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A SHORT NOTE ON THE KOMLÓS - SÓS CONJECTURE.

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Abstract. For positive integers s and k such that $s \geq 2$ and $k > 120(s-1)$, a necessary and sufficient condition for a graph G without $K_{2,s}$ and with a median degree of at least k is given to contain every tree T of order $k+1$. A special case of the Komlós- Sós conjecture that states the following: *Let $k \in N$. If at least half the vertices of a graph G have degree at least k, then G contains as subgraphs, all trees of size $k+1$* is then verified.

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1. Introduction

The famous Komlós- Sós conjecture states that

Let $k \in N$. If at least half the vertices of a graph G have degree at least k, then G contains as subgraphs all trees of size $k+1$.

This is analogous to the well known Erdős - Sós conjecture [9] that states that

Every graph of average degree at least $k-1$ contains every

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tree of order $k+1$.
While approximate versions of both conjectures were proved by Ajtai et al. [1] using the Regularity Lemma, the Erdős - Sós Conjecture has been verified for several classes of graphs. In addition, papers by Fan [10], Fan and Sun [12], Fan and Huo [11] verified the conjecture for Spiders of large size, Spiders and Spiders with four legs respectively. Dobson [8] proved both conjectures for graphs whose complement has no $K_{2,s}$. Brandt and Dobson [5] showed that the Erdős-Sós conjecture holds for graphs with girth at least 5 and Haxell proved the conjecture for graphs with no $K_{2,s}$, where $s = \lfloor \frac{k}{18} \rfloor$. The author and Dobson [2] then showed that the Erdős-Sós conjecture is true for graphs with no $K_{2,s}$ for $k > 12(s-1)$.

Soffer [19] showed that the Komlós- Sós Conjecture holds for graphs with girth at least 7 while Piguet and Stein [18] showed that the Komlós- Sós Conjecture holds for trees of diameter 5 and for certain Caterpillars. Cooley et.al. [7] and Hladký and Piguet [17] verified the above conjecture for dense graphs. Furthermore, in a series of four papers [13], [14], [15] and [16], Hladký, Komlós, Piguet, Szemerédi, Simonovitz and Stein proved an approximate version of the above conjecture for sparse graphs using an analogue of Szemerédi's regularity lemma.

In this paper we use and or extend the ideas similar to those found in the author and Dobson's paper [2] to show that for $s \geq 2$, $k > 120(s-1)$, every graph with median degree at least k, having no $K_{2,s}$ contains every tree of order $k+1$. In the remainder of the paper, we will look at relevant preliminaries and results based on the limits to the maximum degree of the given tree T.

2. Preliminaries and Results

The terms defined below are relatively standard and for the rest of the terms that are not defined here, refer to [4] and [6]. A complete bipartite graph $K_{m,n}$ is a graph whose $m+n$ vertices are divided into two sets of vertices of size m and n respectively such that no two vertices in the same set are adjacent while every pair of vertices one from each of the two sets are adjacent. For a graph G and $x \in V(G)$, $N_G(x)$ will denote the set of all neighbors of x in G. Let the highest, second highest, and the smallest degree of a graph G be denoted by $\Delta(G)$, $\Delta_2(G)$ and $\delta(G)$, respectively. Let $t(G)$ and $\mu(G)$ denote the average degree and the median degree of G.

Let $s > 2$ and $k > 120(s-1)$. Let G be a graph containing no $K_{2,s}$ that is minimal with respect to the property that $\mu(G) \geq k$. Let $L = \{x \in V(G) \colon \deg_G(x) \geq k\}$ and $S = V(G) - L$. By the minimality of G with respect to

$$\mu(G) \geq k, \tag{1}$$

S is an independent set. Also by the minimality of G with respect to (1), if $\deg_G(x) > k$, then $N_G(x) \subset L$.

In this paper, we will assume that since G contains no $K_{2,s}$, for distinct vertices $x, y \in V(G)$, $|N_G(x) \cap N_G(y)| \leq s-1$ and this will be used throughout.

We need the following Lemma [2]

Lemma 2.1. *Let T be any tree of order $k+1$. Then the length of any path joining two distinct non-leaf vertices is at most $k - \Delta(T) - \Delta_2(T) + 2$.*

We will also assume the following and fairly easy lemma that can be proved using the greedy algorithm,

Lemma 2.2. *Let G be a graph with $\delta(G) \geq k$ and $u \in V(G)$. Let T be a tree of order $k+1$ with $v \in V(T)$. Then there exists an inclusion $f: T \to G$ such that $f(v) = u$.*

In addition, we shall be in need of the following lemma [19] as well.

Lemma 2.3. *Let $k, n \in \mathbb{N}$ and let G be a graph with n vertices satisfying $\mu(G) \geq k$. Then G contains a subgraph H with $t(G) \geq \frac{2k}{3}$ and $\delta(H) \geq \frac{k}{3}$.*

The following Lemma [19] on bipartite graphs will be useful.

Lemma 2.4. *Let H be a graph with $\delta(H) > l$ and let S be an independent set of vertices in H. Then; either H has a path of length l with both endpoints outside S or H is bipartite with one color-class S.*

Clearly, any graph G satisfying (1) contains every star of order $k+1$. Hence, for all our purposes here, T is not a star of order $k+1$.

Lemma 2.5. *Let $s \geq 2$, $k > 18(s-1)$. Let G be a graph that is minimal with respect to (1) such that G contains no $K_{2,s}$ and $\delta(G) > \frac{k}{3}$. Let P be any maximal path of length at most k in G connecting vertices $u, v \in G$. Then any vertex $w \in L$ that is on P has at least 9 neighbors in G that are not on P.*

Proof. Let s, k and G be defined as above. Let P be a maximal path of length at most k in G connecting distinct vertices $u, v \in G$. As $\deg_G(x) \geq k$ for every $x \in L$, the maximality of P in conjunction with $|P| \leq k$ implies that both u and v are in S. Hence, all of the at least $\delta(G) \geq \frac{k}{3} + 1$ neighbors of u and v respectively are on P. As S is an independent set, there exists a vertex $w \in L$ that is on the path P. Now, if at most 8 neighbors of w are on P, then at least $\deg_G(w) - 8 \geq k - 8$ neighbors of w are on P. As G contains no $K_{2,s}$, the number of vertices on P is then, at least

$$\begin{aligned} k-8+k/3+1+k/3+1-3(s-1) &= k+2k/3-6-3(s-1) \\ &\geq k+2(18(s-1))/3-6-3(s-1) \\ &= k+9(s-1)-6 \\ &\geq k+1, \end{aligned}$$

a contradiction to $|P| \leq k$. Hence w has at lest 9 neighbors in G that are not on P. □

Lemma 2.6. *Let $s \geq 2$, $k > 18(s-1)$. Let G be a graph that is minimal with respect to* (1) *such that G contains no $K_{2,s}$ and $\delta(G) > \frac{k}{3}$. Then G contains a path of length k.*

Proof. Let s, k and G be defined as above. Suppose every path in G has length at most $k-1$. Let P be such a path joining distinct vertices u, v that is of maximal length in G.

Then, as P is a maximal path, all the neighbors of u and v in G are on P. If either u or v are in L, then, as G contains no $K_{2,s}$, the number of vertices on P are at least $k+\delta(G)-(s-1) \geq k+k/3-(s-1) \geq k+18(s-1)/3-(s-1) = k+5(s-1) \geq k+1$, a contradiction to $|P| \leq k-1$. Hence both u and v are in S. Since S is an independent set, there exists an $x \in L$, such that x is on the uv path P. Since $u, v \in S$, assume without loss of generality, that $x \in N_G(v)$ on P. By Lemma 2.5, x has at least 9 neighbors in G that lie outside P. Let x_1, x_2, be any two neighbors of x in G that are not on P. If both x_1 and x_2 have all their $\deg_G(x_i)$, $i = 1, 2$ neighbors on P, then as G contains no $K_{2,s}$ and $\delta(G) \geq k/3$, the number of vertices on P is at least $\deg_G(u) + \deg_G(v) + \deg_G(x_1) + \deg_G(x_2) - 6(s-1) \geq k/3+1+k/3+1+k/3+1+k/3+1-6(s-1) \geq k+18(s-1)/3+4-6(s-1) \geq k+4 \geq k$, a contradiction to $|P| \leq k-1$. Hence at least one of x_1, x_2 has a neighbor y_1 in G that is not on P. Assuming that x_1 is the vertex, the path $P_1 = P - wx + xx_1 + x_1y_1$ is of length $|P_1| = |P| - 1 + 1 + 1 > |P|$, contradicting the maximality of P. Hence G contains a path of length k. □

Lemma 2.7. *Let $s \geq 2$, $k > 21(s-1)$. Let G be a graph that is minimal with respect to* (1) *such that G contains no $K_{2,s}$ and $\delta(G) > \frac{k}{3}$. Then G contains a path of length k joining two vertices in L or a path of length k joining a vertex in L and a vertex in S.*

Proof. Let s, k and G be defined as above. Then by Lemma 2.6, G contains a path of length k. If G contains a path of length k joining distinct vertices $u \in L$, $v \in L$, or a vertex $u \in L$ and a vertex $v \in S$, then we are done.
Otherwise, let P be a maximal path of length k joining distinct vertices $u, v \in S$. By the maximality of P, all of the at least $\delta(G)$ neighbors of u and v are on P. Let $u_1 \in N_G(u)$ and $v_1 \in N_G(v)$ be adjacent to u and v respectively on P. Note that as $|P| = k > 18(s-1) \geq 18$, $u_1 \neq v_1$. Also, as S is an independent set, u_1, $v_1 \in L$. Hence, by Lemma 2.5, u_1 and v_1 have at least nine neighbors in G that lie outside P. Let x_i, $i = 1, \ldots, 9$, be the nine neighbors of v_1 that lie outside P. If any one of the x_i, $i = 1, \ldots, 9$, say x_1 is in L, then the path $P_1 = P - vv_1 + v_1x_1$ is a path of length $|P_1| = |P| - 1 + 1 = k$ joining vertices $u \in S$, $x_1 \in L$, and we are done. Otherwise, if each of these 9 neighbors have all of their at least $\delta(G)$ neighbors on P, then, as G contains no $K_{2,s}$, G has at least $2k/3 + 9k/3 - 55(s-1) + 1 = k + 8k/3 - 55(s-1) \geq k + 8(21(s-1))/3 - 55(s-1) + 1 \geq k + 1$, a contradiction to $|P| \leq k$. Therefore, there exists a vertex x_1 (say) that has at least one vertex $y_1 \in N_G(x_1)$, $y_1 \neq u$, $y_1 \neq v$ that lies outside P. Hence the path $P_1 = P - uu_1 - vv_1 + v_1x_1 + x_1y_1$ of length k joining $u_1 \in L$ and $y_1 \in L$ or S and we are done. □

Lemma 2.8. *Let $s \geq 2$, $k > 18(s-1)$. Let G be a graph containing no $K_{2,s}$ such that $\delta(G) > \frac{k}{3}$. Let T be a tree with $|V(T)| \leq k$. Suppose $f : T \to H$ where $H \subset G$ is a graph such that $\delta(H) \geq \frac{k}{3}$, is an inclusion. Then at most three vertices in $V(f(T))$ can have all of their at least $\frac{k}{3}$ neighbors in $V(f(T))$.*

Proof. Suppose there are at least 4 distinct vertices w, x, y, z in $V(T)$ such that $f(w)$, $f(x)$, $f(y)$ and $f(z)$ have all of their at least $\frac{k}{3}$ neighbors in $V(f(T))$, Then, as G contains no $K_{2,s}$, the number of vertices in $V(f(T))$ is at least $\frac{k}{3}+\frac{k}{3}+\frac{k}{3}+\frac{k}{3}-6(s-1) = k+\frac{k}{3}-6(s-1) > k$, a contradiction to the cardinality of $V(f(T))$. Thus, at most three distinct vertices in $V(f(T))$ can have all of their at least $\frac{k}{3}$ neighbors in $V(f(T))$. ◻

Lemma 2.9. *Let G be a graph that is minimal with respect to* (1) *and $\delta(G) > k/3$. Suppose T is a tree of order $k+1$ such that $\Delta(T) > \frac{k}{3}$ and $\Delta_2(T) \geq (s+1)$. Suppose $f : T \to G$ is an inclusion. Then at most one vertex in $V(f(T))$ can have all of its neighbors in any path joining two non leaf vertices in $V(f(T))$.*

Proof. Let G, T and f be defined as above. Suppose at least two vertices have all of its neighbors on any path joining two non leaf vertices in $V(f(T))$. Then, as G has no $K_{2,s}$ and $\delta(G) \geq \frac{k}{3}$, the number of vertices on the path is at least $\frac{2k}{3}-(s-1)$. As the number of vertices on any such path is at most $k-\Delta(T)-\Delta_2(T)+2$ (by Lemma 2.1), we then have

$$\frac{2k}{3}-(s-1) \leq k-\Delta(T)-\Delta_2(T)+2$$

$$\Delta(T)+\Delta_2(T) \leq \frac{k}{3}+(s-1)+2,$$

a contradiction. ◻

Lemma 2.10. *Let $s \geq 2$, $k > 36(s-1)$. Let G be a graph containing no $K_{2,s}$ that is minimal with respect to* (1) *and $\delta(G) \geq \frac{k}{3}$. Let T be a tree of order $k+1$ such that $\Delta(T) \geq \frac{k}{2}$. Then G contains every tree of order $k+1$.*

Proof. Let G, T be defined as above. As $\mu(G) \geq k$, by Lemma 2.3, G contains every tree of order $\frac{k}{3}$. As G is minimal with respect to (1), G contains every star of order $k+1$ and by Lemma 2.7 a path of length k joining vertices in L or vertices in L and S respectively as well.
Hence, let $T' \subset T$ be a maximal subtree with $\Delta(T') = \Delta(T)$ such that there exists an inclusion $f\colon T' \to G$ with $f(x) \in L$, where $\deg_{T'}(x) = \Delta(T') = \Delta(T)$.

Assume without loss of generality that $T' \neq T$. Let $w \in V(T')$ such that $\deg_{T'}(w) < \deg_T(w)$. As T' is maximal, all of the $\deg_G(f(w))$ neighbors of $f(w)$ in G must be contained in $V(f(T'))$. Choose $y \in V(f(T'))$ such that $yf(w) \in E(G)$ and $dist_{f(T')}(f(x), y)$ is maximal. Let C be the component of $f(T') - y$ that contains $f(w)$. By choice of y, all of the $\deg_G(f(w)) - 1$ neighbors of $f(w)$ in G except y are contained in C. Let $f(w_1) \in N_{f(T')}(y)$ be such that $f(w_1)$ lies on the unique $yf(w)$ path in $f(T')$ (observe that $f(w_1) \in V(C)$).

Let $C_1 = f^{-1}(C) + w_1 f^{-1}(y)$. Then there exists an inclusion $g\colon C_1 \to G$ such that $g(u) = f(u)$ for every $u \in C_1$, $\deg_{C_1}(w) = \deg_C(w) - 1$ and $g(C_1)$ contains all of the $\deg_G(g(w)) = \deg_G(f(w))$ neighbors in G.

Let T'' be a maximal subtree of T contained in G such that $C_1 \subset T''$. if $T'' \neq T$, then there exists a $w' \in V(T'')$, such that every neighbor of w' is a vertex of T''. As every neighbor of $f(w)$ is also contained in T'' and $f(x)$ has $\frac{k}{2}$ neighbors in T'', we must have that

$$\frac{k}{3} + \frac{k}{3} + \frac{k}{2} - 6(s-1) \leq k,$$

or, equivalently $k \leq 36(s-1)$, a contradiction.Hence G contains the required tree T.

□

Lemma 2.11. *Let $s \geq 2$, $k > 120(s-1)$. Let G be a graph containing no $K_{2,s}$ that is minimal with respect to* (1) *and $\delta(G) \geq \frac{k}{3}$. Let T be a tree of order $k+1$ such that $\frac{k}{3} \leq \Delta(T) \leq \frac{k}{2}$ and $\Delta_2(T) \leq \frac{k}{4}$. Then G contains every tree of order $k+1$.*

Proof. Let G, T be defined as above. As G is minimal with respect to (1), G contains every star of order $k+1$ and by Lemma 2.7, a path of length k joining vertices in L or vertices in L and S respectively as well. Hence let $T' \subseteq T$ be a maximal subtree such that there exists an inclusion $f: T' \to G$. Assume without loss of generality that $T' \neq T$. Let $w \in V(T')$ such that $\deg_{T'}(w) < \deg_T(w)$. As T' is maximal, all of the $\deg_G(f(w))$ neighbors of $f(w)$ in G must be contained in $V(f(T'))$. Choose $y \in V(f(T'))$ such that $yf(w) \in E(G)$ and $dist_{f(T')}(f(w), y)$ is maximal. Let C be the component of $f(T')-y$ that contains $f(w)$. By choice of y, all of the $\deg_G(f(w))-1$ neighbors of $f(w)$ in G except y are contained in C. Let $f(w_1) \in N_{f(T')}(y)$ such that $f(w_1)$ lies on the unique $yf(w)$ path in $f(T')$ (note that $f(w_1) \in V(C)$). Note that by the maximality of T', $f(w) \in S$ which then implies that $y \in L$. Let $C_1 = f^{-1}(C) + wf^{-1}(y)$. Then there exists an inclusion $g: C_1 \to G$ such that $g(u) = f(u)$ for every $u \in C_1$, $\deg_{C_1}(w_1) = \deg_C(w_1) - 1$ and $g(C_1)$ contains all of the $\deg_G(g(w)) = \deg_G(f(w))$ neighbors in G. Furthermore, $\deg_{C_1}(w) = \deg_{T'}(w) + 1$. Let $T_1 \subset T$ be a maximal subtree such that $C_1 \subset T_1$ and there exists an inclusion $g_1: T_1 \to G$ such that $g_1(u) = g(u)$ for every $u \in C_1$ and $g_1(x) \in L$, where $\deg_{T_1}(x) = \Delta(T_1) = \Delta(T)$. Suppose there exists $z_1 \in \{V(T_1) - V(C_1)\} \cup \{w_1\}$ such that $\deg_{T_1}(z_1) < \deg_T(z_1)$. If there exists a $y_1 \in N_G(g_1(z_1)) - N_{g_1(T_1)}(g_1(z_1))$, then the tree $g_1(T_1)+g_1(z_1)y_1$ is a subgraph of G containing $g_1(T_1)$ and is isomorphic to a subtree of T containing C_1 larger than T_1, contradicting the maximality of T_1. Hence every neighbor in G of $g_1(z_1)$ is contained in $V(g_1(T_1))$. Also by the maximality of T_1, $g(z_1) \notin L$, and hence $g_(z_1) \neq g_1(x)$. Let $z_2 \in V(T_1)$ be such that $g_1(z_2) \in N_{g_1(T_1)}(z_1)$ is on the unique $g_1(z_1)g_1(w)$ path. Let C' be the component of $g_1(T_1 - z_1)$ that contains $g_1(w)$. Note that by the choice of z_2, all the neighbors of $g_1(w) = g(w)$ and $g_1(x)$ are in C'. As $z_2 \in L$, by the maximality of T_1, there exists a neighbor $g_1(z_1')$ of $g_1(z_2)$ in G not in C'. Note that by the choice of C', all of the at least $\deg_G(g_1(z_1)) - \deg_{g_1(T_1)}(g_1(z_1)) + 1$ neighbors of $g_1(z_1)$ are in C'. Also, by the construction of C', $g_1(x)$ has all of it's $\Delta(T)$ neighbors in C'. Let $T_2 \subseteq T$ be a subtree containing C'' where, C'' is the component of $g_1(T_1)$ containing C' together with the component containing the neighbors of $g_1(z_1')$ not in C' such that there exists an inclusion $g_2\colon T_2 \to G$ with $g_2(u) = g_1(u)$ for $u \neq z_1'$ and $\deg_{g_2(T_2)}(g_2(z_1')) < \deg_{g_1(T_1)}(g_1(z_1))$. Note that all the neighbors of $g_2(z_1')$ in G are in C'' and hence in $g_2(T_2)$. As G contains no $K_{2,s}$, and $k > 120(s-1)$, the number of vertices in $V(T_2)$ is at least $\frac{k}{3} + \frac{k}{3} + \Delta(T) + \deg_G(g_2(z_1)) - \deg_{g_2(T_2}(g_2(z_2)) + 1 - 10(s-1) \geq k + 1 + \frac{k}{3} - \Delta_2(T_2) - 10(s-1) \geq k + 1 + \frac{k}{3} - \frac{k}{4} - 10(s-1) \geq k+1$, a contradiction to $|V(T_2)| \leq k$. Hence $\deg_{T_2}(g_2(z_1')) = \deg_{g_1(T_1)}(g_1(z_1))$ which then implies that we have constructed a subgraph in G isomorphic to a subtree $T_2 \subseteq T$ that contains T_1, a contradiction to the maximality of T_1. □

Lemma 2.12. *Let $s \geq 2$ and $k > 120(s-1)$. Let T be a tree of order $k+1$ such that $\Delta(T) < \frac{k}{3}$ and $\Delta_2(T) < \frac{k}{5}$. Let G be a graph that is minimal with respect to* (1) *such that $|V(G)| \geq$*

$k+1$, $\delta(G) \geq k/3$ and G contains no $K_{2,s}$. Then G contains a subgraph isomorphic to T.

Proof. For G and T defined above, since G is minimal with respect to (1), G contains every star of order $k+1$ and by Lemma 2.7, a path of length k joining vertices in L or vertices in L and S respectively as well. Hence let $T' \subseteq T$ be a maximal subtree such that there exists an inclusion $f: T' \to G$. Assume without loss of generality that $T' \neq T$. Let $w \in V(T')$ such that $\deg_{T'}(w) < \deg_T(w)$. As T' is maximal, all of the $\deg_G(f(w))$ neighbors of $f(w)$ in G must be contained in $V(f(T'))$. Choose $y \in V(f(T'))$ such that $yf(w) \in E(G)$ and $dist_{f(T')}(f(w), y)$ is maximal. Let C be the component of $f(T') - y$ that contains $f(w)$. By choice of y, all of the $\deg_G(f(w)) - 1$ neighbors of $f(w)$ in G except y are contained in C. Let $f(w_1) \in N_{f(T')}(y)$ such that $f(w_1)$ lies on the unique $yf(w)$ path in $f(T')$ (note that $f(w_1) \in V(C)$).

Let $C_1 = f^{-1}(C) + wf^{-1}(y)$. Then there exists an inclusion $g: C_1 \to G$ such that $g(u) = f(u)$ for every $u \in C_1$, $\deg_{C_1}(w_1) = \deg_C(w_1) - 1$ and $g(C_1)$ contains all of the $\deg_G(g(w)) = \deg_G(f(w))$ neighbors in G. Furthermore, $\deg_{C_1}(w) = \deg_{T'}(w) + 1$. Let $T_1 \subset T$ be a maximal subtree such that $C_1 \subset T_1$ and there exists an inclusion $g_1: T_1 \to G$ such that $g_1(u) = g(u)$ for every $u \in C_1$.

Suppose there exists $z_1 \in \{V(T_1) - V(C_1)\} \cup \{w_1\}$ such that $\deg_{T_1}(z_1) < \deg_T(z_1)$. If there exists a $y_1 \in N_G(g_1(z_1)) - N_{g_1(T_1)}(g_1(z_1))$, then the tree $g_1(T_1) + g_1(z_1)y_1$ is a subgraph of G containing $g_1(T_1)$ and is isomorphic to a subtree of T containing C_1 larger than T_1, contradicting the maximality of T_1. Hence every neighbor in G of $g_1(z_1)$ is contained in $V(g_1(T_1))$. Let C' be the component of $g_1(T_1 - z_1)$ that contains $g_1(w)$.

Suppose that $g_1(z_1)$ is adjacent to some vertex $a \in V(g_1(T_1)) - V(C') \subseteq V(G)$ that is not a neighbor of $g_1(z_1)$ in $g_1(T_1)$. Choose such an a to be of maximal distance in $g_1(T_1)$ from $g_1(z_1)$. Let T_2 be the subtree of $T_1 - g_1^{-1}(a)$ that contains z_1 (and hence w as well) and $g_2 : T_2 \to G$ be the inclusion induced by g_1. Note that $\deg_{T_2}(w) = \deg_{T_1}(w) = \deg_{T'}(w) + 1$ and $\deg_{T_2}(z_1) = \deg_{T_1}(z_1)$. By choice of a, every neighbor in G of $g_2(z_1) = g_1(z_1)$ is contained in $g_2(T_2)$ with the exception of a. Also, as a is not a neighbor of $g_1(w_1) = f(w_1)$ in $g_1(T_1)$, we have that every neighbor of $g_1(w_1)$ is also contained in T_2. Then the tree $g_2(T_2) + g_2(z_1)a$ is isomorphic to a subtree of T' contained in G. Let $T_3 \subseteq T$ be a maximal subtree of T such that there exists an inclusion $g_3 : T_3 \to G$ and $g_2(T_2) + g_2(z_1)a \subseteq g_3(T_3)$. Hence $\deg_{T_3}(w) = \deg_{T_1}(w) = \deg_{T'}(w) + 1$ and $\deg_{T_3}(z_1) > \deg_{T_1}(z_1)$. Then every neighbor of $g_3(z_1) = g_2(z_1)$ is contained in $g_3(T_3)$ and every neighbor of $g_3(w) = g_2(w)$ is also contained in $g_3(T_3)$. If there is an $x \in V(T_3)$ such that $\deg_{T_3}(x) < \deg_T(x)$, by the maximality of T_3, all of the $\deg_G(g_3(x))$ in $S \subset V(G)$ is in $g_3(T_3)$. Let $g_3(x_1) \in N_{g_3(T_3)}(g_3(x)$ lie on the unique $g_3(x)g_3(w)$ path. By the maximality of T_3, there exists an $x_2 \in V(T) - V(T_3)$, $x \neq x_2$ such that there exists a maximal subtree $T_3' \subseteq T$ with an inclusion $g_3'\colon T_3' \to G$ such that $g_3'(T_3')$ contains the component $g_3(T_3 - x_1x + x_2x)$. As G contains no $K_{2,s}$, $\deg_{g_3'(T_3')}(g_3'(w)) \geq \deg_{g_3(T_3)}(g_3(w) - (s-1))$ and $\deg_{g_3'(T_3')}(g_3'(z)) \geq \deg_{g_3(T_3)}(g_3(z) - (s-1))$.

Suppose $b \in T_3'$ such that $\deg_{T_3'}(b) < \deg_T(b)$. If $b = x_2$, then by the maximality of T_3', all of the $\deg_G(g_3'(x_2))$ neighbors of $g_3'(x_2)$ are in T_3'. Moreover all of the $\deg_G(g_3(x_1)) - \deg_{g_3(T_3)}(g_3(x_1)) + 1$ neighbors of $g_3(x_1)$ are in $g_3'(T_3')$. As G contains no $K_{2,s}$, the number of vertices in $g_3'(T_3')$ is at least $\deg_G(g_3(x_1)) - \deg_{g_3(T_3)}(g_3(x_1)) + 1 + \deg_G(g_3(w)) - (s-1) +$

$\deg_{g_3(T_3)}(g_3(z)) - (s-1) + \frac{k}{3} - 4(s-1) \geq k+1+\frac{k}{3} - 6(s-1) - \deg_{g_3(T_3)}(g_3(x_1)) \geq k+1+\frac{k}{3} - 6(s-1) - \Delta_2(T_3) \geq k+1+\frac{k}{3} - 6(s-1) - \frac{k}{5} \geq k+1$, a contradiction.

If $b \neq x_2$, then by the maximality of T_3', all the neighbors of $g_3'(b) \in S \subseteq V(G)$ are in $g_3'(T_3')$. As G contains no $K_{2,s}$, the number of vertices in $g_3'(T_3')$ is at least $\deg_G(g_3(w)) - (s-1) + \deg_{g_3(T_3)}(g_3(z)) - (s-1) + \frac{k}{3} + \deg_G(g_3(x_1)) - \deg_{g_3(T_3)}(g_3(x_1)) + 1 + \frac{k}{3} + \frac{k}{3} - 8(s-1) \geq k+1$, a contradiction.

Thus in either case, we arrive at a contradiction to $|V(T_3')| \leq k+1$ which then implies that we have constructed a maximal subtree $T_3' \subseteq T$ containing T_3 that is isomorphic to a subgraph of G contradicting the maximality of T_3.

□

Assume that in general, $T(k,l,c)$ is the class of all trees with k edges that is obtained from a path P of length $k-l$ and two stars S_1 and S_2 whose centers are at distance c from each other on P. We also assume that a caterpillar is a tree T where every vertex is of length at most 1 to some central path P.

For the rest of this paper any Caterpillar $T = C(a,b,c,d,e)$ is a tree connecting at most two vertices v_1, v_2 of degree at least 2 by a path P such that

(1) P has length $a+c+e$,
(2) v_1 and v_2 are the $(a+1)th$ and $(a+c+1)th$ vertex on P and have respectively b, d neighbors outside P.

Here, we let $T(k,l,c)$ denote the class of all Caterpillars $C(a,b,c,d,e)$ with $b+d = l$, and $a+b+c+d+e = k$. We need the following Theorems [3], [18] respectively regarding Caterpillars.

Theorem 2.13. *Let $k, l, c \in \mathbb{N}$, and let $T = C(a, 0, c, d, e)$ be a tree from $T(k, l, c)$. Let G be a graph that is minimal with respect to* (1)*. Then T is a subgraph of G.*

Clearly if T is a tree of order $k+1$ such that $\Delta(T) \leq \frac{k}{3}$, $\frac{k}{5} \leq \Delta_2(T) \leq \Delta(T)$ and $T(k, l, c)$ contains all caterpillars with stars S_1 and S_2 of orders $\Delta(T)$ and $\Delta_2(T)$ respectively, then, by the above theorem, any graph G that is minimal with respect to (1) contains T as a subgraph.

Theorem 2.14. *Let $k, l, c, n \in \mathbb{N}$ such that $l \geq c$. Let $T \in T(k, l, c)$, and let G be a graph of order n so that at least $\frac{n}{2}$ vertices of G have degree at least k. If c is even or $l + c \geq \lfloor n/2 \rfloor + 1$ (or both), then T is a subgraph of G.*

Conclusion

We now are in a position to provide a partial solution to the Loebl-Komlós-Sós Conjecture as follows. Combining Theorem 2.13 with Lemmas 2.11 and 2.10, we have:

Corollary 2.15. *Let $s \geq 2$ and $k > 120(s-1)$. If G is a graph with no $K_{2,s}$ that is minimal with respect to* (1)*, then G contains as subgraphs all trees of size $k+1$.*

References

[1] A. M. Ajtal, J. Komlós, E. Szemerédi, On a conjecture of Loebl,*Graph Theory, Combinatorics, and Applications in: Y. Alavi, A. Schwent* (*Eds.*), *Proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs*, (*on the occasion of Paul Erdős's 80th birthday*), (1992) 1135-1146.

[2] S. Balasubramanian, E. Dobson, Constructing Trees in Graphs with no $K_{2,s}$,*Journal of Graph Theory*, **56** (4) (2007) 301-310.

[3] C. Bazgan, H. Li, M. Wozniak, On the Loebl-Komlós-Sós conjecture, *Journal of Graph Theory*,**34** (4) (2000) 269-276.

[4] B. Bollobas, Modern Graph Theory, *Graduate Texts in Mathematics*, *Springer Series*, *New York* (1998) x+408.

[5] S. Brandt, E. Dobson, The Erdős-Sós conjecture for graphs of girth 5,*Discrete Mathematics* **150** (1-3) (1996) 411-414.

[6] G. Chartrand, L. Lesniak, Graphs and Digraphs, *Chapman and Hall CRC Press*, *Boca Raton*, *Florida*, *3rd Ed.* (2000).

[7] O. Cooley, J. Hladký, D. Piguet, Loebl-Komlós- Sós conjecture: dense case,*Electronic Notes in Discrete Mathematics* **34** (2009) 609-613.

[8] E. Dobson, Constructing trees in graphs whose complement has no $K_{2,s}$, *Combinatorics*, *Probability and Computing*,**11** (4) (2002) 343-347.

[9] P. Erdős, Extremal problems in graph theory,*M. Fiedler (Ed.)*, *Theory of Graphs and its Applications*, *Proceedings of the Symposium Smolenice*, (1965) 29-36.

[10] G. Fan, The Erdős-Sós conjecture for spiders of large size, *Discrete Mathematics*, (2013) 2513-2517.

[11] G. Fan, Z. Huo, Erdős-Sós conjecture for spiders of four legs, *J. Combin.*, **7** (2016) 271-283.

[12] G.Fan, L. Sun, The Erdős-Sós conjecture for spiders, *Discrete Mathematics*,**307** (2007) 3055-3062.

[13] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, E. Szemerédi, The approximate loebl–komlós–sós conjecture i:the sparse decomposition,*Siam Journal on Discrete Mathematics*,**31** (2) (2017) 845-982.

[14] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, E. Szemerédi, The approximate loebl– komlós–sós conjecture ii:the rough structure of lks graphs,*Siam Journal on Discrete Mathematics*,**31** (2) (2017) 983-1016.

[15] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, E. Szemerédi, The approximate loebl– komlós–sós conjecture iii:the finer structure of lks graphs,*Siam Journal on Discrete Mathematics*,**31** (2) (2017) 1017-1071.

[16] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, E. Szemerédi, The approximate loebl– komlós–sós conjecture iv: embedding techniques and the proof of the main result,*Siam Journal on Discrete Mathematics*,**31** (2) (2017) 1072-1148.

[17] J. Hladký, D. Piguet, Loebl-komlós-sós conjecture: dense case, *Journal of Combinatorial Theory, Series B*,**116** (2016) 123-190.

[18] D. Piguet, M.J. Stein, The Loebl-Komlós-Sós conjecture for trees of diameter 5 and certain caterpillars, *The Electronic Journal of Combinatorics*,**15** (1) (2008) 1-11.
[19] S.N. Soffer, The Komlós - Sós conjecture for graphs of girth 7,*Discrete Mathematics*, **214** (2000) 279-283.

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