Advanced Studies in Contemporary Mathematics
36 (2026), No. 3, pp. 157 - 159

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2026.36.3.001

PURE PSEUDOREPRESENTATIONS AS FIXED POINTS OF AN AVERAGING PROCESS FOR QUASIREPRESENTATIONS

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Abstract. We prove that pure pseudorepresentations are fixed points of the averaging process used in the definition of pseudorepresentations.

§ 1. Introduction

For the information concerning quasirepresentations, pseudorepresentations, and pure pseudorepresentations on groups and concerning the averaging process defining a pseudorepresentation close to a bounded quasirepresentation of a group, see [1]–[3].

§ 2. Preliminaries

Thanks to 1988 theorem by B. E. Johnson [4] and its explicit form for quasirepresentations of locally compact groups in dual Banach spaces [1], the definition of a pseudorepresentation, of an amenable locally compact group in a dual Banach space, close to a given bounded quasirepresentation π of the group uses the averaging process involving every cyclic subgroup H of the given group G. The closure $\overline{H}$ of this group is locally compact and commutative, and hence amenable.

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25 22E25.
Submitted May 23, 2026.
Key words and phrases. Quasirepresentations, pseudorepresentations, pure pseudorepresentations, averaging process..

Applying some invariant mean on this subgroup to the function

$$k \mapsto \pi(hk)\pi(k^{-1}), \qquad h, k \in H,$$

in a special topology, we obtain an operator-valued function of $h \in H$, which is a pseudorepresentation of G. However, a pseudorepresentation need not be a fixed point of this averaging process.

The main objective of this paper is to show that every continuous pure pseudorepresentation is a fixed point of the averaging process defining a pseudorepresentation approximating a given quasirepresentation.

This fact is of importance because continuous finite-dimensional quasirepresentations of connected Lie groups admit approximations by pure pseudorepresentations.

§ 3. Main Results

Theorem 1. *Let G be a group and let π be a bounded continuous pure pseudorepresentation of G. Then the averaging process used to define a pseudorepresentation approximating π gives π itself, i.e., π is a fixed point of the averaging process.*

Proof. By the definition of the averaging process, we are to consider the function

$$k \mapsto \pi(hk)\pi(k^{-1}), \qquad h, k \in H,$$

where H is the closure of a cyclic subgroup generated by some element $a \in G$. However, by the assumptions concerning the pure pseudorepresentation π, namely, the fact that the restriction of π to every cyclic subgroup is an ordinary representation, the pseudorepresentation π is bounded, and π is continuous, we see that the restriction of π to H is an ordinary representation of H.

This means that the function under consideration is constant and equal to $\pi(h)$ for all $h \in H$. Hence, for every mean (which need not be invariant), the mean value of this function is $\pi(h)$ and, therefore, the value of the averaged function at h is equal to $\pi(h)$ for all $h \in H$.

Since, by the continuity of π, this holds for every $h \in G$ belonging to the closure of an arbitrary cyclic subgroup, including the subgroup generated by h itself, it follows that the averaging process in question leaves every bounded pure pseudorepresentation fixed, as was to be proved.

This completes the proof of the theorem.

§ 4. Discussion

It should be noted that, for a sufficiently small defect, a bounded one-dimensional pseudorepresentation is pure and an unbounded one-dimensional pseudorepresentation is an ordinary representation of the group in question.

It looks probable that all fixed points of the averaging process under consideration are pure pseudorepresentations.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Advanced Studies of Contemporary Mathematics.

Funding

The research was supported by the Scientific Research Institute for System Analysis of the National Research Centre "Kurchatov Institute" according to the project FNEF-2024-0001.

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