\sqrt{-\lambda}(\xi + h)) \mp \operatorname{csch}(\sqrt{-\lambda}(\xi + h)) \right)$$

$$w_{8,9} = \frac{ba_0}{a} + b\varepsilon \sqrt{-\lambda} \left(\coth(\sqrt{-\lambda}(\xi + h)) \mp \operatorname{csch}(\sqrt{-\lambda}(\xi + h)) \right)$$

Case 2. When $\lambda > 0$ (trigonometric function solutions)

Substituting Eq. (20) into Eq. (19), using Eq. (6) and Eq. (10) the left-hand side of Eq. (19) becomes a polynomial in ψ and ϕ . Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \beta, \gamma, \mu$ and λ . After solving the algebraic equations by Maple we get,

set 1.

$$a_0 = a_0, a_1 = -2a\lambda\varepsilon, b_1 = -2a\mu\varepsilon, c = \frac{a^2\alpha a_0 + b^2\gamma a_0 + \beta a_0 - a}{a},$$

set 2.

$$\begin{aligned} a_0 &= a_0, a_1 = -a\lambda\varepsilon, b_1 = a\varepsilon \left(-\mu \pm \lambda \sqrt{(c_2^2 + c_1^2)\lambda} \right), \\ c &= \frac{a^2\alpha a_0 + b^2\gamma a_0 + \beta a_0 - a}{a}. \end{aligned}$$

Putting the set (1) into Eq. (20), we get

$$u_1 = a_0 + \frac{2a\varepsilon\sqrt{\lambda} \left(-C_1 \sin \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \cos \left(\sqrt{\lambda} (\xi + h) \right) \right)}{C_1 \cos \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \sin \left(\sqrt{\lambda} (\xi + h) \right)},$$

$$v_1 = \frac{a_0}{a} + \frac{2\varepsilon\sqrt{\lambda} \left(-C_1 \sin \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \cos \left(\sqrt{\lambda} (\xi + h) \right) \right)}{C_1 \cos \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \sin \left(\sqrt{\lambda} (\xi + h) \right)},$$

$$w_1 = \frac{ba_0}{a} + \frac{2\varepsilon b\sqrt{\lambda} \left(-C_1 \sin \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \cos \left(\sqrt{\lambda} (\xi + h) \right) \right)}{C_1 \cos \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \sin \left(\sqrt{\lambda} (\xi + h) \right)}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_2 = a_0 - 2a\varepsilon\sqrt{\lambda} \tan \left(\sqrt{\lambda} (\xi + h) \right),$$

$$v_2 = \frac{a_0}{a} - 2\varepsilon\sqrt{\lambda} \tan \left(\sqrt{\lambda} (\xi + h) \right),$$

$$w_2 = \frac{ba_0}{a} - 2\varepsilon b\sqrt{\lambda} \tan \left(\sqrt{\lambda} (\xi + h) \right).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_3 = a_0 + 2a\varepsilon\sqrt{\lambda} \cot \left(\sqrt{\lambda} (\xi + h) \right),$$

$$v_3 = \frac{a_0}{a} + 2\varepsilon\sqrt{\lambda} \cot \left(\sqrt{\lambda} (\xi + h) \right),$$

$$w_3 = \frac{ba_0}{a} + 2\varepsilon b\sqrt{\lambda} \cot \left(\sqrt{\lambda} (\xi + h) \right).$$

Putting the set (2) into Eq. (20), we get

$$u_{4,5} = a_0 + \frac{a\varepsilon \left(-\sqrt{\lambda}C_1 \sin \left(\sqrt{\lambda} (\xi + h) \right) + \sqrt{\lambda}C_2 \cos \left(\sqrt{\lambda} (\xi + h) \right) \mp \sqrt{(c_2^2 + c_1^2)\lambda} \right)}{C_1 \cos \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \sin \left(\sqrt{\lambda} (\xi + h) \right)},$$

$$v_{4,5} = \frac{a_0}{a} + \frac{\varepsilon \left(-\sqrt{\lambda}C_1 \sin \left(\sqrt{\lambda} (\xi + h) \right) + \sqrt{\lambda}C_2 \cos \left(\sqrt{\lambda} (\xi + h) \right) \mp \sqrt{(c_2^2 + c_1^2)\lambda} \right)}{C_1 \cos \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \sin \left(\sqrt{\lambda} (\xi + h) \right)},$$

$$w_{4,5} = \frac{ba_0}{a} + \frac{\varepsilon b \left(-\sqrt{\lambda} C_1 \sin \left(\sqrt{\lambda} (\xi + h) \right) + \sqrt{\lambda} C_2 \cos \left(\sqrt{\lambda} (\xi + h) \right) \mp \sqrt{(c_2^2 + c_1^2) \lambda} \right)}{C_1 \cos \left(\sqrt{\lambda} (\xi + h) \right) + C_2 \sin \left(\sqrt{\lambda} (\xi + h) \right)}.$$

If $C_1 \neq 0$ and $C_2 = 0$,

$$u_{6,7} = a_0 - a\varepsilon\sqrt{\lambda} \left(\tan \left(\sqrt{\lambda} (\xi + h) \right) \mp \sec \left(\sqrt{\lambda} (\xi + h) \right) \right),$$

$$v_{6,7} = \frac{a_0}{a} - \varepsilon\sqrt{\lambda} \left(\tan \left(\sqrt{\lambda} (\xi + h) \right) \mp \sec \left(\sqrt{\lambda} (\xi + h) \right) \right),$$

$$w_{6,7} = \frac{ba_0}{a} - \varepsilon b\sqrt{\lambda} \left(\tan \left(\sqrt{\lambda} (\xi + h) \right) \mp \sec \left(\sqrt{\lambda} (\xi + h) \right) \right).$$

If $C_2 \neq 0$ and $C_1 = 0$,

$$u_{8,9} = a_0 + a\varepsilon\sqrt{\lambda} \left(\cot \left(\sqrt{\lambda} (\xi + h) \right) \mp \csc \left(\sqrt{\lambda} (\xi + h) \right) \right),$$

$$v_{8,9} = \frac{a_0}{a} + \varepsilon\sqrt{\lambda} \left(\cot \left(\sqrt{\lambda} (\xi + h) \right) \mp \csc \left(\sqrt{\lambda} (\xi + h) \right) \right),$$

$$w_{8,9} = \frac{ba_0}{a} + \varepsilon b\sqrt{\lambda} \left(\cot \left(\sqrt{\lambda} (\xi + h) \right) \mp \csc \left(\sqrt{\lambda} (\xi + h) \right) \right).$$

where $\xi = x + ay + bz - \left(\frac{a^2\alpha a_0 + b^2\gamma a_0 + \beta a_0 - a}{a} \right) t$.

Case 3. When $\lambda = 0$ (rational function solutions)

Substituting Eq. (20) into Eq. (19), using Eq. (6) and Eq. (12) the left-hand side of Eq. (19) becomes a polynomial in ψ and ϕ . Setting each coefficient of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, a, b, c, \alpha, \beta, \gamma, \mu$ and λ . After solving the algebraic equations by Maple we get,

$$a_0 = 0, a_1 = 0, b_1 = -2a\mu\varepsilon, c = -1.$$

Putting the values of the constants into Eq. (20), we get

$$u_1 = \frac{-2a\varepsilon\mu}{-\mu(x + ay + bz + t) + C_1},$$

$$v_1 = \frac{-2\varepsilon\mu}{-\mu(x + ay + bz + t) + C_1},$$

$$w_1 = \frac{-2b\varepsilon\mu}{-\mu(x + ay + bz + t) + C_1}.$$

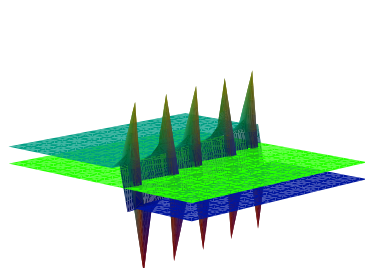
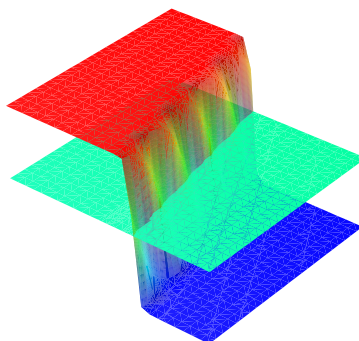
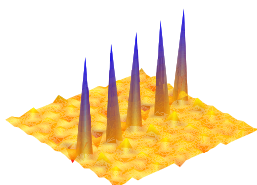
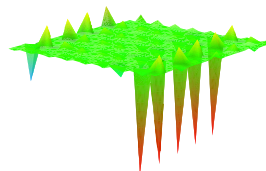

 $u_1(x, y, z, t)$ if $\lambda < 0$

 $u_2(x, y, z, t)$ if $\lambda < 0$

 $u_1(x, y, z, t)$ if $\lambda > 0$

 $u_2(x, y, z, t)$ if $\lambda > 0$

Fig. 3D-plot for $u_1(x, y, z, t)$, $u_2(x, y, z, t)$ when $\lambda < 0$, $u_1(x, y, z, t)$, $u_2(x, y, z, t)$ when $\lambda > 0$ and $\varepsilon = a = b = a_0 = y = 1, \alpha = \beta = \gamma = 2, c_1 = 2, c_2 = 3, z = 0.5, h = 0$.

5 Conclusion

In this paper, we established two equations, namely, the $(2 + 1)$ -dimensional Clannish Random Walker's Parabolic equation and the $(3 + 1)$ -dimensional Clannish Random Walker's Parabolic equation, which we believe that these two equations are introduced to the first time. The $\left(\frac{1}{G''} \bigg/ \frac{G'}{G''}\right)$ -expansion method has been successfully implemented to find the new two traveling waves solutions. The results show that this method is a powerful Mathematical tool for obtaining

exact solutions for our equations. It is also a promising method to solve other nonlinear partial differential equations.

Declarations

Competing interests: No

Authors' contributions: The research problem was jointly formulated by Prof. Dr. Mohammed Al-Amry and M.P. Eman Al-Abdali. The manuscript was prepared by M.P. Eman Al-Abdali under the supervision of Prof. Dr. Mohammed Al-Amry.

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