Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 2, pp. 359 – 366
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.2.015

BOUNDARY CHROMATIC NUMBER IN GRAPHS

[1]G.PALANI MURUGAN, [2]S. ANGALEESWARI, [3]R. SUNDARESWARAN, AND [4]V.SWAMINATHAN

ABSTRACT. In this paper, we introduce and investigate new graph parameters based on boundary relationships among vertices. In a finite, simple, connected and undirected graph G, consider any two vertices u and v. A vertex v is a boundary vertex of u if the distance from u to v is greater than or equal to the distance from u to the neighbor of v. A Partition $\Pi = \{V_1, V_2, \cdots, V_k\}$ is called a boundary independent partition of G if each V_i $(1 \leq i \leq k)$ is independent and for any two vertices $u, v \in V_i$, either u is a boundary vertex of v or v is a boundary vertex of u. The minimum cardinality of a boundary independent partition of $V(G)$ is called the boundary chromatic number of G and it is denoted by $\chi_b(G)$. A subset S of $V(G)$ is called a boundary chromatic preserving $bdom$-setof G if S is a $bdom$-setof G and $\chi_b(\langle S\rangle) = \chi_b(G)$. The minimum (maximum) cardinality of a minimal boundary chromatic preserving $bdom$-setof G is called the boundary chromatic preserving boundary domination number of G (upper boundary chromatic preserving boundary domination number of G) and it is denoted by $\gamma_{bchpb}(G)$. $(\Gamma_{bchpb}(G))$. In this paper, we introduce and study the boundary chromatic number and boundary Chromatic preserving boundary domination number in graphs. .

2010 MATHEMATICS SUBJECT CLASSIFICATION 05C12,05C15,05C69 .
KEYWORDS AND PHRASES. Boundary vertex, Boundary chromatic number, Boundary chromatic preserving boundary domination number.

Submission Date: 18 February 2024

1. INTRODUCTION

For the following basic definitions, refer [1–4]. Let G be a non trivial connected graph. The distance $d(u,v)$ between two vertices u and v of G is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is called a $u-v$ geodesic, that is, it is a shortest path connecting the vertices u and v in a graph G. Let u be a vertex in G. A vertex v is an eccentric vertex of u if $d(u,v)$ is denoted by $e(u)$, if v is at maximum distance from u in G. In this case, a vertex x is an eccentric vertex of G if x is an eccentric vertex of some vertex of G. In this case, a vertex x is an $ecc-$ vertex of G if x is an $ecc-$ vertex of some vertex of G. Therefore, if v is an $ecc-$ vertex of u and w is a neighbour of v, then $d(u,w) \leq d(u,v)$. A vertex v is a boundary vertex of u if $d(u,w) < d(u,v)$ for all $w \in N(v)$. Further results, refer [5–7].

In this paper, the first section deals with the concept of boundary chromatic partition in graphs is discussed in the second section. In the second

*R.Sundareswaran.

section, boundary domination ($bdom$) with respect to the vertex exchange is studied. Boundary chromatic preserving boundary dom-set is discussed.

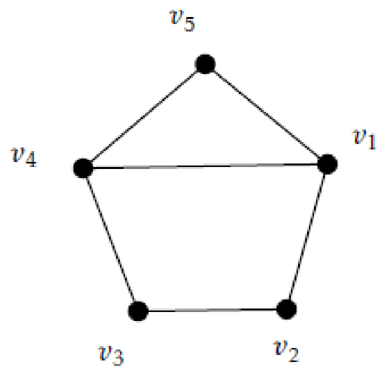
2. Boundary Chromatic Partition in graphs

Definition 2.1. *Suppose G represents the simple connected graph. Let $\Pi = \{V_1, V_2, \cdots, V_k\}$ be an independent partition of $V(G)$. Given any two vertices $u, v \in V_i$ $(1 \leq i \leq k)$ one of the vertices u or v; serves as a boundary vertex form the other. Then Π is called a boundary independent partition of $V(G)$.*

Remark 2.2. *A partition in which every element is a singleton is clearly a boundary independent partition of $V(G)$.*

Definition 2.3. *The minimum cardinality of a boundary independent partition of $V(G)$ is called the boundary chromatic number of G and it is denoted by $\chi_b(G)$.*

Example 2.1. *Let $G =$*



In the above graph $G, \Pi = \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}$ and $\chi_b(G) = 3$.

Remark 2.4. *Provided that G is a connected graph and u is a full degree vertex, then every vertex of G different from u is a boundary vertex of u.*

Theorem 2.5. *Let G be a connected graph. $\chi_b(G) = n$ if and only if $G = K_n$.*

Proof. If $G = K_n$ then $\chi_b(G) = n$.

Conversely, let $\chi_b(G) = n$. Suppose $G \neq K_n$. Then there exist vertices u and v such that u and v are not adjacent. Suppose w is an eccentric vertex of u. Since $d(u, v) \geq 2$. $d(u, w) \geq 2$. Therefore $\Pi = \{\{u, v\}, \{x_i\}\}$, where x_i is an element of $V(G) - \{u, v\}$, $1 \leq i \leq n-2$. Π is a boundary independent partition of G. Therefore, $\chi_b(G) \leq n-1$, which is a contradiction.Hence $G = K_n$. □

Remark 2.6. *A graph G is complete if and only if any two vertices of G are boundary vertices of each others.*

Proposition 2.1.

(i) If $G = K_n - e$ then $\chi_b(G) = n-1$. (ii) If $V(G) = \{u_1, u_2, \cdots, u_n\}$ and u_1 is not adjacent with exactly two vertices say u_i and u_j and the vertices $V(G) - \{u_1, u_i\}$ and $V(G) - \{u_1, u_j\}$ from a cliques, then $\chi_b(G) = n-1$. Conversely, if $\chi_b(G) = n-1$ then G is one of the types mentioned in (i) or (ii).

Proof. Let the hypothesis hold. Then $\{\{u_1u_i\}, \{u_2\}, \cdots, \{u_j\}, \cdots, \{u_n\}\}$ is a boundary independent partition. Therefore, $\chi_b(G) = n-1$. Since $(n-2)$ vertices from a clique, $\chi_b(G) \geq n-2$. Hence $\chi_b(G) = n-1$. Suppose $\chi_b(G) = n-1$. Suppose G is not one of the types mentioned in (i)and (ii). Then there exist two edges u_iu_j and u_ru_s such that i, j, r, s are different then $\chi_b(G) \leq n-2$, a contradiction therefore G is of type (i)or (ii). □

Proposition 2.2. *For any graph G, $\chi_b(K_n \circ K_1) = n$.*

Proof. Let $V(K_n \circ K_1) = \{u_1, u_2, \cdots, u_n, u_1', u_2', \cdots, u_n'\}$ where u_i' is the pendent attached with u_i. let $\Pi = \{\{u_1, u_2', u_3', \cdots, u_n'\}, \{u_2, u_1'\}, \{u_3\}, \cdots, \{u_n\}\}$ is a boundary independent partition. Therefore $\chi_b(K_n \circ K_1) \leq n$. Since $\chi_b(K_n \circ K_1)$ contains a clique of order n, $\chi_b(K_n \circ K_1) \geq n$. But $\chi_b(K_n \circ K_1) \geq \chi(K_n \circ K_1) \geq n$. Therefore $\chi_b(K_n \circ K_1) = n$. □

Definition 2.7. *Let G be the graph obtained from K_m by attaching a_i pendent vertices at u_i Where $V(K_m) = \{u_1, u_2, \cdots, u_m\}$. G is denoted by $K_m(a_1, a_2, \cdots, a_m)$.*

Proposition 2.3. *For any graph G, $\chi_b(K_m(a_1, a_2, \cdots, a_m)) = m$.*

Proof. π is a boundary independent partition of $K_m(a_1, a_2, \cdots, a_m)$. Therefore, $\chi_b(K_m(a_1, a_2, \cdots, a_m)) \leq m$. Since $K_m(a_1, a_2, \cdots, a_m)$ contains a clique of order $m, \chi_b(K_m(a_1, a_2, \cdots, a_m)) \geq m$.

But $\chi_b(K_m(a_1, a_2, \cdots, a_m)) \geq \chi(K_m(a_1, a_2, \cdots, a_m)) \geq m$.

Therefore, $\chi_b(K_m(a_1, a_2, \cdots, a_m)) = m$. □

Proposition 2.4. *$\chi_b(G)$ for some known classes of graphs:*

(1) $\chi_b(K_{1,n}) = 2, \quad \forall n \geq 1$.
(2) $\chi_b(K_n) = n$.
(3) $\chi_b(K_{m,n}) = 2, \quad \forall m, n$.
(4) $\chi_b(P_n) = n-2$.
(5) $\chi_b(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & if\ n \geq 4 \\ 3 & if\ n = 3 \end{cases}$
(6) $\chi_b(W_n) = \begin{cases} 2\lfloor \frac{n}{3} \rfloor + 1, & if\ n \equiv 0, 1 (mod\ 3) \\ 2\lfloor \frac{n}{3} \rfloor + 2, & if\ n \equiv 2 (mod\ 3) \end{cases}$
(7) $\chi_b(D_{r,s}) = 2, \forall r, s \geq 0$.
(8) $\chi_b(P) = 4$,*where P is the Petersen graph*

Theorem 2.8. *Let G be a connected graph with $diam(G) = 2$ then $\chi_b(G) = 2$ if and only if G is bipartite.*

Proof. Let G be a partite graph with $diam(G) = 2$. Let $\Pi = \{V_1, V_2\}$ be such that V_1 and V_2 are independent. Let $u, v \in V_i (1 \leq i \leq 2)$. Let $w \in N(u)$. Since $diam(G) = 2$ and u and v are independent, $d(u, v) = 2$. $d(w, v) \leq 2$, since $diam(G) = 2$. Therefore, u is a boundary of v. Also v is a boundary of u. Therefore, $\chi_b(G) = 2$. The converse is obvious. □

Theorem 2.9. *Let G be a connected graph. Then $\chi_b(G) = 2$ if and only if G is bipartite with bi-partition V_1, V_2 and for any two vertices $u, v \in V_i (1 \leq i \leq 2)$, $d(u, v) \geq d(w, v)$ for any $w \in N(v)$.*

Proof. Obvious. □

Definition 2.10. *A subset S of G is called an independent boundary set of G if S is independent and for any $u, v \in S$, either u is a boundary vertex of v or v is a boundary vertex of u. The maximum cardinality of an independent boundary set of G is denoted by $\beta_{ib}(G)$.*

Theorem 2.11. *For any graph G, $\frac{n}{\beta_{ib}(G)} \leq \chi_b(G) \leq n - \beta_{ib}(G) + 1$.*

Proof. Let $\chi_b(G) = k$. Let $\Pi = \{V_1, V_2, \cdots, V_k\}$ be a boundary, independent partition of G. Then $n = \sum_{i=1}^{k} |V_i| \leq k\beta_{ib}(G)$ (Since $|V_i| \leq \beta_{ib}(G)$.) Therefore, $\frac{n}{\beta_{ib}(G)} \leq k = \chi_b(G)$. Let S be a maximum independent boundary set of G. Consider $H = \langle V - S \rangle$. Let $\chi_b(H) \leq \chi_b(G) - 1$. Let $\Pi = \{V_1, V_2, \cdots, V_t\}$ where $\chi_b(H) = t$ be a boundary, independent partition of H. Then $t \leq \chi_b(G) - 1$. That is, $t \leq \chi_b(G) - 2$. Let $\Pi^{'} = \{V_1, V_2, \cdots, V_t, S\}$. Then Π' is a boundary, independent partition of G. Therefore, $\chi_b(G) \leq t+1 \leq \chi_b(G)-2+1 = \chi_b(G)-1$, a contradiction. Therefore, $\chi_b(H) \geq \chi_b(G) - 1$. Since H has $n - \beta_{ib}(G)$ vertices, $\chi_b(H) \leq n-\beta_{ib}(G)$. Therefore, $\chi_b(G)-1 \leq \chi_b(H) \leq n-\beta_{ib}(G)$. Hence $\chi_b(G) \leq n - \beta_{ib}(G) + 1$. □

Conjecture: $\chi_b(G) \leq 1 + \max\{Delta_b(H)\}$, where H is an induced subgraph of G.

3. Boundary chromatic preserving *bdom*-set in graphs

Definition 3.1. *A subset S of $V(G)$ is termed as boundary chromatic preserving set of G if $\chi_b(\langle S \rangle) = \chi_b(G)$. The smallest possible size of a boundary chromatic preserving set of G is symbolized by $bcpn(G)$ (boundary chromatic preserving number of G).*

Definition 3.2. *Let $R \subseteq V(G)$. Then R is called a boundary chromatic preserving bdom-set of G if R is a bdom-set of G and $\chi_b(\langle R \rangle) = \chi_b(G)$. The smallest (largest) size of a minimal boundary chromatic preserving bdom-set of G is called the boundary chromatic preserving boundary domination number of G (upper boundary chromatic preserving bdom-number of G) and it is denoted by $\gamma_{bchpb}(G)$. ($\Gamma_{bchpb}(G)$).*

Proposition 3.1.

$\gamma_{bchpb}(G)$ for some known graphs:

(1) $\gamma_{bchpb}(K_{1,n}) = 2$, $\forall n \geq 1$.
(2) $\gamma_{bchpb}(K_n) = n$.
(3) $\gamma_{bchpb}(K_{m,n}) = 2$, $\forall m, n$.
(4) $\gamma_{bchpb}(P_n) = n - 2$, $\forall n \geq 4$, 2 if n = 2, 3.
(5) $\gamma_{bchpb}(C_n) = \lceil \frac{n}{2} \rceil$, $\forall\ n \geq 4$, 3 if n = 3.
(6) $\gamma_{bchpb}(W_n) = \begin{cases} 4, & \text{if n is even} \\ 3, & \text{if n is odd} \end{cases}$
(7) $\gamma_{bchpb}(D_{r,s}) = 2$, $\forall\ r, s \geq 0$.
(8) $\gamma_{bchpb}(P) = 4$, Where P is the Petersen graph.
(9) $\gamma_{bchpb}(K_m(a_1, a_2, \cdots, a_m)) = m$

Remark 3.3. *For any graph G, the following holds:*

(1) *$bcpn(G) \leq \gamma_b chpb(G)$.*
(2) *Boundary chromatic preserving bdom-set exists in any graph. Since $V(G)$ is a trivial boundary chromatic preserving bdom-set.*
(3) *If S is a $\gamma_{bchpb}(G)$ set of G then each vertex in $V-S$ is not adjacent with atleast one vertex of S. For : Suppose there exists u in $V-S$ such that u is adjacent with every vertex of S. Then $\chi_b(G) > \chi_b(\langle S\rangle)$, a contradiction.*

Proposition 3.2. *A boundary chromatic preserving bdom-set of a graph is a dominating set of $\bar{G}$.*

Proof. Let S be a boundary chromatic preserving $bdom$-set of G. Then for any vertex u in $V-S$ there exist a vertex $v \in S$ such that v is not adjacent with u. Therefore v is adjacent with u in $\bar{G}$. Therefore, S is a dominating set of $\bar{G}$. □

Remark 3.4. *(i) $\gamma(\bar{G}) \leq \gamma_{bchpb}(G)$.*
(ii) If S is a boundary chromatic preserving bdom-set of G which is also dominating then S is a global dom-set of G. Hence $\gamma_g(G) \leq \gamma^d_{bchpb}(G)$.

Example 3.1. *Consider $G = K_{m,n}$. Let V_1, V_2 be the partite sets of G. Let $u \in V_1$ and $v \in V_2$. Then $S = \{u, v\}$ is a boundary chromatic preserving bdom-set which is also dominating. Therefore, $\gamma^d_{bchpb}(K_{m,n}) = 2$, $\gamma_g(G) = 2$, $\gamma(G) = 2$, $\gamma(\bar{G}) = 2$.*

Proposition 3.3. *For a χ_b - critical graph, V is the only boundary chromatic preserving bdom-set of G.*

Proof. If G is a χ_b - critical graph (that is, $\chi_b(G-u) < \chi_b(G)$ for any u in $V(G,)$) then for any minimum $bdom$-set D of G, SD is a proper subset of G and hence $\chi_b(\langle S\rangle) < \chi_b(G)$. Therefore, $\chi_{bchpb}(G) = |V(G)|$. □

Proposition 3.4. *In the case of a connected graph G, it follows that $\chi_{bchpb}(G) = p-q$ is necessary and sufficient condition for $G = K_1$.*

Proof. If $G = K_1$ then $p = 1, q = 0$. Therefore, $p-q = 1.\gamma_{bchpb}(k_1) = 1 = p-q$. Conversely, suppose $\gamma_{bchpb}(G) = p-q$. Since $\gamma_{bchpb}(G) \geq 1, p-q \geq 1$. As G is connected, $q \geq p-1$. Therefore $q = p-1$. That is, G is k_1. □

Definition 3.5. *Let $v \in V(G)$. The boundary degree of v in G is the number of vertices u such that v is a boundary of u, and it is denoted by $deg_b(v)$. Also $N_b(v) = \{u \in V(G) : v \text{ is a boundary of } u\}$.*

Proposition 3.5. *Let S be a χ_b - preserving bdom-setof G. Then $|V-S| = \sum\limits_{u\in S} deg_b(u)$ if and only if $G = p.K_1, p \geq 1$.*

Proof. Suppose $G = pk_1$, $p \geq 1$ then $S = V$. and $deg_b(u) = 0$ for every $u \in S$, $|V-S| = 0$. Since $S = V$. $\sum_{u\in S} deg_b(u) = 0$. Hence $|V-SD| = \sum\limits_{u\in S} deg_b(u)$.

Conversely, suppose $|V-S| = \sum\limits_{u\in S} deg_b(u) = k$ (say).

Claim: $k = 0$. Suppose $k \geq 1$.

Case (i): G is connected. Since $k \geq 1, q \geq 1$. Therefore $\chi_b(G) \geq \chi(G) \geq 2$. Let $V-S = \{u_1, u_2, \cdots, u_k\}$. Since S is a bdom-set, each u_i is boundary dominated by some vertex of S and hence contributes atleast one to the boundary degree of D. Since $\chi_b(|S|) \geq 2$, in a boundary chromatic partition of S, there exists atleast two independent sets, where in each set any two vertices are such that one is the boundary for another or vice versa. Hence in S, there exists two vertices say u_1, u_2 and another two vertices u_3, u_4 such that u_1 is a boundary vertex of u_2 or vice versa and u_3 is a boundary vertex of u_4 or vice versa. Hence these vertices contribute to $N_b(S)$. Therefore $\sum\limits_{u\in S} deg_b(u) \geq k+2$, a contradiction, since $\sum_{u\in SD} deg_b(u) = k$. Therefore $k = 0$.

Case (ii): Suppose G is disconnected.

Subcase (i): G is totally disconnected. Therefore $S = V$ and hence $V - S$ is empty. Therefore $|V - S| = 0$. Therefore $k = 0$.

Subcase (ii): G is not totally disconnected. Therefore, G has a non-trivial component. This non-trivial component contributes atleast 1 to $\sum\limits_{u\in S} deg_b(u)$. That is, $k \geq 1$. Using case (i), we get a contradiction. Therefore $k = 0$. Hence the claim. Therefore $|V - D| = 0$ therefore $D = V$ and $\sum\limits_{u\in S} deg_b(u) = 0$. Therefore, $deg(u) = 0$ for every $u \in D$. Therefore G is totally disconnected. Hence, $G = pK_1$. □

Corollary 3.6. *For any non-trivial connected graph with a boundary chromatic Preserving boundary dominating set D,*
$\sum\limits_{u\in S} deg_b(u) \geq |V - S| + 2.$

Proof. Suppose G is χ_b - critical. Then $V = D$. Therefore $|V - S| = 0$. Since $(\langle V\rangle)$ is connected and non-trivial, there exists two vertices u, v such that u_1 is a boundary vertex of u and v_1 is a boundary vertex of v. Therefore, $\sum\limits_{u\in S} deg_b(u) \geq 2$. Suppose G is not χ_b - critical. Since G is non-trivial, $\chi(G) \geq 2$. Therefore, $\chi_b(G) \geq 2$. Using case (i) of previous theorem, $\sum\limits_{u\in S} deg_b(u) = |V - D|$. □

Proposition 3.6. *For any graph* $G, \lfloor \frac{p}{(1+\Delta_b(G))} \rfloor \leq \gamma_{bchpb}(G)$ *and equality holds if and only if* $G = pk_1$, $p \geq 1$.

Proof. In any graph G, $\lfloor \frac{p}{(1+\Delta_b(G))} \rfloor \leq \gamma_b(G) \leq \gamma_{bchpb}(G)$. Therefore, $\lfloor \frac{p}{(1+\Delta_b(G))} \rfloor \leq \gamma_{bchpb}(G)$. Suppose $\lfloor \frac{p}{(1+\Delta_b(G))} \rfloor = \gamma_{bchpb}(G) = k$. (say)

Case(i): G is connected. Let S be a $\gamma_{bchpb}(G)$ set of G. If $k \geq 2$, then G has atleast two vertices. By corollary $\sum\limits_{u\in D} deg(u) \geq |V - S| + 2$. Therefore, $|V - S| \leq \sum_{u\in D} deg(u) - 2$. Therefore, $|V - D| < \sum_{u\in S} deg(u)$. That is, $p - k < \sum\limits_{u\in D} deg_b(u)$. Therefore, $p - k < k.\Delta_b(G)$. Therefore, $p < k + k.\Delta_b(G)$. $P < k(1 + \Delta_b(G)$. Hence $\frac{p}{(1+\Delta_b(G))} < k$. Therefore, $\lfloor \frac{p}{(1+\Delta_b(G))} \rfloor < k$. But by the hypothesis $\lfloor \frac{p}{(1+\Delta_b(G))} \rfloor = k$, a contradiction. Therefore $k = 1$. That is, $\gamma_{bchpb}(G) = 1$. Since D is χ_b presering, $\chi_b(G) = \chi_b(\langle S\rangle) = 1$. Therefore, G is disconnected unless $p = 1$. Since G is connected, $G = K_1 = 1$. $K_1 = p.K_1$, since $p = 1$.

Case (ii): G is disconnected. Suppose G is totally disconnected. $\gamma_{bchpb}(G) = |V(G)| = p.\Delta_b(G) = 0$. Therefore, $\lfloor \frac{p}{(1+\Delta_b(G))} \rfloor = p = \gamma_{bchpb}(G)$. Suppose G is not totally disconnected. Then G has atleast one non trivial component. By proceeding as in case (i), a contradiction arises. Therefore, $G = pK_1$. □

Proposition 3.7. *A boundary dominating boundary chromatic preserving set S is minimal if and only if for each $u \in S$, any of the following holds.*
(i) $\chi_b(\langle S - \{u\rangle\}) < \chi_b(G)$.
(ii) $S - \{u\}$ *is not a* $bdom$*-setof* G.

Proof. Boundary dominating boundary chromatic preserving property is super hereditary. Let S be a boundary dominating boundary chromatic preserving set of G. Let T be a super set of S. Then T is boundary dominating, since for any vertex u in $V - T, u \in V - S$ and hence there exists $v \in S$ such that v boundary dominates u. Since $S \subseteq T, v \in T$ and hence v boundary dominates u in $V - T$. Since S is boundary chromatic preserving, $\chi_b(\langle S\rangle) = \chi_b(G)$. Since $S \subseteq T \subseteq G$, $\chi_b(\langle S\rangle) \leq \chi_b(\langle T\rangle) \leq \chi_b(G)$. But $\chi_b(\langle S\rangle) = \chi_b(G)$. Therefore, $\chi_b(T) = \chi_b(G)$. Therefore T is boundary chromatic preserving set of G. Since boundary dominating boundary chromatic preserving property is super hereditary, a boundary dominating boundary chromatic preserving set S is minimal if and only if it is 1-minimal. That is, S is minimal iff $S - \{u\}$ is not a boundary dominating boundary chromatic preserving set of G for every $u \in S$. Suppose S is minimal. Let $u \in S$. Then $S - \{u\}$ is not a boundary dominating boundary chromatic preserving set of G. Therefore $S - \{u\}$ is not a $bdom$-setor $S - \{u\}$ is not a boundary chromatic preserving set or both. That is, $\langle S - \{u\rangle\}$ is not a $bdom$-setor $\chi_b(\langle S - \{u\rangle\}) < \chi_b(G)$.

Conversely, suppose for any $u \in S$, condition (i) holds. $\chi_b(\langle S - \{u\rangle\}) < \chi_b(G)$. That is, $S - \{u\}$ is not a boundary chromatic preserving set. If condition (ii) holds then $\langle S - \{u\}\rangle$ is not a bdom-set. Therefore, S is a minimal boundary dominating boundary chromatic preserving set of G. □

Conclusion

In this work, we defined boundary chromatic partition set as well as boundary chromatic preserving bdom-setin the non trivial connected graphs. There is a structural dependency between $\chi_b(G)$ and $\gamma_{bchpb}(G)$. While they measure different aspects (partitioning vs domination), they intersect conceptually through how coloring constraints are preserved in boundary-dominating subsets. In practice, understanding one can help constrain or lestimate the other.

Future Work

The future work is to Exploring how $\chi_b(G)$ relates to classical chromatic number $\chi(G)$, domination number $\gamma(G)$, and other variants like total domination or independent domination could help unify boundary-based concepts within broader graph theory.

References

[1] F. Harary, *Graph Theory*, Narosa Publishing house (1988).
[2] F. Buckley and F.Harary, *Distance in Graphs*, Addition - Wesley Reading (1990).
[3] G. Chartrand, David Erwin, G.L. Johns and P. Zhang, Boundary vertices in graphs, *Discrete Mathematics*, **263**(2003), 25-34.

[4] G. Chartrand, David Erwin, G.L. Johns and P. Zhang, On boundary vertices in graphs, *J.Combin. Math. Combin. Comput.*,**48** (2004) 39-53.

[5] K.M. Kathiresan, G. Marimuthu and M. Sivanandha Saraswathy, Boundary Dominations in Graphs, *Kragujevac J. Math.* **33**(2010),63-70.

[6] G. Marimuthu and M. Sivanandha Sarawathy, Characterization of Boundary Graphs, *International Journal of Latest Engineering and Management Research (IJLEMR)*,**1(6)** (2016) , 08-15.

[7] M. Sivanandha Saraswathy, A study on boundary vertices in graphs, Ph.D thesis, The Madura College, Affiliated to Madurai Kamaraj University, Madurai.

[8] G. Palani Murugan, K. Angaleeswari, R.Sundareswaran and V. Swaminathan, Bondary degree equitable dom-set in Graphs, Indian Journal of Natural sciences(tnsroindia.org.in) 15(85) (2024) 78719 - 78725.

Research Scholar, Ramanujan Research Centre, Saraswathi Narayanan College, Madurai, Tamilnadu, India.,
Email address: gpnmaths@gmail.com

araswathai Narayan College, Madurai, Tamilnadu, India.
Email address: aadeshangal@gmail.com

Sri Sivasuramaniya Nadar College of Engineering,, Kalavakkam,Tamilnadu,India.
Email address: sundareswaranr@ssn.edu.in

Ramanujan Research Center in Mathematics,, Saraswathai Narayan Colelge,, Madurai, Tamilnadu, India.
Email address: lalitha.sulanesri@gmail.com