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GOWER'S PSEUDOINVERSE VIA FADDEEV-SOMINSKY'S PROCESS

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ABSTRACT. We exhibit that the Faddeev-Sominsky's algorithm gives the Gower's pseudoinverse for any non-defective singular square matrix. If the matrix under study is symmetric, then this pseudoinverse coincides with the Moore-Penrose generalized inverse matrix.

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1. INTRODUCTION

For any matrix $\mathbf{A}_{n\times n} = (A^i_j)$ its characteristic equation [1-3]:

$$\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n = 0, \tag{1}$$

can be obtained, through several procedures [1, 4-7] directly from the condition $\det(A^i_j - \lambda\delta^i_j) = 0$. The approach of Leverrier-Takeno [4, 8-13] is a simple and interesting technique to construct (1) based in the traces of the powers $\mathbf{A}^r, r = 1, ..., n$.

On the other hand, it is well known that an arbitrary matrix satisfies (1), which is the Cayley-Hamilton-Frobenius theorem [1-3]:

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + ... + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}. \tag{2}$$

If $\mathbf{A}$ is non-singular (that is, $\det\,\mathbf{A} \neq 0$), then (2) gives the inverse matrix:

$$\mathbf{A}^{-1} = -\frac{1}{a_n}\Big(\mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + ... + a_{n-1}\mathbf{I}\Big) \tag{3}$$

where $a_n \neq 0$ because $a_n = (-1)^n \det\,\mathbf{A}$.

Faddeev-Sominsky [13-19] proposed an algorithm to obtain $\mathbf{A}^{-1}$ in terms of $\mathbf{A}^r$ and their traces, which is equivalent [17] to the Cayley-Hamilton-Frobenius identity (2) plus the Leverrier-Takeno's method to construct the characteristic polynomial of a matrix $\mathbf{A}$, to see Sec. 2. The condition $\det\,\mathbf{A} \neq 0$ means $p \equiv rank\ \mathbf{A} = n$, then it is immediate the quest of an inverse of $\mathbf{A}$ when $1 \leq p \leq (n-1)$, in fact, in Sec. 3 we exhibit a pseudoinverse $\mathbf{A}^+$ with the properties [20, 21]:

$$\mathbf{A}\ \mathbf{A}^+\mathbf{A} = \mathbf{A}, \tag{4}$$

$$\mathbf{A}^+\mathbf{A}\ \mathbf{A}^+ = \mathbf{A}^+, \tag{5}$$

which coincides with the Gower's generalized inverse matrix [22, 23].

2. Leverrier-Takeno and Faddeev-Sominsky algorithms

If we define the quantities:

$$a_0 = 1, \quad s_k = tr\ \mathbf{A}^k, \quad k = 1, 2, \ldots, n \tag{6}$$

then the technique of Leverrier-Takeno [4, 8-13] implies (1) wherein the a_i are determined with the recurrence relation:

$$r a_r + s_1 a_{r-1} + s_2 a_{r-2} + \cdots + s_{r-1} a_1 + s_r = 0, \quad r = 1, 2, \ldots, n \tag{7}$$

therefore:

$$a_1 = -s_1, \quad 2!\ a_2 = (s_1)^2 - s_2, \quad 3!\ a_3 = -(s_1)^3 + 3s_1 s_2 - 2s_3,$$

$$4!\ a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1 s_3 + 3(s_2)^2 - 6s_4, \quad etc. \tag{8}$$

in particular, det $\mathbf{A} = (-1)^n a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that $\mathbf{A}$ and its transpose have the same determinant. In [24, 25] we find the general expression:

$$a_k = \frac{(-1)^k}{k!} \begin{vmatrix} s_1 & k-1 & 0 & \ldots & 0 \\ s_2 & s_1 & k-2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ s_{k-1} & s_{k-2} & \ldots & \ldots & 1 \\ s_k & s_{k-1} & \ldots & \ldots & s_1 \end{vmatrix}, \qquad k = 1, \ldots, n, \tag{9}$$

The Faddeev-Sominsky's procedure [13-19, 26] to obtain $\mathbf{A}^{-1}$ is a sequence of algebraic computations on the powers $\mathbf{A}^r$ and their traces, in fact, this algorithm is given via the instructions:

$$\begin{array}{lll} \mathbf{B}_1 = \mathbf{A}, & q_1 = \mathrm{tr}\ \mathbf{B}_1, & \mathbf{C}_1 = \mathbf{B}_1 - q_1 \mathbf{I}, \\ \mathbf{B}_2 = \mathbf{C}_1\ \mathbf{A}, & q_2 = \frac{1}{2}\ \mathrm{tr}\ \mathbf{B}_2, & \mathbf{C}_2 = \mathbf{B}_2 - q_2 \mathbf{I}, \\ \vdots & \vdots & \vdots \\ \mathbf{B}_{n-1} = \mathbf{C}_{n-2}\ \mathbf{A}, & q_{n-1} = \frac{1}{n-1}\ \mathrm{tr}\ \mathbf{B}_{n-1}, & \mathbf{C}_{n-1} = \mathbf{B}_{n-1} - q_{n-1} \mathbf{I}, \\ \mathbf{B}_n = \mathbf{C}_{n-1}\ \mathbf{A}, & q_n = \frac{1}{n}\ \mathrm{tr}\ \mathbf{B}_n, & \mathbf{C}_n = \mathbf{B}_n - q_n \mathbf{I}, \end{array} \tag{10}$$

then:

$$\mathbf{A}^{-1} = \frac{1}{q_n}\ \mathbf{C}_{n-1}. \tag{11}$$

For example, if we apply (10) for $n = 4$, then it is easy to see that the corresponding q_r imply (6) with $q_j = -a_j$, and besides (11) reproduces (3). By mathematical induction one can prove that (10) and (11) are equivalent to (3), (4) and (5), showing [17] thus that the Faddeev-Sominsky's technique has its origin in the Leverrier-Takeno method plus the Cayley-Hamilton-Frobenius theorem.

From (10) we can see that [26]:

$$\mathbf{C}_k = \mathbf{A}^k + a_1\mathbf{A}^{k-1} + a_2\mathbf{A}^{k-2} + \cdots + a_{k-1}\mathbf{A} + a_k\mathbf{I}, \quad k = 1, 2, \ldots, n, \tag{12}$$

and for $k = n - 1$:

$$\mathbf{C}_{n-1} = \mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + a_2\mathbf{A}^{n-3} + \cdots + a_{n-2}\mathbf{A} + a_{n-1}\mathbf{I} \ = -a_n\mathbf{A}^{-1}$$

in harmony with (11) because $a_n = -q_n$. The property $\mathbf{C}_k = \mathbf{0}$ is equivalent to (2). If $\mathbf{A}$ is singular, the process (10) gives the adjoint matrix of $\mathbf{A}$ [2, 3], in fact, Adj $\mathbf{A} = (-1)^{n+1}\mathbf{C}_{n-1}$.

If the roots of (1) have distinct values, then the Faddeev-Sominsky's algorithm allows obtain the corresponding eigenvectors of $\mathbf{A}$ [6]:

$$\mathbf{A}\vec{u}_k = \lambda_k\vec{u}_k, \qquad k = 1, 2, ..., n, \tag{13}$$

because for a given value of k, each column of:

$$\mathbf{Q}_k \equiv \lambda_k^{n-1}\mathbf{I} + \lambda_k^{n-2}\mathbf{C}_1 + ... + \mathbf{C}_{n-1}. \tag{14}$$

satisfies (13), and therefore all columns of $\mathbf{Q}_k$ are proportional to each other, that is, $rank\ \mathbf{Q}_k = 1$ [22, 27].

3. Gower's pseudoinverse

Here we consider the situation $1 \leq p \leq (n-1)$ with the condition $a_p \neq 0$, thus the multiplicity of the eigenvalue zero is $(n-p)$ and from (1) we deduce that [22]:

$$a_j = 0, \qquad j = p + 1, \ldots, n, \tag{15}$$

$$\lambda_k^p + a_1\lambda_k^{p-1} + \cdots + a_{p-1}\lambda_k + a_p = 0, \qquad k = 1, \ldots, p, \tag{16}$$

which allows to show the property:

$$\mathbf{A}^{p+1} + a_1\mathbf{A}^p + \cdots + a_{p-1}\mathbf{A}^2 + a_p\mathbf{A} = \mathbf{0}, \tag{17}$$

if $\mathbf{A}$ is non-defective [1], that is, it has n independent eigenvectors.

From (10) and (12) we observe that (17) means:

$$\mathbf{C}_p\ \mathbf{A} = \mathbf{B}_{p+1} = \mathbf{0} \tag{18}$$

that is:

$$\mathbf{B}_j = \mathbf{C}_j = \mathbf{0}, \qquad j = p + 1, \ldots, n. \tag{19}$$

Thus, from (18) we have that:

$$\mathbf{0} = \mathbf{A}\ \mathbf{C}_p = \mathbf{A}(\mathbf{B}_p + \gamma\mathbf{C}_p + a_p\mathbf{I}) \quad \therefore \quad \mathbf{A}\Big[-\frac{1}{a_p}\Big(\mathbf{C}_{p-1} + \gamma\mathbf{C}_p\Big)\Big]\mathbf{A} = \mathbf{A},$$

whose comparison with (4) suggests the pseudoinverse:

$$\mathbf{A}^+ = -\frac{1}{a_p}\Big(\mathbf{C}_{p-1} + \gamma\mathbf{C}_p\Big) \tag{20}$$

and we must determine the parameter γ; thus:

$$\mathbf{A}\ \mathbf{A}^+ = \mathbf{A}^+\mathbf{A} = -\frac{1}{a_p}\mathbf{B}_p \tag{21}$$

Then from (4), (5), (12), (18), (20) and (21):

$$\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}\ \mathbf{A}^+ = -\frac{1}{a_p}\mathbf{A}^+\mathbf{A}\Big(\mathbf{A}^{p-1} + a_1\mathbf{A}^{p-2} + \cdots + a_{p-2}\mathbf{A} + a_{p-1}\mathbf{I}\Big)$$

(22)

$$= -\frac{1}{a_p}\mathbf{A}^+\Big(\mathbf{A}^p + a_1\mathbf{A}^{p-1} + \cdots + a_{p-2}\mathbf{A}^2 + a_{p-1}\mathbf{A}\Big) = -\frac{1}{a_p}\mathbf{A}^+\Big(\mathbf{C}_p - a_p\mathbf{I}\Big) = \mathbf{A}^+ - \frac{1}{a_p}\mathbf{A}^+\mathbf{C}_p,$$

hence from (12), (18), (20) and (22):

$$\mathbf{0} = \mathbf{A}^+\mathbf{C}_p = -\frac{1}{a_p}\Big(\mathbf{C}_{p-1} + \gamma\mathbf{C}_p\Big)\mathbf{C}_p = -\frac{1}{a_p}\Big(a_{p-1} + \gamma a_p\Big)\mathbf{C}_p,$$

therefore $\gamma = -(a_{p-1}/a_p)$ and (20) gives the Gower's pseudoinverse verifying (4) and (5) [22, 23]:

$$\mathbf{A}^+ = -\frac{1}{a_p}\Big(\mathbf{C}_{p-1} - \frac{a_{p-1}}{a_p}\mathbf{C}_p\Big), \tag{23}$$

We emphasize that (23) was obtained under the constraints $1 \leq p \leq (n-1)$, $a_p \neq 0$ and $\mathbf{A}$ non-defective.

Remark 1.- The Leverrier-Faddeev's technique, according to Householder [28, 29], was rediscovered and improved by Souriau [30] and Frame [31].

Remark 2.- If $p = n$ then $\mathbf{C}_n = \mathbf{0}$ and (23) implies (11).

Remark 3.- If $\mathbf{A}$ is symmetric, then (23) also satisfies the conditions:

$$\Big(\mathbf{A}^+\mathbf{A}\Big)^T = \mathbf{A}^+\mathbf{A}, \qquad \Big(\mathbf{A}\ \mathbf{A}^+\Big)^T = \mathbf{A}\ \mathbf{A}^+ \tag{24}$$

thus (4), (5) and (24) are the four properties of Penrose [20, 21, 32, 33] and therefore the Gower's inverse coincides with the corresponding Moore-Penrose pseudoinverse [34-41].

Remark 4.- From (10) it is immediate that $tr\ \mathbf{C}_k = (n-k)a_k$, then (23) implies the relation:

$$tr\mathbf{A}^+ = -\ \frac{a_{p-1}}{a_p}. \tag{25}$$

Remark 5.- (Ben-Israel)-Charnes [42] and Decell [43] construct the polynomial:

$$\varphi(\lambda) = -\frac{1}{a_p}\Big(\lambda^{p-1} + a_1\lambda^{p-2} + \cdots + a_{p-2}\lambda + a_{p-1}\Big), \tag{26}$$

which generates a pseudoinverse via the following expression [36]:

$$\mathbf{A}^+ = \varphi(\mathbf{A}) + \varphi(0)\Big[\mathbf{A}\ \varphi(\mathbf{A}) - \mathbf{I}\Big], \tag{27}$$

then:

$$\varphi(\mathbf{A}) = -\frac{1}{a_p}\ \mathbf{C}_{p-1}, \qquad \mathbf{A}\ \varphi(\mathbf{A}) - \mathbf{I} = -\frac{1}{a_p}\ \mathbf{C}_p, \qquad \varphi(0) = -\frac{a_{p-1}}{a_p},$$

thus (27) gives the Gowers matrix (23).

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