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ON MULTIDIMENSIONAL CYCLIC CODES OVER A FINITE CHAIN RING

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Abstract. Multidimensional (nD) cyclic codes are a significant class of error-correcting codes that find applications in communication systems, data storage and systems involving multidimensional data. In this paper, we present a structured method for determining the generators of an nD cyclic code of arbitrary length over a finite chain ring. The key idea behind our approach is the sequential construction of generators of higher-dimensional cyclic codes based on the generators of lower-dimensional cyclic codes. Specifically, using the generators of cyclic codes (1D), the generators for two-dimensional (2D) cyclic codes have been determined, which have been used as a foundation to determine the generators for three-dimensional (3D) cyclic codes. This framework is extendable to higher dimensions, allowing us to construct the generators for nD cyclic codes from those of $(n-1)$D cyclic codes. The paper also provides a few illustrative examples to elaborate the theoretical results.

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1. Introduction

The class of multidimensional (nD) cyclic codes is an important class of error-correcting codes. Cyclic codes, which are initially explored in the one-dimensional context, have been extended to multidimensional arrays to address the challenges posed by data structures in various applications such as communication systems and data storage, where multidimensional data representation is prevalent. One-dimensional cyclic codes, a foundational class, are widely used in barcodes, where they ensure accurate data scanning and error correction despite any physical damage. Extending to two dimensions, cyclic codes become essential in QR codes, enabling data recovery even when sections of the QR code are corrupted. As we move to three dimensions, 3D cyclic codes find significant applications in fields like medical imaging, where these codes help to maintain the integrity of volumetric data, such as CT or MRI scans, during transmission or storage. Beyond three dimensions, higher-dimensional cyclic codes are becoming more important in upcoming technologies. For example, higher dimensional cyclic codes have importance in virtual reality (VR), which uses multiple layers of data to create immersive real-time experiences. The growing importance of multidimensional cyclic codes motivates us to study their

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structure over finite chain rings, as they provide a mathematical foundation for addressing the challenges of modern data representation and error correction. In this paper, we propose a systematic method for deriving generators of nD cyclic codes of arbitrary length over a finite chain ring. Our method is based on the construction of generators of higher-dimensional cyclic codes from lower-dimensional ones. Starting with the well-established structure of one-dimensional (1D) cyclic codes, we first derive the generators of two-dimensional (2D) cyclic codes. The generators of 2D cyclic codes then serve as a foundation for constructing the generators for three-dimensional (3D) cyclic codes. This framework is extendable to higher dimensions, allowing us to construct the generators for nD cyclic codes from those of $(n-1)$D cyclic codes.

In the past few decades, many researchers explored nD cyclic codes over finite fields. The basic theory of 2D cyclic codes was introduced by H. Imai [1], in 1977. In 2016, Z. Sepasdar and K. Khashyarmanesh [2] characterized 2D cyclic codes of length $n = s2^k$ over the finite field $\mathbb{F}_{p^m}$ as ideals of the quotient ring $\mathbb{F}_{p^m}[x,y]/\langle x^s - 1, y^{2^k} - 1\rangle$ for an odd prime p. Using a similar approach, Z. Rajabai and K. Khashyarmanesh [3] characterized the algebraic structure of repeated root 2D constacyclic codes of length $2p^s2^k$ over the finite field $\mathbb{F}_{p^m}$. O. Prakash and S. Patel [4] determined the structure of repeated root two dimensional (μ, ν)-constacyclic codes of length $4p^t2^r$ over $\mathbb{F}_{p^m}$. Subsequently, they [5] determined the structure of two-dimensional cyclic codes of length $3l$ and characterized the structure of two-dimensional (λ_1, λ_2)-constacyclic codes and their duals over $\mathbb{F}_{p^m}$. Z. Sepasdar [6], determined the generator matrix of 2D cyclic codes of arbitrary length. R. M. Lalasoa et al. [7] generalized these findings to 3D and nD cyclic codes. S. Bhardwaj and M. Raka [8] established a new form of generator polynomials for multidimensional constacyclic codes of arbitrary length using the concept of central primitive idempotents. Recently, D. Garg and S. Dutt [10] have determined the structure of a two-dimensional cyclic code of length mn over a finite chain ring, where m is arbitrary and n is co-prime to the cardinality of the residue field.

The structure of this paper is as follows: Section 2 introduces fundamental definitions and preliminary results on nD cyclic codes over a finite chain ring, specifically for $n = 2$. In section 3, we derive the generators of an nD cyclic code of arbitrary length over a finite chain ring for $n = 2$ alongwith illustrative examples. In Section 4, we extend these results to nD cyclic codes of arbitrary length. Finally, in Section 5, we conclude our work and summarize the key findings of the study.

2. Preliminaries

Consider a finite commutative ring R. A linear code C of length m over R is a sub-module of R^m over R. A linear code C of length $m_1 m_2 \dots m_n$ over R is called an nD cyclic code if its codewords are represented as n-dimensional arrays $\left[c_{i_1 i_2 \dots i_n}\right]$, $0 \leq i_k \leq m_k - 1$, $0 \leq k \leq n-1$, $c_{i_1 i_2 \dots i_n} \in R$ and C is closed under cyclic shifts along all the n dimensions. Specifically, for $n = 1$, a 1-D cyclic code C of length m_1 is the classical cyclic code, where codewords are closed under the usual cyclic shift operator σ defined as $\sigma(r_0, r_1, \ldots, r_{m_1-1}) = (r_{m_1-1}, r_0, r_1, \ldots, r_{m_1-2})$ for every $(r_0, r_1, \ldots, r_{m_1-1}) \in C$. It is well established that a cyclic code C of length m_1 over R can be viewed as an ideal of $R[x]/\langle x^{m_1} - 1\rangle$. For $n = 2$, the codewords of a 2D cyclic code of a length $m_1 m_2$ are $m_1 \times m_2$ arrays of the form $c = \left[r_{ij}\right]$, $0 \le i \le m_1 - 1$, $0 \le j \le m_2 - 1$, $r_{ij} \in R$ and C is closed under both row and column cyclic shifts. The row and column cyclic shift denoted by σ_r and σ_c respectively, are defined by $\sigma_r \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{m_1-1} \end{pmatrix} = \begin{pmatrix} r_{m_1-1} \\ r_0 \\ \vdots \\ r_{m_1-2} \end{pmatrix}$, where r_i denotes the i^{th} row of c for $0 \le i \le m_1 - 1$ and $\sigma_c(c_0, c_1, \ldots, c_{m_2-1}) = (c_{m_2-1}, c_0, \ldots, c_{m_2-2})$, where c_j denotes the j^{th} column of c for $0 \le j \le m_2 - 1$ respectively. It is easy to verify that a 2D cyclic code of length $m_1 m_2$ over R can be viewed as an ideal of the ring $R[x,y]/\langle x^{m_1} - 1, y^{m_2} - 1\rangle$ which is isomorphic to $(\aleph[x]/\langle x^{m_1} - 1\rangle)[y]/\langle y^{m_2} - 1\rangle$ under the map $\sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} a_{ij} x^i y^j \longrightarrow \sum_{j=0}^{m_2-1} \Big(\sum_{i=0}^{m_1-1} a_{ij} x^i \Big) y^j$. For a natural number n, an nD cyclic code of length $m_1 m_2 \ldots m_n$ over R can be defined as an ideal of the ring $R[x_1, x_2, \ldots, x_n]/\langle x_1^{m_1} - 1, x_2^{m_2} - 1, \ldots, x_n^{m_n} - 1\rangle$.

Let $\mathcal{C}$ be an nD cyclic code of length $m_1 m_2 \ldots m_n$ over R. Then, the Hamming distance between two codewords of $\mathcal{C}$ is the number of positions, where the corresponding elements in two codewords differ. The minimum Hamming distance $d_H(\mathcal{C})$ of $\mathcal{C}$ is defined as the smallest Hamming distance among distinct codewords of $\mathcal{C}$. The Hamming weight of a codeword is defined as the number of non-zero elements in that codeword. The minimum Hamming weight $w_H(\mathcal{C})$ of $\mathcal{C}$ is defined as the smallest Hamming weight among all non-zero codewords in $\mathcal{C}$. It is clear that $w_H(\mathcal{C})$ is always equal to $d_H(\mathcal{C})$. The rank of $\mathcal{C}$ is defined as the cardinality of the minimal spanning set of $\mathcal{C}$. The code $\mathcal{C}$ is called MHDR if $d_H(\mathcal{C}) = m_1 m_2 \ldots m_n - rank(\mathcal{C}) + 1$.

A finite commutative ring R is called a finite chain ring if all its ideals form a chain under the inclusion operation. A finite chain ring always has a unique maximal ideal. Let $\aleph$ denote a finite chain ring and let $\langle \gamma \rangle$ represent its maximal ideal. Suppose ν is the nilpotency index of γ and $\aleph/\langle \gamma \rangle = \mathbb{F}_q$, where q is the prime power.

Monika et al. [9] have determined the generators of a cyclic code C over $\aleph$. Let us recall the relevant results from [9], which are required for later use.

Theorem 2.1. *[9] A cyclic code $\mathcal{C}$ of length n over $\aleph$ is generated by $\mathcal{C} = \langle p_0(x),\ p_1(x), \ldots,\ p_r(x)\rangle$, where $p_j(x) = \gamma^{s_j} q_j(x)$ is the polynomial of minimal degree among all polynomials in $\mathcal{C}$ whose leading coefficient is γ^{s_j} for $0 \le j \le r$. Also, $q_j(x) \in F_q + \gamma \mathbb{F}_q + \gamma^2 \mathbb{F}_q + \cdots + \gamma^{\nu - s_j - 1} \mathbb{F}_q[x]/\langle x^n - 1\rangle$ is a monic polynomial for $0 \le j \le r$. Also, $s_0 > s_1 > \cdots > s_r$ and $t_0 < t_1 < \cdots < t_r$, where $t_j = deg(p_j(x))$.*

Corollary 2.2. *[9] The maximum number of generators of $\mathcal{C}$ of length n over $\aleph$ is $k = min\{\nu, t_r + 1\}$.*

Let us also recall the following result from [10], which is required for later use.

Theorem 2.3. *[10] Consider a 2D cyclic code $\mathcal{C}$ of length m_1m_2 over $\mathcal{R}$. Then $\mathcal{C}_{\nu-1} = \{f(x,y) \in \mathbb{F}_q[x,y]/\langle x^m-1, y^n-1\rangle$ such that $\gamma^{\nu-1}f(x,y) \in \mathcal{C}\}$ is a 2D cyclic code of length m_1m_2 over $\mathbb{F}_q$. Also, $w_H(\mathcal{C}) = w_H(\mathcal{C}_{\nu-1})$.*

3. Generators of 2D cyclic codes of length m_1m_2 over a finite chain ring

In this section, we determine the generators of a 2D cyclic code of arbitrary length m_1m_2 over a finite chain ring $\mathcal{R}$.

Let $\mathcal{C}$ be a 2D cyclic code of length m_1m_2 over a finite chain ring $\mathcal{R}$. Let $f(x,y) \in \mathcal{C}$ be any element. Then, $f(x,y)$ can be uniquely written as $\sum_{j=0}^{m_2-1} f_j(x)y^j$, where $f_j(x) \in \mathcal{R}[x]/\langle x^{m_1}-1\rangle$ for $0 \le j \le m_2-1$. Define the sets,

$$I_j = \left\{g_j(x) \in \mathcal{R}[x]/\langle x^{m_1}-1\rangle \mid \exists\ g(x,y) \in \mathcal{C} \text{ such that } g(x,y) = \sum_{k=0}^{m_2-1-j} g_{m_2-1-k}(x)y^k\right\}$$

for $0 \le j \le m_2-1$. It can be easily verified that each I_j, $0 \le j \le m_2-1$, is an ideal of the ring $\mathcal{R}[x]/\langle x^{m_1}-1\rangle$ and therefore a cyclic code of length m_1 over $\mathcal{R}$. By Theorem 2.1, we can find a set of polynomials $p_0^{(j)}(x),\ p_1^{(j)}(x)\ldots,\ p_{r_j}^{(j)}(x) \in \mathcal{R}[x]/\langle x^{m_1}-1\rangle$ such that

$$I_j = \langle p_0^{(j)}(x),\ p_1^{(j)}(x)\ldots,\ p_{r_j}^{(j)}(x)\rangle,\ 0 \le j \le m_2-1.$$

Also, by Corollary 2.2, $r_j+1 \le min(\nu,\ t_{r_j}+1)$ for each j, $0 \le j \le m_2-1$, where $t_{r_j} = deg(p_{r_j}^{(j)}(x))$.

Theorem 3.1. *Let $\mathcal{C}$ be a 2D cyclic code of length m_1m_2 over $\mathcal{R}$. Then the set $\{P_i^{(j)}(x,y) \in \mathcal{R}[x,y] \mid 0 \le i \le r_j,\ 0 \le j \le m_2-1\}$ generates $\mathcal{C}$ where*

$$P_i^{(j)}(x,y) = \sum_{k=0}^{m_2-1-j} a_{ik}^{(j)}(x)y^k, \text{ for all } 0 \le j \le m_2-1$$

and $a_{ik}^{(j)}(x) \in I_j$ for every $0 \le k \le m_2-1-j$. Also, $a_{i(m_2-1-j)}^{(j)}(x) = p_i^{(j)}(x)$, $0 \le i \le r_j$, $0 \le j \le m_2-1$.

Proof. Let $\mathcal{C}$ be a 2D cyclic code of length m_1m_2 over $\mathcal{R}$. Let $f(x,y) = \sum_{j=0}^{m_2-1} f_j(x)y^j$ with $f_j(x) \in \mathcal{R}[x]/\langle x^{m_1}-1\rangle$, $0 \le j \le m_2-1$, be any element of $\mathcal{C}$.

Clearly, $f_{m_2-1}(x) \in I_0$. Therefore, $f_{m_2-1}(x) = \sum_{i=0}^{r_0} p_i^{(0)}(x)t_i^{(0)}(x)$, where $t_i^{(0)}(x) \in \mathcal{R}[x]/\langle x^{m_1}-1\rangle$ for all $0 \le i \le r_0$. Also, $p_i^0(x) \in I_0$; $0 \le i \le r_0$, therefore by definition of I_0, there exists $P_i^{(0)}(x,y) \in \mathcal{C}$ such that $P_i^{(0)}(x,y) = \sum_{k=0}^{m_2-1} a_{ik}^{(0)}(x)y^k$, where $a^{(0)}_{i(m_2-1)}(x) = p^{(0)}_i(x),\ 0 \le i \le r_0$. Since $\mathcal{C}$ is an ideal of $\mathfrak{R}$, $y^j P^{(0)}_i(x,y) \in \mathcal{C}$. Therefore, $a^{(0)}_{ik}(x) \in I_0$ for all $0 \le k \le m_2-1,\ 0 \le i \le r_0$. Let

$$\begin{aligned} h_1(x,y) &= f(x,y) - \sum_{i=0}^{r_0} P^{(0)}_i(x,y) t^{(0)}_i(x) \qquad (3.1)\\ &= \sum_{k=0}^{m_2-2} f_k(x) y^k - \sum_{i=0}^{r_0} t^{(0)}_i(x) \Big(\sum_{k=0}^{m_2-2} a^{(0)}_{ik}(x) y^k \Big) \\ &= \sum_{k=0}^{m_2-2} h^{(1)}_k(x) y^k;\ h^{(1)}_k(x) \in \mathfrak{R}[x]/\langle x^{m_1}-1\rangle. \end{aligned}$$

Clearly, $h_1(x,y) \in \mathcal{C}$ since $f(x,y), P^{(0)}_i(x,y) \in \mathcal{C}$ for every $0 \le i \le r_0$. Also, by definition of I_1, $h^{(1)}_{m_2-2}(x) \in I_1$. So, $h^{(1)}_{m_2-2}(x) = \sum_{i=0}^{r_1} p^{(1)}_i(x) t^{(1)}_i(x)$, where $t^{(1)}_i(x) \in \mathfrak{R}[x]/\langle x^{m_1}-1\rangle$, for all $0 \le i \le r_1$. Now, $p^{(1)}_i(x) \in I_1;\ 0 \le i \le r_1$, therefore there exists $P^{(1)}_i(x,y) \in \mathcal{C}$ such that $P^{(1)}_i(x,y) = \sum_{k=0}^{m_2-2} a^{(1)}_{ik}(x) y^k$, where $a^{(1)}_{i(m_2-2)}(x) = p^{(1)}_i(x),\ 0 \le i \le r_1$. Also, since $\mathcal{C}$ is an ideal of $\mathfrak{R}$, $y^j P^{(1)}_i(x,y) \in \mathcal{C}$. Therefore $a^{(1)}_{ik}(x) \in I_1$ for all $0 \le k \le m_2-2,\ 0 \le i \le r_1$. Let

$$\begin{aligned} h_2(x,y) &= h_1(x,y) - \sum_{i=0}^{r_1} P^{(1)}_i(x,y) t^{(1)}_i(x) \qquad (3.2)\\ &= \sum_{k=0}^{m_2-3} h^{(1)}_k(x) y^k - \sum_{i=0}^{r_1} t^{(1)}_i(x) \Big(\sum_{k=0}^{m_2-3} a^{(1)}_{ik}(x) y^k \Big) \\ &= \sum_{k=0}^{m_2-3} h^{(2)}_k(x) y^k;\ h^{(2)}_k(x) \in \mathfrak{R}[x]/\langle x^{m_1}-1\rangle. \end{aligned}$$

Clearly, $h_2(x,y) \in \mathcal{C}$ since $h_1(x,y),\ P^{(1)}_i(x,y) \in \mathcal{C};\ 0 \le i \le r_1$. Continuing in this fashion, we obtain polynomials $h_1(x,y),\ h_2(x,y), \ldots, h_{m_2-1}(x,y)$, where

$$h_{m_2-1}(x,y) = h_{m_2-2}(x,y) - \sum_{i=0}^{r_{m_2-2}} P^{(m_2-2)}_i(x,y) t^{(m_2-2)}_i(x) = h^{(m_2-2)}_0(x). \qquad (3.m\text{-}1)$$

Clearly, by definition of I_{m_2-1}, $h^{(m_2-2)}_0(x) \in I_{m_2-1}$, which implies that $h^{(m_2-2)}_0(x) = \sum_{i=0}^{r_{m_2-1}} p^{(m_2-1)}_i(x) t^{(m_2-1)}_i(x)$, where $t^{(m_2-1)}_i(x) \in \mathfrak{R}[x]/\langle x^{m_1}-1\rangle;\ 0 \le i \le r_{m_2-1}$. Now $p^{(m_2-1)}_i(x) \in I_{m_2-1}$, therefore there exist $P^{(m_2-1)}_i(x,y) \in \mathcal{C}$ such that $P^{(m_2-1)}_i(x,y) =$

$p_i^{(m_2-1)}(x)$. Therefore,

$$h_{m_2-1}(x) = h_0^{(m_2-2)}(x) = \sum_{i=0}^{r_{m_2-1}} P_i^{(m_2-1)}(x,y)t_i^{(m_2-1)}(x). \qquad (3.m_2)$$

It follows from equations (3.1) to ($3.m_2$) that

$$f(x,y) = \sum_{i=0}^{r_0} P_i^{(0)}(x,y)t_i^{(0)}(x) + \sum_{i=0}^{r_1} P_i^{(1)}(x,y)t_i^{(1)}(x) + \cdots + \sum_{i=0}^{r_{m_2-2}} P_i^{(m_2-2)}(x,y)t_i^{(m_2-2)}(x) + \sum_{i=0}^{r_{m_2-1}} P_i^{(m_2-1)}(x,y)t_i^{(m_2-1)}(x).$$

Thus, the set $\{P_i^{(j)}(x,y) \in \mathfrak{R}[x,y] \mid 0 \le i \le r_j,\ 0 \le j \le m_2-1\}$ generates $\mathcal{C}$, where

$$P_i^{(j)}(x,y) = \sum_{k=0}^{m_2-1-j} a_{ik}^{(j)}(x)y^k, \text{ for all } 0 \le j \le m_2-1$$

and $a_{ik}^{(j)}(x) \in I_j$ for every $0 \le k \le m_2-1-j$. Also, $a_{i(m_2-1-j)}^{(j)}(x) = p_i^{(j)}(x)$, $0 \le i \le r_j$, $0 \le j \le m_2-1$. □

The following result directly follows from Theorem 3.1 and Corollary 2.2.

Corollary 3.2. *The maximum number of generators for a 2D cyclic code $\mathcal{C}$ over $\mathfrak{R}$ is km_2, where* $k = \sum_{j=0}^{m_2-1}(r_j+1)$.

Next, we present a few examples to illustrate our theoretical results.

Example 3.1. *Let $\mathcal{C}$ be a 2D cyclic code of length m_1m_2 over the finite chain ring $\mathfrak{R} = \mathbb{F}_4 + \gamma\mathbb{F}_4$, where $\gamma^2 = 0$, $m_1 = 8$ and $m_2 = 3$. Consider the following cyclic codes of length* 8 *over $\mathfrak{R}$ given by*

$$I_0 = \langle p_0^{(0)}(x),\ p_1^{(0)}(x)\rangle = \langle \gamma(x^3-3x^2+3x-1),\ x^4-4x^3+6x^2-4x+1\rangle$$
$$I_1 = \langle p_0^{(1)}(x)\rangle = \langle \gamma(x^4-4x^3+6x^2-4x+1)\rangle$$
$$I_2 = \langle p_0^{(2)}(x)\rangle = \langle x^6-6x^5+15x^4-20x^3+15x^2-6x+1\rangle$$

By Theorem 3.1, the set $\{P_0^{(0)}(x,y),\ P_1^{(0)}(x,y),\ P_0^{(1)}(x,y),\ P_0^{(2)}(x,y)\}$ generates $\mathcal{C}$, where

$$P_0^{(0)}(x,y) = p_{00}^{(0)}(x) + p_{01}^{(0)}(x)y + \gamma(x^3-3x^2+3x-1)y^2$$
$$P_1^{(0)}(x,y) = p_{10}^{(0)}(x) + p_{11}^{(0)}(x)y + (x^4-4x^3+6x^2-4x+1)y^2$$
$$P_0^{(1)}(x,y) = p_{00}^{(1)}(x) + \gamma(x^4-4x^3+6x^2-4x+1)y$$
$$P_0^{(2)}(x,y) = (x^6-6x^5+15x^4-20x^3+15x^2-6x+1)$$

such that $p_{00}^{(0)}(x)$, $p_{01}^{(0)}(x)$, $p_{10}^{(0)}(x)$, $p_{11}^{(0)}(x) \in I_0$, $p_{00}^{(1)}(x) \in I_1$. *More specifically, taking* $p_{00}^{(0)}(x) = p_{01}^{(0)}(x) = p_{10}^{(0)}(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$ *and* $p_{11}^{(0)}(x) = \gamma(x^3 - 3x^2 + 3x - 1)$, $p_{00}^{(1)}(x) = \gamma(x^4 - 4x^3 + 6x^2 - 4x + 1)$ *we get, the set* $\Big\{(x^4 - 4x^3 + 6x^2 - 4x + 1)(1 + y) + \gamma(x^3 - 3x^2 + 3x - 1)y^2, (x^4 - 4x^3 + 6x^2 - 4x + 1) + \gamma(x^3 - 3x^2 + 3x - 1)y + (x^4 - 4x^3 + 6x^2 - 4x + 1)y^2, \gamma(x^4 - 4x^3 + 6x^2 - 4x + 1) + \gamma(x^4 - 4x^3 + 6x^2 - 4x + 1)y, (x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1)\Big\}$, *which generates a 2D cyclic code of length* $m_1 m_2 = 24$ *over* $\mathfrak{R}$.

Example 3.2. *Let* $\mathcal{C}$ *be a 2D cyclic code of length* $m_1 m_2$ *over the finite chain ring* $\mathfrak{R} = \mathbb{Z}_9$, *where* $m_1 = 3$ *and* $m_2 = 2$. *Consider the following cyclic codes of length* 3 *over* $\mathfrak{R}$ *given by*

$$I_0 = \langle p_0^{(0)}(x) \rangle = \langle 3(x-1) \rangle$$
$$I_1 = \langle p_0^{(1)}(x) \rangle = \langle x^2 - 2x + 1 \rangle.$$

By Theorem 3.1, the set $\{P_0^{(0)}(x,y),\ P_0^{(1)}(x,y)\}$ *generates* $\mathcal{C}$, *where*

$$P_0^{(0)}(x,y) = p_{00}^{(0)}(x) + 3(x-1)y$$
$$P_0^{(1)}(x,y) = p_{00}^{(1)}(x) + (x^2 - 2x + 1)y$$

such that $p_{00}^{(0)}(x) \in I_0$ *and* $p_{00}^{(1)}(x) \in I_1$. *More specifically, taking* $p_{00}^{(0)}(x) = 3(x-1)$ *and* $p_{00}^{(1)}(x) = 8(x^2 - 2x + 1)$ *we get, the set* $\Big\{3(x-1)(1+y), (x^2 - x + 1)(8 + y)\Big\}$, *which generates a 2D cyclic code* $\mathcal{C}$ *of length* $m_1 m_2 = 6$ *over* $\mathfrak{R}$. *It is easy to verify that the set* $\{3(x-1)(1+y), 3x(x-1)(1+y), (x^2 - 2x + 1)(8 + y)\}$ *is the minimal spanning set of* $\mathcal{C}$ *which implies that rank*$(C) = 3$. *Clearly, the set* $\{(x-1)(1+y), (x^2 - 2x + 1)(2 + y)\}$ *generates the 2D cyclic code* $\mathcal{C}_{\nu-1}$ *of length* 6 *over the residue field* $\mathbb{Z}_3$. *The generator matrix of this code is:*

$$G = \begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

The minimum Hamming distance of this code, computed using MAGMA software, is 3. By Theorem 2.3, $d_H(\mathcal{C}) = 3$.

Example 3.3. *Let* $\mathcal{C}$ *be a 2D cyclic code of length* $m_1 m_2$ *over the finite chain ring* $\mathfrak{R} = \mathbb{Z}_{49}$, *where* $m_1 = 3$ *and* $m_2 = 2$. *Consider the following cyclic codes of length* 3 *over* $\mathfrak{R}$ *given by*

$$I_0 = \langle p_0^{(0)}(x) \rangle = \langle x^2 + x + 1 \rangle$$
$$I_1 = \langle p_0^{(1)}(x) \rangle = \langle x - 1 \rangle.$$

By Theorem 3.1, the set $\{P_0^{(0)}(x,y),\ P_0^{(1)}(x,y)\}$ *generates* $\mathcal{C}$, *where*

$$P_0^{(0)}(x,y) = p_{00}^{(0)}(x) + (x^2 + x + 1)y$$
$$P_0^{(1)}(x,y) = p_{00}^{(1)}(x) + (x-1)y$$

such that $p_{00}^{(0)}(x) \in I_0$ *and* $p_{00}^{(1)}(x) \in I_1$. *More specifically, taking* $p_{00}^{(0)}(x) = 48(x^2 + x + 1)$ *and* $p_{00}^{(1)}(x) = (x-1)$, *we get the set* $\Big\{(x^2 + x + 1)(48 + y), (x-1)(1+y)\Big\}$, *which* *generates a 2D cyclic code $\mathcal{C}$ of length $m_1m_2 = 6$ over $\mathfrak{R}$. It is easy to verify that the set $\{(x^2+x+1)(48+y),(x-1)(1+y),x(x-1)(1+y)\}$ is the minimal spanning set of $\mathcal{C}$ which implies that $rank(C) = 3$. Clearly, the set $\{(x^2+x+1)(6+y),(x-1)(1+y)\}$ generates the 2D cyclic code $\mathcal{C}_{\nu-1}$ of length 6 over the residue field $\mathbb{Z}_7$. The generator matrix of this code is:*

$$G = \begin{pmatrix} 6 & 1 & 6 & 1 & 6 & 1 \\ 6 & 6 & 1 & 1 & 0 & 0 \\ 0 & 0 & 6 & 6 & 1 & 1 \end{pmatrix}.$$

The minimum Hamming distance of this code, computed using MAGMA software, is 4. By Theorem 2.3, $d_H(\mathcal{C}) = 4$, which implies that $\mathcal{C}$ is a 2D MHDR cyclic code over $\mathbb{Z}_{49}$.

Example 3.4. *Let $\mathcal{C}$ be 2D cyclic code of length mm_2 over the finite chain ring $\mathfrak{R} = \mathbb{Z}_{25}$, where $m = 5$ and $m_2 = 2$. Consider the following cyclic codes of length 5 over $\mathfrak{R}$ given by*

$$I_0 = \langle p_0^{(0)}(x)\rangle = \langle 5,(x-1)\rangle$$
$$I_1 = \langle p_0^{(1)}(x)\rangle = \langle x-1\rangle.$$

By Theorem 3.1, the set $\{P_0^{(0)}(x,y),\ P_1^{(0)}(x,y),\ P_0^{(1)}(x,y)\}$ generates $\mathcal{C}$, where

$$P_0^{(0)}(x,y) = p_{00}^{(0)}(x) + 5y$$
$$P_1^{(0)}(x,y) = p_{10}^{(0)}(x) + (x-1)y$$
$$P_0^{(1)}(x,y) = p_{00}^{(1)}(x) + (x-1)y$$

such that $p_{00}^{(0)}(x), p_{10}^{(0)}(x) \in I_0$ and $p_{00}^{(1)}(x) \in I_1$. More specifically, taking $p_{00}^{(0)}(x) = 5(24)$ and $p_{10}^{(0)}(x) = p_{00}^{(1)}(x) = (x-1)$, we get the set $\left\{5(24+y),(x-1)(1+y)\right\}$, which generates a 2D cyclic code $\mathcal{C}$ of length $mm_2 = 10$ over $\mathfrak{R}$. It is easy to verify that the set $\{5(24+y),\ 5x(24+y),\ 5x^2(24+y),\ 5x^3(24+y),\ 5x^4(24+y),\ (x-1)(1+y),\ x(x-1)(1+y),\ x^2(x-1)(1+y),\ x^3(x-1)(1+y)\}$ is the minimal spanning set of $\mathcal{C}$ which implies that $rank(C) = 9$. Clearly, the set $\{(24+y),(x-1)(1+y)\}$ generates the 2D cyclic code $\mathcal{C}_{\nu-1}$ of length 10 over the residue field $\mathbb{Z}_5$. The generator matrix of this code is:

$$G = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 4 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 1 & 1 \end{pmatrix}.$$

The minimum Hamming distance of this code, computed using MAGMA software, is 2. By Theorem 2.3, $d_H(\mathcal{C}) = 2$, which implies that $\mathcal{C}$ is a 2D MHDR cyclic code over $\mathbb{Z}_{25}$.

4. Generators of a multidimensional cyclic code of arbitrary length over a finite chain ring

In this section, we determine the generators of a 3D cyclic code of length $m_1m_2m_3$ over $\aleph$. In addition, we generalize this result to determine the generators of a multidimensional cyclic code of arbitrary length over $\aleph$.

Let $\mathcal{C}'$ be a 3D cyclic code of length $m_1m_2m_3$ over a finite chain ring $\aleph$. Then, $\mathcal{C}'$ can be viewed as an ideal of the ring $\aleph[x,\ y,\ z]/\langle x^{m_1}-1,\ y^{m_1}-1, z^{m_3}-1\rangle$ which can be easily seen to be isomorphic to $\aleph[x,\ y]/\langle x^{m_1}-1,\ y^{m_2}-1\rangle/\langle z^{m_3}-1\rangle$. Let $f(x,y,z)\in\mathcal{C}'$ be any element. Then, $f(x,y,z)$ can be uniquely written as $\sum_{k=0}^{m_3-1} f_k(x,y)z^k$, where $f_k(x,y)\in\aleph[x,y]/\langle x^{m_1}-1, y^{m_2}-1\rangle$ for $0\le k\le m_3-1$. Define the sets, $I'_k=\Big\{g_k(x,y)\in\aleph[x,y]/\langle x^{m_1}-1,\ y^{m_2}-1\rangle\mid \exists\ g(x,y,z)\in\mathcal{C}'$ such that $g(x,y,z)=\sum_{t=0}^{m_3-1-k} g_{m_3-1-t}(x,y)z^t\Big\}$, $0\le k\le m_3-1$. It can be easily verified that each I'_k, $0\le k\le m_3-1$, is an ideal of the ring $\aleph[x,y]/\langle x^{m_1}-1,\ y^{m_2}-1\rangle$ and therefore a cyclic code of length m_1m_2 over $\aleph$. By Theorem 3.1, I'_k is generated by a set $\{P_i^{(j,k)}(x,y)\mid 0\le i\le r_{j,k},\ 0\le j\le m_2-1\}$.

Theorem 4.1. *Let $\mathcal{C}'$ be a 3D cyclic code of length $m_1m_2m_3$ over $\aleph$. Then the set $\{\mathfrak{P}_i^{(j,k)}(x,y,z)\in\aleph[x,y,z]\mid 0\le i\le r_{j,k},\ 0\le j\le m_2-1,\ 0\le k\le m_3-1\}$ generates $\mathcal{C}'$, where*

$$\mathfrak{P}_i^{(j,k)}(x,y,z)=\sum_{t=0}^{m_3-1-k} a_{it}^{(j,k)}(x,y)z^t,\ \textit{for all}\ \ 0\le k\le m_3-1$$

and $a_{it}^{(j,k)}(x,y)\in I'_k$ for every $0\le t\le m_3-1-k$. Also, $a_{i(m_3-1-k)}^{(j,k)}(x,y)=P_i^{(j,k)}(x,y)$, $0\le i\le r_{j,k},\ 0\le j\le m_2-1,\ 0\le k\le m_3-1$.

Proof. Let $\mathcal{C}'$ be a 3D cyclic code of length $m_1m_2m_3$ over $\aleph$. Let $f(x,y,z)=\sum_{k=0}^{m_3-1} f_k(x,y)z^k$ with $f_k(x,y)\in\aleph[x,y]/\langle x^{m_1}-1,y^{m_2}-1\rangle$, $0\le k\le m_3-1$, be any element of $\mathcal{C}'$. Clearly, $f_{m_3-1}(x)\in I'_0$. Therefore, $f_{m_3-1}(x)=\sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,0}} P_i^{(j,0)}(x,y)t_i^{(j,0)}(x,y)$, where $t_i^{(j,0)}(x,y)\in\aleph[x,y]/\langle x^{m_1}-1,y^{m_2}-1\rangle$ for all $0\le i\le r_{j,0},\ 0\le j\le m_2-1$. Also, $P_i^{(j,0)}(x,y)\in I'_0$; $0\le i\le r_{j,0},\ 0\le j\le m_2-1$, therefore by definition of I'_0, there exists $\mathfrak{P}_i^{(j,0)}(x,y,z)\in\mathcal{C}'$ such that $\mathfrak{P}_i^{(j,0)}(x,y,z)=\sum_{t=0}^{m_3-1} a_{it}^{(j,0)}(x,y)z^t$, where $a_{i(m_3-1)}^{(j,0)}(x,y)=P_i^{(j,0)}(x,y)$, $0\le i\le r_{j,0},\ 0\le j\le m_2-1$. Since $\mathcal{C}'$ is an ideal of $\aleph[x,y,z]/\langle x^{m_1}-1,y^{m_2}-1,z^{m_3}-1\rangle$, $z^k\mathfrak{P}_i^{(j,0)}(x,y,z)\in\mathcal{C}'$. Therefore,

$a_{it}^{(j,0)}(x,y) \in I_0'$ for all $0 \le t \le m_3-1,\ 0 \le i \le r_{j,0},\ 0 \le j \le m_2-1$. Let

$$h_1(x,y,z) = f(x,y,z) - \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,0}} \mathfrak{p}_i^{(j,0)}(x,y,z)t_i^{(j,0)}(x,y) \tag{4.1}$$
$$= \sum_{t=0}^{m_3-2} f_t(x,y)z^t - \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,0}} t_i^{(j,0)}(x,y)\Big(\sum_{t=0}^{m_3-2} a_{it}^{(j,0)}(x,y)z^t\Big)$$
$$= \sum_{t=0}^{m_3-2} h_t^{(j,1)}(x,y)z^t;\ h_t^{(j,1)}(x,y) \in \mathfrak{R}[x,y]/\langle x^{m_1}-1, y^{m_2}-1\rangle.$$

Clearly, $h_1(x,y,z) \in \mathcal{C}'$ since $f(x,y,z), \mathfrak{p}_i^{(j,0)}(x,y,z) \in \mathcal{C}'$ for every $0 \le i \le r_{j,0},\ 0 \le j \le m_2-1$. Also, by definition of I_1', $h_{m_3-2}^{(j,1)}(x,y) \in I_1'$. So, $h_{m_3-2}^{(j,1)}(x,y) = \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,1}} P_i^{(j,1)}(x,y)t_i^{(j,1)}(x,y)$, where $t_i^{(j,1)}(x,y) \in \mathfrak{R}[x,y]/\langle x^{m_1}-1, y^{m_2-1}\rangle$, for all $0 \le i \le r_{j,1},\ 0 \le j \le m_2-1$. Now, $P_i^{(j,1)}(x,y) \in I_1';\ 0 \le i \le r_{j,1},\ 0 \le j \le m_2-1$, therefore there exists $\mathfrak{p}_i^{(j,1)}(x,y,z) \in \mathcal{C}'$ such that $\mathfrak{p}_i^{(j,1)}(x,y,z) = \sum_{t=0}^{m_3-2} a_{it}^{(j,1)}(x,y)z^t$, where $a_{i(m_3-2)}^{(j,1)}(x,y) = P_i^{(j,1)}(x,y),\ 0 \le i \le r_{j,1},\ 0 \le j \le m_2-1$. Also, since $\mathcal{C}'$ is an ideal of $\mathfrak{R}[x,\ y,\ z]/\langle x^{m_1}-1,\ y^{m_1}-1, z^{m_3}-1\rangle$, $z^k\mathfrak{p}_i^{(j,1)}(x,y,z) \in \mathcal{C}'$. Therefore $a_{it}^{(j,1)}(x,y) \in I_1'$ for all $0 \le t \le m_3-2,\ 0 \le i \le r_{j,1},\ 0 \le j \le m_2-1$. Let

$$h_2(x,y,z) = h_1(x,y,z) - \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,1}} \mathfrak{p}_i^{(j,1)}(x,y,z)t_i^{(j,1)}(x,y) \tag{4.2}$$
$$= \sum_{t=0}^{m_3-3} h_t^{(j,1)}(x,y)z^t - \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,1}} t_i^{(j,1)}(x,y)\Big(\sum_{t=0}^{m_3-3} a_{it}^{(j,1)}(x,y)z^t\Big)$$
$$= \sum_{t=0}^{n-3} h_t^{(j,2)}(x,y)z^t;\ h_t^{(j,2)}(x,y) \in \mathfrak{R}[x,y]/\langle x^{m_1}-1, y^{m_2-1}\rangle.$$

Clearly, $h_2(x,y,z) \in \mathcal{C}'$ since $h_1(x,y,z),\ \mathfrak{p}_i^{(j,1)}(x,y,z) \in \mathcal{C}';\ 0 \le i \le r_{j,1},\ 0 \le j \le m_2-1$. Continuing in this fashion, we obtain polynomials $h_3(x,y,z)$, $h_4(x,y,z), \ldots, h_{m_3-1}(x,y,z)$, where

$$h_{m_3-1}(x,y,z) = h_{m_3-2}(x,y,z) - \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,m_3-2}} \mathfrak{p}_i^{(j,m_3-2)}(x,y,z)t_i^{(j,m_3-2)}(x,y)$$
$$= h_0^{(j,m_3-2)}(x,y). \tag{4.m_3-1}$$

Clearly, by definition of I_{m_3-1}', $h_0^{(j,m_3-2)}(x,y) \in I_{n-1}'$, therefore $h_0^{(j,m_3-2)}(x,y) = \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,m_3-1}} P_i^{(j,m_3-1)}(x,y)t_i^{(j,m_3-1)}(x,y)$, where $t_i^{(j,m_3-1)}(x,y) \in \mathfrak{R}[x,y]/\langle x^{m_1}-1, y^{m_2}-$

$1\rangle$; $0 \le i \le r_{j,m_3-1}$. Now $P_i^{(j,m_3-1)}(x,y) \in I'_{n-1}$, therefore there exist $\mathfrak{p}_i^{(j,m_3-1)}(x,y,z) \in \mathcal{C}'$ such that $\mathfrak{p}_i^{(j,m_3-1)}(x,y,z) = P_i^{(j,m_3-1)}(x,y)$ Thus, we have

$$h_{m_3-1}(x,y,z) = h_0^{(j,m_3-2)}(x,y) = \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,m_3-1}} \mathfrak{p}_i^{(j,m_3-1)}(x,y,z)t_i^{(j,m_3-1)}(x). \quad (4.m_3)$$

It follows from equations (4.1) to $(4.m_3)$ that

$$\begin{aligned} f(x,y,z) &= \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,0}} \mathfrak{p}_i^{(j,0)}(x,y,z)t_i^{(j,0)}(x,y) + \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,1}} \mathfrak{p}_i^{(j,1)}(x,y,z)t_i^{(j,1)}(x,y) \\ &\quad + \cdots + \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,m_3-2}} \mathfrak{p}_i^{(j,m_3-2)}(x,y,z)t_i^{(j,m_3-2)}(x,y) \\ &\quad + \sum_{j=0}^{m_2-1}\sum_{i=0}^{r_{j,m_3-1}} \mathfrak{p}_i^{(j,m_3-1)}(x,y,z)t_i^{(j,m_3-1)}(x,y). \end{aligned}$$

Thus, the set $\{\mathfrak{p}_i^{(j,k)}(x,y,z) \in \mathfrak{R}[x,y,z] \mid 0 \le i \le r_{j,k},\ 0 \le j \le m_2-1,\ 0 \le k \le m_3-1\}$ generates $\mathcal{C}'$, where

$$\mathfrak{p}_i^{(j,k)}(x,y,z) = \sum_{t=0}^{m_3-1-k} a_{it}^{(j,k)}(x,y)z^t, \text{ for all } 0 \le k \le m_3-1$$

and $a_{it}^{(j,k)}(x,y) \in I'_k$ for every $0 \le t \le m_3-1-k$. Also, $a_{i(m_3-1-k)}^{(j,k)}(x,y) = P_i^{(j,k)}(x,y)$, $0 \le i \le r_{j,k}$, $0 \le j \le m_2-1$, $0 \le k \le m_3-1$. □

The following result directly follows from Theorem 4.1 and Corollary 3.2.

Corollary 4.2. *The maximum number of generators for a 3D cyclic code $\mathcal{C}'$ of length $m_1m_2m_3$ over $\mathfrak{R}$ is km_2m_3, where $k = \sum_{j=0}^{n-1}(r_j+1)$.*

Remark 4.3. *Clearly, an nD cyclic code viewed as an ideal of $R_n = \mathfrak{R}[x_1,x_2,\dots,x_n]/\langle x_1^{m_1}-1, x_2^{m_2}-1,\dots,x_n^{m_n}-1\rangle$ over a finite chain ring $\mathfrak{R}$. Therefore, continuing in the same fashion as above, suppose that we have obtained the generators of ideals of R_{n-1}. Using the generators of ideals of R_{n-1} and the fact that $R_n \cong R_{n-1}[x_n]/\langle x_n^{m_n}-1\rangle$, the generators of ideals of R_n for any natural number n can be obtained.*

Conclusion

This paper presents a systematic framework for constructing generators of multidimensional (nD) cyclic codes of arbitrary length over a finite chain ring. Specifically, using the generators of 1D cyclic codes, we obtain the generators for two-dimensional (2D) cyclic codes and use them as a foundation to determine the generators for three-dimensional (3D) cyclic codes. This process can similarly be extended to higher dimensions, offering a scalable methodology for constructing generators of nD cyclic codes based on those of $(n-1)$D cyclic codes. The inclusion of illustrative examples highlights the practical significance and utility of the proposed methods to obtain some optimal codes.

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