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TOTAL CHROMATIC NUMBER FOR INDU-BALA PRODUCT AND SYMMETRIC DIFFERENCE PRODUCT OF GRAPHS

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Abstract. Total coloring of a graph is a well-studied concept in graph theory that extends the classical notion of vertex and edge coloring. The study of total coloring has important connections to several other areas of graph theory, including chromatic number, edge coloring, and the study of coloring algorithms. The Total Coloring Conjecture, which posits that the total chromatic number $\chi''(G)$ of a graph G is at most $\Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G, was independently proposed by Bezhad and Vizing. In this paper, we aim to prove the Total Coloring Conjecture for specific categories of the Indu-Bala product and the Symmetric difference product of graphs. Specifically, we aim to establish precise bounds for the total chromatic number within these classes, contributing to a better understanding of the conjecture's validity.

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1. Introduction

One of the main topics of graph theory research is graph coloring. There are a lot of challenging open problems and conjectures in graph coloring. Let G be a simple graph. Its vertices are collectively denoted by $V(G)$, and its edges by $E(G)$. The symbol $v(G)$ indicates the order of G, which is the cardinality $|V|$ of V and $e(G)$ represents the size of G, which is the cardinality $|E|$ of E. A total coloring of a graph G is a coloring assignment in which no two neighboring vertices, adjacent edges, or edges with their incident vertices are assigned the same color. The objective is to achieve

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a total coloring of G with the fewest possible colors. The symbol $\chi''(G)$ represents total chromatic number of G. This number represents the minimum number of colors required for a proper total coloring of the graph.

The study of total coloring is driven by real-world uses, like register allocation in compilers, frequency allocation in radio networks, and channel assignment in wireless communication networks. It also has theoretical significance, as it provides insights into the structural properties and combinatorial complexity of graphs.

One of the intriguing conjectures in graph theory is the Total Coloring Conjecture (TCC), proposed by Behzad [1] and Vizing [2]. This conjecture asserts that the total chromatic number of a graph is no greater than $\Delta(G) + 2$, with $\Delta(G)$ representing the maximum degree of the graph G. Graphs fall into two types based on their total coloring needs: Type I for those requiring just one more color than their maximum degree vertex ($\Delta(G)+1$), and Type II needing two more extra colors ($\Delta(G)+2$). Yap [3] provided a thorough overview, exploring deeply into the evolution and development of total coloring. Additionally, Barodin [4] provided an excellent overview of the total coloring of planar graphs. In 2022, Geetha et al. [5] gave a comprehensive and in-depth survey on the most current findings on total coloring, providing insightful information. Kostochka [6, 7] verified the total coloring conjecture for general graphs with $\Delta(G) \leq 5$ and Rosenfeld [8] and Vijayaditya [9] for $\Delta(G) = 3$. The total coloring problem, which asks that a given graph G have a total coloring with the fewest possible colors, is known to be NP-hard [10]. The problem of figuring out the total coloring of a μ-regular bipartite graph is NP-hard, $\mu \geq 3$, as demonstrated by McDiarmid and Arroyo [11].

Graph products are important classes of graphs. Many attempts have been made to confirm the total coloring conjecture for different graph products, but numerous questions still lack answers. There are four standard graph products, namely, *Cartesian product* ($G\Box H$), *direct product* ($G \times H$), *strong product* ($G \boxtimes H$) and *lexicographic product* ($G \circ H$). Zmasek and Zerovnik [12] demonstrated that TCC holds for the cartesian product $G\Box H$ if it holds for graphs G and H. Also, they demonstrated that the cartesian product is associated with the largest vertex degree factor of Type I. For specific values of m and n, Geetha and Somasundaram [13] shown that $C_m \times C_n$ is Type I, as is the direct product of even complete graphs. Castonguay and colleagues. [14, 15] identified $K_2 \times K_2$ as Type I, but any direct product of two complete graphs was classified as Type I. In order to prove that the three types of corona products (neighborhood, edge, and vertex) of graphs are Type I, Vignesh et al., resolved the conjecture for all corona product types. Sandhiya et al., confirmed TCC for a few lexicographic classes and deleted lexicographic products in [16–18]. TCC remains open for various graph classes.

Lemma 1.1. *[19] Let G be a graph with n vertices. If $\Delta(G) \geq \frac{3}{4}n$, then G is total colorable with $\Delta(G)+2$ colors.*

We will be using the above lemma to prove the theorems in the following sessions.

2. Indu-Bala Product

The graph with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{(i,j) : i \in V(G_1), j \in V(G_2)\}$ is the result of joining two graphs G_1 and G_2, represented by $G_1 \vee G_2$. The Indu-Bala product was introduced by Indulal and Balakrishnan [20]. The Indu-Bala product of two graphs G_1 and G_2 is obtained by taking two disjoint copies of $G_1 \vee G_2$ and joining the corresponding vertices in two copies of G_2. It is denoted by $G_1 \blacktriangledown G_2$. For example, Fig.1 shows the product $C_5 \blacktriangledown P_3$. The maximum degree of $G_1 \blacktriangledown G_2$ is $\Delta(G_1 \blacktriangledown G_2) = \max\{\Delta(G_1)+n_1, \Delta(G_2)+n_2+1\}$, where n_1 and n_2 represent the cardinality of vertex set of G_1 and G_2 respectively. The Indu-Bala product $G_1 \blacktriangledown G_2$ is not commutative. Recently, Somasundaram et al. [21] have discussed the total coloring of Indu-Bala product of graphs with degree constraints. They proved that for any total colorable graphs G and H, $G \blacktriangledown H$ is total colorable whenever $\Delta(G) > \Delta(H)$ and $n \geq m$. Also, $G \blacktriangledown H$ is total colorable for $\Delta(H) > \Delta(G)$ and $m \geq n$, where m and n represents the number of vertices in G and H respectively.

Theorem 2.1. *For any total colorable graphs G_1 and G_2, the Indu-Bala product, $G_1 \blacktriangledown G_2$ is type-I when the cardinality of the vertices of G_1 and G_2 are equal.*

Proof. Let the cardinality of the vertex set of G_1 and G_2 be n. The maximum degree, $\Delta(G_1 \blacktriangledown G_2) = \max\{\Delta(G_1)+n, \Delta(G_2)+n+1\}$. First, we try to assign colors to $G_1 \vee G_2$. Before going to the proof for the two cases, namely $\Delta(G_1) > \Delta(G_2)$ and $\Delta(G_1) \leq \Delta(G_2)$, we need

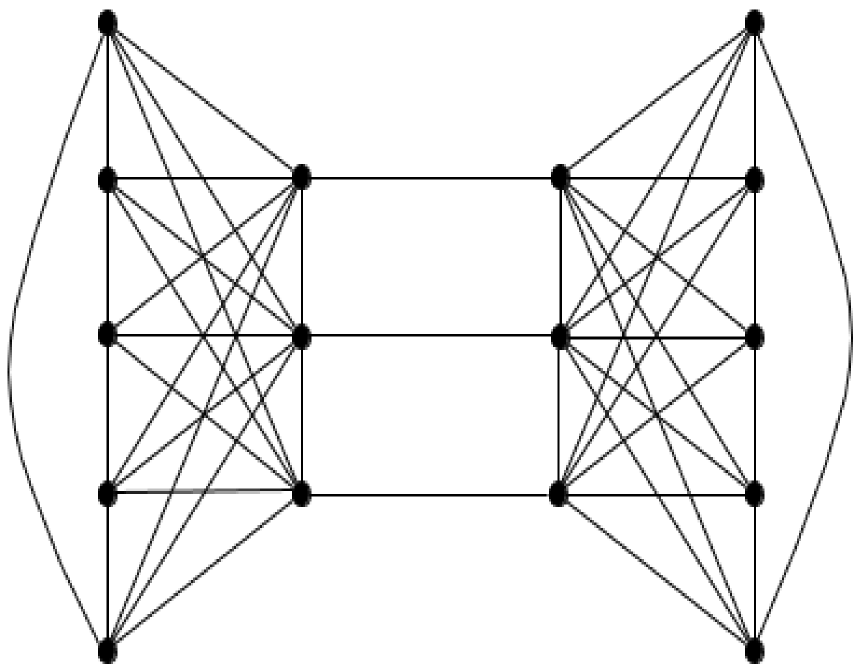

Figure 1. $C_5 \blacktriangledown P_3$.

to show the coloring assignment for $G_1 \vee G_1$.

Claim: $K_2 \circ G_1$ is total colorable.
The graph $K_2 \circ G_1 \cong G_1 \vee G_1$. Let us label the copies of G_1 by $G_1^{'}$ and $G_1^{''}$. The maximum degree, $\Delta(G_1 \vee G_1)$ is given by $\Delta(G_1) + n$, where n is the cardinality of the vertex set of G_1. Let $C_1 = \{1, 2, ..., \Delta(G_1) + 2\}$ and $C_2 = \{c_1, c_2, ..., c_n\}$ be the color sets. Assign a proper total coloring to $G_1^{'}$ using $\Delta(G_1) + 2$ colors from C_1. Moreover, we assign a set of n distinct colors from C_2 for the vertices of $G_1^{''}$, and color the edges of $G_1^{''}$ in accordance with the edge coloring pattern of $G_1^{'}$. This means that corresponding vertices in $G_1^{'}$ and $G_1^{''}$ have common missing colors from the set $\{1, 2, ..., \Delta(G)+2\}$. Within the graph $G_1 \vee G_1$ locate those edges that connect corresponding vertices between the two copies of G_1. This collection of newly colored edges will constitute a 1-factor of $G_1 \vee G_1$. Subsequently color these edges with the missing color at its corresponding vertices. Every vertex in $G_1^{''}$ has a set of $n - 1$ unused colors from C_2. The final step involves coloring the remaining

edges between $G_1^{'}$ and $G_1^{''}$ using the $n-1$ colors from C_2. Hence, the claim.

Case 1: $\Delta(G_1) > \Delta(G_2)$
We first consider $G_1 \vee G_2$. Since the maximum degree of $G_1 \vee G_2$ depends on $\Delta(G_1)$ and $\Delta(G_2)$, and $\Delta(G_1)$ is greater than $\Delta(G_2)$ in this case, the maximum degree of $G_1 \blacktriangledown G_2$ will be $\Delta(G_1) + n + 1$. Construct $G_1 \vee G_1$ from $G_1 \vee G_2$ by adding edges of G_1. We know that $G_1 \vee G_1 \cong K_2 \circ G_1$. Also, from the above claim, $K_2 \circ G_1$ is total colorable. Therefore, we use a set of $\Delta(G_1) + n + 2$ colors and apply the total coloring to $G_1 \vee G_1$. We now remove the additional edges to get back $G_1 \vee G_2$. Now copy the same total coloring to the other copy of $G_1 \vee G_2$. Since we have used $\Delta(G_1) + n + 2$ colors and also $\Delta(G_2) > \Delta(G_1)$, there will be a set of missing colors at each vertex of G_2 in $G_1 \vee G_2$. Take one of the missing colors at each vertex of G_2 to color the edges joining the corresponding vertices of G_2 between the copies of $G_1 \vee G_2$. Here we have used $\Delta(G_1)+n+2 = \Delta(G_1 \blacktriangledown G_2)+1$. Hence, we show that $G_1 \blacktriangledown G_2$ is Type I.

Case 2: $\Delta(G_1) \leq \Delta(G_2)$
The maximum degree $\Delta(G_1 \blacktriangledown G_2) = \Delta(G_2)+n+1$. Construct $G_2 \vee G_2$ from $G_1 \vee G_2$ by adding edges of G_2. We assign the total coloring to $G_2 \vee G_2$ as in the claim with $\Delta(G_2) + n + 2$ colors. We now remove the additional edges to get back $G_1 \vee G_2$. Since we have used $\Delta(G_2)+n+2$ colors and also $\Delta(G_2) > \Delta(G_1)$, at every vertex of G_2 in $G_1 \vee G_2$, there will be at least one color that is not assigned. Take the color which is not assigned at each vertex of G_2 to color the edges connecting the corresponding vertices of G_2 between the copies of $G_1 \vee G_2$. Hence, we prove $G_1 \blacktriangledown G_2$ is Type I. ◻

Corollary 2.2. *If G_1 is total colorable then $G_1 \blacktriangledown G_1$ is Type I.*

Theorem 2.3. *For any total colorable graphs G_1 and G_2, with $\Delta(G_1) = \Delta(G_2)$*

$$\chi''(G_1 \blacktriangledown G_2) \begin{cases} = \Delta(G_1 \blacktriangledown G_2) + 1, & \textit{if } n_1 \geq n_2 \\ \leq \Delta(G_1 \blacktriangledown G_2) + 2, & \textit{otherwise.} \end{cases}$$

Proof. Let the cardinality of vertices of G_1 be n_1 and the cardinality of vertices of G_2 be n_2. The maximum degree, $\Delta(G_1 \blacktriangledown G_2)$ depends on n_1 and n_2.

Case 1: when $n_1 \geq n_2$
Here, the maximum degree of $G_1 \blacktriangledown G_2$ is $\Delta(G_2)+n_1+1$. Assign a set of $\Delta(G_2)+2$ colors for a proper total coloring of G_1. Additionally, select a set of n_1 colors to color the edges connecting G_1 and G_2. Remove the colors of n_1 matching edges(each having a different color) and replace with the missing color at the corresponding vertex of G_1. Shift the colors of the chosen matching edges to the corresponding vertices of G_2. Since $\Delta(G_1) = \Delta(G_2)$, use the same set of $\Delta(G_2)+2$ colors to color the edges of G_2. Copy the same coloring assignment for the other copy of $G_1 \vee G_2$. In the above assignment of colors, at each vertex of G_2 in both copies, there will be at least one color that is missing. At each vertex, assign one of the unused colors to the edges that connect to the corresponding vertices in G_2.
Case 2: when $n_1 < n_2$
Here, $\Delta(G_1 \blacktriangledown G_2) = \Delta(G_1) + n_2$. Using a collection of $\Delta(G_1)+2$ colors properly color all the elements of G_1. Additionally, assign a different collection of n_2 colors to color the edges connecting G_1 and G_2. The edges of G_2 are colored using the same set of $\Delta(G_1)+2$ colors. There will be one or more missing colors from the set of n_2 colors at each vertex of G_2. Take the missing colors to color the vertices of G_2 properly. Also, there will be at least one missing color from $\Delta(G_1)+2$ colors at each vertex of G_2 in both copies of $G_1 \vee G_2$. Take the colors from $\Delta(G_1)+2$ colors to assign the color for the edges joining the corresponding vertices of G_2. Hence, $G_1 \blacktriangledown G_2$ is total colorable.

□

Corollary 2.4. *Let G_1 be a Type I graph and G_2 be any total colorable graph with n_1 and n_2 vertices respectively. When $\Delta(G_1) = \Delta(G_2)$ and $n_1 < n_2$, $G_1 \blacktriangledown G_2$ is Type I.*

Proof. Take a set of $\Delta(G_1)+1$ colors instead of $\Delta(G_1)+2$ and using the proof technique in Case 2 of the theorem 2.3, it is clear that $G_1 \blacktriangledown G_2$ is Type I. □

Theorem 2.5. *Let G_2 be any total colorable graph with n vertices. Then, $K_m \blacktriangledown G_2$ is total colorable.*

Proof. $K_m \blacktriangledown G_2$ has two copies of $K_m \vee G_2$. $\Delta(K_m \vee G_2) = \max\{\Delta(K_m) + n,\ \Delta(G_2) + m\}$. Clearly, $\Delta(K_m \vee G_2) \geq \frac{3}{4}(m+n)$. Hence, by lemma 2.1, $K_m \vee G_2$ is total colorable. We take a set of $\Delta(K_m \vee G_2)+2$ colors and assign total coloring for both copies of $K_m \vee G_2$ in the same way. At every vertex of $K_m \blacktriangledown G_2$, there will be at least one color that is not used. Select one of the unused colors at each vertex of G_2 to assign colors to the edges that connect to the corresponding vertices of G_2 in $K_m \blacktriangledown G_2$. □

Remark 2.6. *We can also prove that $G_1 \blacktriangledown K_n$ is total colorable for any graph G_1 using the same argument as in Theorem 2.3.*

Theorem 2.7. *Let G_1 and G_2 be two total colorable graphs with n_1 and n_2 vertices, respectively. If $\Delta(G_1) > \Delta(G_2)$ and $n_1 > n_2$, then $G_1 \blacktriangledown G_2$ is also total colorable.*

Proof. We know that $\Delta(G_1 \blacktriangledown G_2) = \max\{\Delta(G_1) + n_2, \Delta(G_2) + n_1 + 1\}$.

Case 1: $\Delta(G_1 \blacktriangledown G_2) = \Delta(G_1) + n_2$.
Let S denote the color set $\{a_1, a_2, ..., a_{\Delta(G_1)}, b_1, b_2, ..., b_n, b_{n_2+2}\}$ with $\Delta(G_1) + n_2 + 2$ colors. We first assign the total coloring for $G_1 \vee G_2$. From the given condition $n_1 > n_2$, we choose a set of n_1 colors from S to color the join edges between G_1 and G_2. Clearly $\Delta(G_1) + n_2 > \Delta(G_2) + n_1 + 1$. Hence, there will be at least $\Delta(G_2) + 2$ colors remaining in S. Take this $\Delta(G_2) + 2$ unused colors to properly color all the elements of G_2. Now, at each vertex of G_1, there will be a minimum of $\Delta(G_2)+2$ colors together with a set of $n_1 - n_2$ colors. It is evident that $\Delta(G_2) + 2 + (n_1 - n_2) \geq \Delta(G_1) + 2$. Use these colors to color the elements of G_1. Copy the same coloring scheme for the other copy of $G_1 \vee G_2$. We have used $\Delta(G_2) + 2$ colors in the total coloring of G_2. Hence, choose the available color at each vertex of G_2 to color the edges joining the corresponding vertices between the two copies of G_2.

Case 2 : $\Delta(G_1 \blacktriangledown G_2) = \Delta(G_2) + n_1 + 1$.
Here, the maximum degree vertices will be in the copies of G_2. We choose the color set, $S = \{a_1, a_2, ..., a_{\Delta(G_1)+2}, b_1, b_2, ..., b_{n_1}\}$. First, we assign colors for one of the copies of $G_1 \vee G_2$. Take $\Delta(G_2) + 2$ colors from S and apply the total coloring for G_2. Choose a set of m unused colors to color the join edges between G_1 and G_2. There will be $n_1 - n_2$ colors available at each vertex of G_1 along with $\Delta(G_2) + 2$ colors. Clearly, by the condition, $\Delta(G_2) + 2 + (n_1 - n_2) \geq \Delta(G_1) + 1$. Using these available colors, we can color the elements of G_1. Use the same coloring scheme for coloring the other copy of $G_1 \vee G_2$. There will be a color available at each vertex of G_2. Using this available color, we assign colors to the edges joining the corresponding vertices between the two copies of G_2.

□

Example 2.1. *For example, consider the graphs G_1 and G_2 with vertices $n_1 = 8$ and $n_2 = 7$ respectively. let $\Delta(G_1) = 5$ and $\Delta(G_2) = 2$. Here $\Delta(G_1 \blacktriangledown G_2) = \Delta(G_1) + n_2 = 12$. We first assign colors for $G_1 \vee G_2$. Take a set of $\Delta(G_1) + n_2 + 2 = 14$ colors. Choose a set of $n_1 = 8$ colors to color the edges joining G_1 and G_2. There will be atleast one unassigned color at each vertex of G_1. Use the unassigned colors for the vertex coloring of G_1. There will be 6 unused colors ($> \Delta(G_2) + 2$). Assign the total coloring for G_2 and edge coloring for G_1 with these unused colors. Repeat the same coloring scheme for the other copy of $G_1 \vee G_2$. Since we have used $\Delta(G_2) + 2$ colors for total coloring of G_2, there will be some unassignment missing color at each vertex of G_2. Assign this missing color to the edges connecting the corresponding vertices of G_2 in both copies of $G_1 \vee G_2$.*

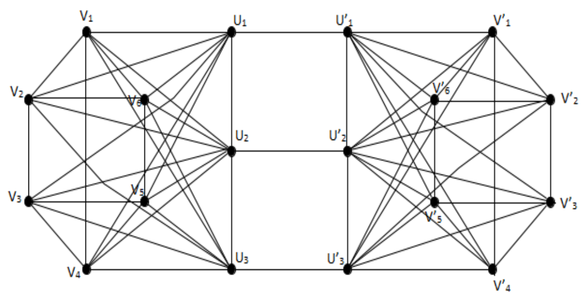

FIGURE 2. $G_1 \blacktriangledown G_2$.

Figure 2 shows $G_1 \blacktriangledown G_2$ with vertices $n_1 = 6$ and $n_2 = 3$ respectively. let $\Delta(G_1) = 3$ and $\Delta(G_2) = 2$. Here $\Delta(G_1 \blacktriangledown G_2) = \Delta(G_2) + n_1 + 1 = 9$. This comes under case 2 of the above theorem. We first assign colors for $G_1 \vee G_2$. Take a set of $\Delta(G_2) + n_1 + 3 = 11$ colors. Choose a set of $\Delta(G_2) + 2 = 4$ colors for the total coloring of G_2. Take a set of $n_1 = 6$ to color the edges connecting G_1 and G_2. There will be $n_1 - n_2 = 3$ colors available at each vertex of G_1 along with $\Delta(G_2) + 2 = 4$. Totally there will be 7 colors available at each vertex of G_1. Use these available colors we can assign total coloring of G_1. Repeat the same coloring scheme for the other copy of $G_1 \vee G_2$. Since we have used $\Delta(G_2) + 2$ colors for total coloring of G_2, there will be missing color at each vertex of G_2. Assign this missing color to the edges connecting the corresponding vertices of G_2 in both copies of $G_1 \vee G_2$.

3. Symmetric Difference Product

The symmetric difference of two graphs G_1 and G_2 denoted by $G_1 \diamond G_2$ is a graph with vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \diamond G_2) = \{(u_1, u_2)(v_1, v_2) | (u_1, v_1) \in E(G_1)$ or $(u_2, v_2) \in E(G_2)$ but not both$\}$. The maximum degree of $G_1 \diamond G_2$ is given by
$\Delta(G_1 \diamond G_2) = \max_{u \in V(G_1), v \in V(G_2)} \{|V(G_2)| d_{G_1}(u) + |V(G_1)| d_{G_2}(v) - 2 d_{G_1}(u) d_{G_2}(v)\}$.
The symmetric difference product $G_1 \diamond G_2$ can be viewed as $G_1 \square G_2 \cup E(\overline{G_1} \times G_2) \cup E(G_1 \times \overline{G_2})$. For example, Fig.3 shows $P_3 \diamond P_4$.

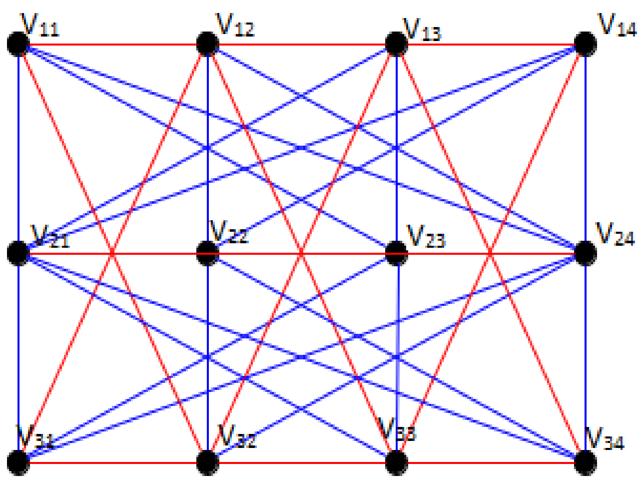

Figure 3. $P_3 \diamond P_4$.

Theorem 3.1. *$P_m \diamond K_n$ is total colorable.*

Proof. It is easy to see that $\Delta(P_m \diamond K_n) = n + (m-2)(n-1)$. In $P_m \diamond K_n$, there will be m copies of K_n. Take a set of $n+1$ colors for assigning the total coloring of K_n. Assign the total coloring for each odd copy of K_n using the same set of $n+1$ colors. There will be a missing color at each vertex of K_n in all the odd copies. Take the missing color at each vertex of K_n(odd copy) and assign it to the corresponding vertex in the next copy of K_n(even copy). In this coloring assignment, vertices of odd copies are assigned with a pattern, and vertices of even copies of K_n are assigned color with a different pattern. For the edges of the even copies of K_n, we follow the same pattern as the edge coloring of odd copies of K_n. If $(a,b) \in E(P_m)$, then there is a set of n edges joining the corresponding vertices in K_n^a and K_n^b. Also, if $(a,b) \notin E(P_m)$, then there will be a set of $(n-1)$ colors joining the vertices of K_n^a and K_n^b. Since $\Delta(P_m) = 2$, we take $m-2$ sets of $n-1$ colors to color the edges between the copies corresponding to non- adjacent vertices of P_m. There will be a set of $n-1$ colors available at each of the copies of K_n corresponding to the middle vertices of P_m. Take a new color, say C_1, and assign to the join edges between K_n^1 and K_n^2. Take the color at each vertex of K_n^2 and assign it to the join edges between K_n^2 and K_n^3. Now replace the vertices of K_n^2 with the set of $n-1$ available colors. Take the color C_1 to color the join edges between K_n^3 and K_n^4. Replace the vertices of K_n^4 with the available colors and shift its colors to the join edges between K_n^4 and K_n^5. Follow the above coloring procedure until all the join edges are properly colored. Here we have totally used $n+1+(n-1)(m-2)+1 = \Delta(P_m \diamond K_n)+2$ colors. □

Remark 3.2. *$P_2 \diamond K_2 \cong C_4$. Since C_4 is Type II, $P_2 \diamond K_2$ is also Type II. Similarly, $P_m \diamond K_2 \cong P_m \Box P_2$ and $P_m \Box P_2$ is Type I, Hence $P_m \diamond K_2$ is Type I.*

Theorem 3.3. *$C_m \diamond K_n$ is total colorable, when m is even.*

Proof. In $C_m \diamond K_n$, there are m copies of K_n. We need a set of n or $n+1$ colors for a proper total coloring of K_n. The maximum degree of $C_m \diamond K_n$ is given by $2n+(m-4)(n-1)$. Let $C_1 = \{c_1, c_2, ..., c_{n+1}\}$ and $C_2 = \{d_1, d_2, ..., d_{n+1}\}$ be two sets of $n+1$ colors. Take the colors from C_1 and assign the total coloring for the first copy of K_n. Use the colors from C_2 for assigning the total coloring to the second copy of K_n. Choose the colors from C_1 and C_2 alternatively to color the remaining copies of K_n. Since we have used $n+1$ colors to color each copy of K_n, there will be missing color at each vertex of K_n. Take the missing color at each vertex of K_n^1 and assign it to the join edges between K_n^1 and K_n^2. Continue this process by taking the missing color at the vertices of K_n^i and assigning it to the join edges between K_n^i and K_n^{i+1}. There are $(n-1)$ edges between K_n^i and K_n^j whenever $(i,j) \notin E(C_m)$. Since C_m is 2-regular, there will be $m-3$ non-adjacent vertices to each vertex of C_m. Correspondingly, we will have $m-3$ sets of join edges from each copy to the other copies of K_n. Hence, $(m-3)$ set of $n-1$ colors are enough to color the join edges. There will be a set of n colors from C_1 or C_2 available at each copy of K_n. We choose $m-4$ sets of

$n-1$ unused colors and the set of n available colors at each copy to assign the proper coloring for the join edges. Hence, $C_m \diamond K_n$ is total colorable.

□

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