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STAR EDGE COLORING OF GRAPHS WITH MAD$(G)< \frac{8}{3}$

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Abstract. A star edge coloring of a graph G is a proper edge coloring such that there is no bi-colored path or cycle of length four. The minimum number of colors needed for a graph G to admit a star edge coloring is called the star chromatic index and it is denoted by $\chi_s^{'}(G)$. The maximum average degree of a graph G, denoted by Mad(G) is the maximum of average degrees of subgraphs of G $(= max_{H\subseteq G,|V(H)|\geq 1}\{\frac{2|E(H)|}{|V(H)|}\})$. In this paper, we consider graphs of maximum degree $\Delta \geq 4$ and show that if the maximum average degree of a graph is less than $\frac{8}{3}$ then $\chi_s^{'}(G) \leq 2\Delta$.

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1. Introduction

For a simple graph G, with vertex set $V(G)$ and edge set $E(G)$, a *proper k-edge coloring* is a mapping $f : E(G) \to \{1, 2, 3, ..., k\}$ such that $f(e) \neq f(e^{'})$, for any two adjacent edges e and $e^{'}$. A *star k-edge coloring* is a proper k-edge coloring such that there is no bi-colored path of length four or cycle of length four. The *star chromatic index* of G, denoted by $\chi_s^{'}(G)$ is the minimum number of colors required for G to admit a star edge coloring. This coloring was introduced by Liu and Deng [7] in 2008.

Dvořák et al. [13], found a bound for the star chromatic index of complete graphs and gave a near linear upper bound for any graph.

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$$(2+o(1))n \leq \chi_s^{'}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{log(n)}}}{(log(n))^{\frac{1}{4}}}.$$

They also showed that for a subcubic graph G (graph with maximum degree at most 3), $4 \leq \chi_s^{'}(G) \leq 7$ and conjectured that for such graphs $\chi_s^{'}(G) \leq 6$. Xuling H. et. al. [3] gave a tight upper bound for the star chromatic index of cubic Halin graph G and showed that $\chi_s^{'}(G) \leq 6$. Pradeep K. et al. [4] proved that if G is a subcubic graph with maximum average degree, Mad(G) $< \frac{8}{3}$, then $\chi_s^{'}(G) \leq 6$. Bezegová L. et al. [6] obtained some tight bounds for the star chromatic index of trees and subcubic outerplanar graphs. Wang Y. et al. [12] showed that if G is a graph with $\Delta = 4$, then $\chi_s^{'}(G) \leq 14$ and if G is a bipartite graph with $\Delta = 4$, then $\chi_s^{'}(G) \leq 13$. Pradeep K. [5] showed that $\chi_s^{'}(G) \leq 2\Delta + 1$ for the graphs G with $\Delta \geq 4$ and Mad(G) $< \frac{14}{5}$. Lei H. et al. [2] proved that it is NP-complete to determine whether $\chi_s^{'}(G) \leq 3$ for an arbitrary graph.

The list version of star edge coloring is investigated in [8], [9], [10] and list star chromatic index $ch_s^{'}(G)$ is found for various graphs in terms of their maximum average degrees. The maximum average degree of a graph G, denoted by Mad(G) is the maximum of average degrees of subgraphs of G ($= max_{H\subseteq G,|V(H)|\geq 1}\{\frac{2|E(H)|}{|V(H)|}\}$). Kerdjoudj S. et al. proved that for a subcubic graph G,
(i) if Mad(G) $< \frac{7}{3}$, then $ch_s^{'}(G) \leq 5$,
(ii) if Mad(G) $< \frac{5}{2}$, then $ch_s^{'}(G) \leq 6$,
(iii) if Mad(G) $< \frac{30}{11}$, then $ch_s^{'}(G) \leq 7$.

Lužar B. et al. [1] have shown that seven colors suffice for the list star edge coloring of a subcubic graph G. Apart from subcubic graphs, the following results were proved in [9] and [10] for a graph G with maximum degree $\Delta \geq 4$,
(i) if Mad(G) $< \frac{7}{3}$, then $ch_s^{'}(G) \leq 2\Delta - 1$,
(ii) if Mad(G) $< \frac{5}{2}$, then $ch_s^{'}(G) \leq 2\Delta$,
(iii) if Mad(G) $< \frac{8}{3}$, then $ch_s^{'}(G) \leq 2\Delta + 1$,
(iv) if Mad(G) $< \frac{14}{5}$, then $ch_s^{'}(G) \leq 2\Delta + 2$,
(v) if Mad(G) < 3, then $ch_s^{'}(G) \leq 2\Delta + 3$.

Using the concept of edge partitions, Y.Wang et al. [11] showed that, (i) for a planar graph G with maximum degree Δ, $\chi_s^{'}(G) \leq$

$2.75\Delta+18$, and (ii) for a planar graph G of girth at least 8, $\chi_s'(G) \leq \lfloor 1.5\Delta \rfloor + 3$.

In this paper, we consider the graphs G of maximum degree $\Delta \geq 4$ with $\text{Mad}(G) < \frac{8}{3}$ and prove the following theorem.

Theorem 1.1. *Let G be a graph with maximum degree $\Delta \geq 4$ and $Mad(G) < \frac{8}{3}$. Then $\chi_s'(G) \leq 2\Delta$.*

2. Definitions and notations

For a graph G, let $d_G(v)$ denote the degree of a vertex v in G. If G is clear from the content, we may omit the subscript. Let $N(v)$ be the set of neighbors of v. A vertex of degree k is called a k-vertex. A k^+-vertex is a vertex of degree at least k. A k-vertex adjacent to a vertex v is a k-neighbor of v. A 3_k-vertex is a 3-vertex adjacent to exactly k $(0 \leq k \leq 3)$ 2-vertices. A 3_1-vertex adjacent to two 3-vertices is called a *bad* 3_1-vertex. A 2-vertex adjacent to a 2-vertex is called a *bad* 2-vertex. An edge incident to a vertex of degree one is called a *pendant* edge. For an edge coloring φ of the graph G, $\varphi(v)$ denotes the set of colors used on the edges incident with the vertex $v \in V(G)$, in the coloring φ. Similarly, for an edge $uv \in E(G)$, $\varphi(uv)$ denotes the color used on the edge uv. We say that a color c is an *available color* for an edge uv if c is not assigned to any of its neighbors and there is no bi-colored path of length four or cycle of length four involving uv when uv is colored with c. Otherwise, it is said to be a *forbidden color* for the edge uv. The set of forbidden colors for a given edge uv is denoted by $F(uv)$.

3. Proof of the Theorem 1.1

The proof is by the method of contradiction. For some integer k, let G_k be the class of graphs with maximum degree at most k and maximum average degree less than $\frac{8}{3}$. Let for the smallest k, $H \in G_k$ be a counterexample to this theorem minimizing $|E(H)| + |V(H)|$. That is, $\text{Mad}(H) < \frac{8}{3}$ and $\chi_s'(H) > 2k$ and for any edge $e \in E(H)$, $\chi_s'(H \backslash \{e\}) \leq 2k$. By minimality of H, we can assume that H is connected. Otherwise, we can star color independently the edges of each connected component of H with $2k$ colors.

Structure of minimal counterexample

Claim 3.1. *H does not contain a vertex u adjacent to* $d(u)$*-1 vertices of degree 1.*

Suppose H contains a p-vertex u with $N(u) = \{u_1, ..., u_p\}$ and $d(u_i) = 1$ for $i \in \{1, ..., p-1\}$ $(p \leq k)$. Let $H^{'} = H \setminus \{uu_1\}$. By minimality of H, $H^{'}$ has a star edge coloring φ with $2k$ colors. As $d(u_p) \leq k$, $|\varphi(u_p)| \leq k$ and $|F(uu_1)| \leq |\varphi(u) \cup \varphi(u_p)| \leq (k-1) + (k-1) = 2k-2$. So, there are at least three colors available for uu_1. Hence, φ can be extended to H, a contradiction.

Now, consider $G^{'} = H \setminus \{v : v \in V(H), d_H(v) = 1\}$. If H does not contain a 1-vertex, then $G^{'} = H$. It can be observed that by Claim 3.1, $G^{'}$ does not contain 1-vertices. Since $G^{'} \subseteq H$, we have, $\text{Mad}(G^{'}) < \frac{8}{3}$.

Claim 3.2. *For any integer* $p \geq 2$*, if* $G^{'}$ *contains a p-vertex u adjacent to p 2-vertices then u has no 1-neighbors in H.*

Suppose $G^{'}$ contains a p-vertex u adjacent to p 2-vertices u_i, $1 \leq i \leq p$ in $G^{'}$ and a 1-vertex u_{p+1} in H. The vertices u_1, u_2, ..., u_p may also have 1-neighbors in H. Let $H^{'} = H \setminus \{uu_{p+1}\}$ as shown in Figure 1. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. For $1 \leq i \leq p$, let $u_i^{'}$ be the neighbors of u_i other than u. It is easy to see that, $|F(uu_{p+1})| \leq |\varphi(u_i u_i^{'}) \cup \varphi(u)| \leq p + p = 2p$ for $1 \leq i \leq p$. Since u has 1-neighbor in H, $p < k$. As $k \geq 4$, there are at least two colors available for the edge uu_{p+1}. The colors used on the pendant edges incident to u_i, $1 \leq i \leq p$ may also be available for uu_{p+1}. So, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H with $2k$ colors.

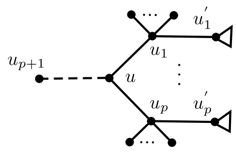

Figure 1. Configuration of Claim 3.2

Claim 3.3. *If* $G^{'}$ *contains a path xuvy, where u and v are 2-vertices, then* $d_H(u) = d_H(v) = 2$ *and* $d_{G'}(x) = d_{G'}(y) = k$.

Suppose $G^{'}$ contains a path $xuvy$, where u and v are 2-vertices. If u and v have 1-neighbors in H, let them be denoted by u_i and v_j respectively for $i, j \in \{1, ..., k-2\}$. Let $H^{'} = H \setminus \{uu_1\}$ as shown in Figure 2(a). By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. As $|F(uu_1)| \leq |\varphi(x) \cup \varphi(u) \cup \varphi(v)| \leq k + (k-2) + 1 = 2k - 1$, at least one color is available for uu_1. The colors used on the pendant edges incident to v may be available for uu_1. Hence, $d_H(u) = 2$. Similarly, we can show that $d_H(v) = 2$.

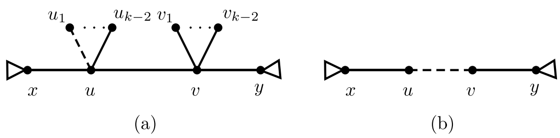

Figure 2. Configuration of Claim 3.3

Now, suppose that $d_{G'}(x) < k$. The vertex x may have 1-neighbors in H. Let $H^{'} = H \setminus \{uv\}$ as shown in Figure 2(b). By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. Observe that $|F(uv)| \leq |\varphi(x) \cup \varphi(y)|$. Since $d_{G'}(x) < k$, $|F(uv)| < 2k$, there is at least one color available for uv. The colors used on the pendant edges incident to x or y may be available for uv. So, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H with $2k$ colors. Hence, $d_{G'}(x) = d_H(x) = k$. Using similar arguments, we can show that $d_{G'}(y) = d_H(y) = k$.

Claim 3.4. *$G^{'}$ does not contain a 3-vertex adjacent to three 2-vertices.*

Suppose $G^{'}$ contains a 3-vertex u adjacent to three 2-vertices v, w and x. By Claim 3.2, $d_H(u) = 3$. If v, w and x have 1-neighbors in H, let them be denoted by v_i, w_j and x_l respectively, where $i, j, l \in \{1, ..., k-2\}$. Let $H^{'} = H \setminus \{uv, uw, ux, vv_i, ww_j, xx_l\}$ for $i, j, l \in \{1, ..., k-2\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. Let $v^{'}$, $w^{'}$ and $x^{'}$ be the other neighbors of v, w and x respectively. By Claim 3.3, the vertices $v^{'}$, $w^{'}$ and $x^{'}$ have degree greater than 2. To extend this coloring, we color the edges uv, uw and ux in order. Color the edge uv with a color c_1 such that $c_1 \notin \varphi(v^{'}) \cup \{\varphi(ww^{'}), \varphi(xx^{'})\}$. Then, color the edge uw

with a color c_2 such that $c_2 \notin \varphi(w^{'}) \cup \{\varphi(vv^{'}), \varphi(xx^{'}), c_1\}$. Clearly, c_1 and c_2 are available colors for these edges. Now, color the edge ux with a color c_3 such that $c_3 \notin \varphi(x^{'}) \cup \{\varphi(vv^{'}), c_1, c_2\}$. It can be observed that, there is no bi-colored path or cycle of length four involving the color c_3 of the edge ux, as $\varphi(vv^{'}) \neq c_3$ and $c_3 \notin \varphi(x^{'})$ also $\varphi(xx^{'}) \neq c_2$ and $c_2 \notin \varphi(w^{'})$.

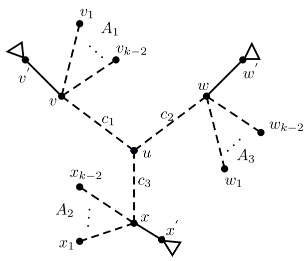

Figure 3. Illustration of Claim 3.4

Next, we choose three sets of colors, say, A_1,A_2 and A_3, such that (i) each color in A_1 is not in $\varphi(v^{'}) \cup \{c_1, c_3\}$, (ii) each color in A_2 is not in $\varphi(x^{'}) \cup \{c_2, c_3\}$ and (iii) each color in A_3 is not in $\varphi(w^{'}) \cup \{c_1, c_2\}$. As $\varphi(ww^{'}) \neq c_1$, c_2 can be in A_1 and $|A_1| \geq 2k-(k+2) = k-2$, we color the edges vv_i, $i \in \{1,...,k-2\}$ with the colors from A_1. As $c_3 \notin \varphi(v)$, c_1 can be in A_2 and $|A_2| \geq 2k-(k+2) = k-2$, we color the edges xx_l, $l \in \{1,...,k-2\}$ with the colors from A_2. As $c_2 \notin \varphi(x)$, c_3 can be in A_3 and $|A_3| \geq 2k-(k+2) = k-2$, we color the edges ww_j, $j \in \{1,...,k-2\}$ with the colors from A_3. This coloring is depicted in Figure 3. Hence, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H with $2k$ colors.

Claim 3.5. *$G^{'}$ does not contain a 3-vertex adjacent to two 3_2-vertices.*

Suppose $G^{'}$ contains a 3-vertex v with $N(v) = \{u, v^{'}, w\}$, where u and w are 3_2-vertices. Let $N(w) = \{v, x_1, x_2\}$ and $N(u) = \{v, x_3, x_4\}$. For $i \in \{1,...,4\}$ let x_i be the 2-vertices in $G^{'}$. Let z_i be the other neighbors of x_i. The vertices u, v, w and x_i's may have 1-neighbors in H. Let them be denoted by u_j, v_l, w_m and y^i_n respectively for $i \in \{1,...,4\}$, $n \in \{1,...,k-2\}$ and $j,m,l \in \{1,...,k-3\}$.

Let $H^{'} = H \setminus \{ww_1\}$ as shown in Figure 4. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. As

$|F(ww_1)| \leq |\varphi(v) \cup \varphi(x_1) \cup \varphi(x_2) \cup \varphi(w)| \leq 3+2+2+(k-4) = k+3$ and $k \geq 4$, there is at least one color available for ww_1. The colors used on the pendant edges incident to the vertices v, x_1 and x_2 may be available for ww_1. So, we can extend the coloring φ of $H^{'}$ to a star edge coloring of H with $2k$ colors. Therefore, $d_H(w) = 3$. Using similar arguments we can conclude that $d_H(u) = 3$. Hence, by Claim 3.3, any of the 2-vertices x_i, $i \in \{1,...,4\}$ are not adjacent.

If $x_1 = x_3$, then let y_1^i, $i \in \{1,...,k-2\}$ be the 1-vertices adjacent to x_1. Consider $H^{'} = H \setminus \{x_1y_1^1\}$, which by minimality of H, has a star edge coloring φ with $2k$ colors. As $|F(x_1y_1^1)| \leq |\varphi(u) \cup \varphi(w) \cup \varphi(x_1)| \leq 3+3+(k-3) = k+3$ and $k \geq 4$, there is at least one color available for $x_1y_1^1$. So, the coloring φ of $H^{'}$ can be extended to H. Therefore, $d_H(x_1) = 2$.

Now, consider $H^{'} = H \setminus \{x_1u\}$, which by minimality of H, has a star edge coloring φ with $2k$ colors. As $|F(x_1u)| \leq |\varphi(x_4) \cup \varphi(v) \cup \varphi(w)| \leq 7$ and $k \geq 4$, there is at least one color available for x_1u. Therefore, $x_1 \neq x_3$. Similarly, we can show that $x_1 \neq x_4$, $x_2 \neq x_3$ and $x_2 \neq x_4$.

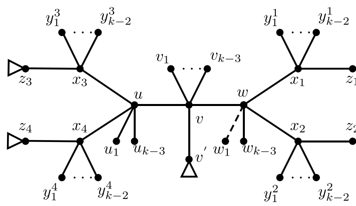

FIGURE 4. Illustration for $d_H(w) = d_H(u) = 3$

Now, let $H^{'} = H \setminus \{wx_1, wx_2, x_iy_n^i\}$, for $i \in \{1,2\}$, $n \in \{1,...,k-2\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. Suppose we are not able to extend this coloring to all the edges of H. In order to extend this coloring, we recolor the edge vw as follows. Before recoloring, let $\varphi(vw) = a$. As $d_H(u) = 3$, there is at least one color, say, $b \notin \varphi(u) \cup \varphi(v^{'})$. If b do not appear on any of the pendant edges incident to v, we set $\varphi(vw) = b$. Otherwise, we swap the colors a and b. So, the edge vw is colored with a color that do not appear in $\varphi(u) \cup \varphi(v^{'}) \cup \{\varphi(vv_l)\}$ for $l \in \{1,...,k-3\}$. After this recoloring, let $\varphi(vw) = c_1$. Now, if $d_H(v) = 3$, we can easily extend the coloring to H. So, we assume

that v is incident with pendant edges. (Situation (*))
We extend the coloring φ of $H^{'}$ to H in following two cases.
Case 1: Either $|\varphi(v) \cap \varphi(z_1)| \geq 1$ or $|\varphi(v) \cap \varphi(z_2)| \geq 1$. By symmetry, we can assume that $|\varphi(v) \cap \varphi(z_1)| \geq 1$. We first color the edges wx_1 and wx_2 as follows.
(i) When $|\varphi(v) \cap \varphi(z_1)| \geq 1$ and $|\varphi(v) \cap \varphi(z_2)| > 1$, (there is at least one color unused at v and z_1 and at least two at v and z_2) we first color the edge wx_1 with a color c_2 such that $c_2 \notin \varphi(v) \cup \varphi(z_1)$. Then, we color wx_2 with a color c_3 such that $c_3 \notin \varphi(v) \cup \varphi(z_2) \cup \{c_2\}$. There is at least one such color for wx_2.
(ii) When $|\varphi(v) \cap \varphi(z_1)| > 1$ and $|\varphi(v) \cap \varphi(z_2)| = 1$, (there are at least two colors unused at v and z_1 and exactly one at v and z_2) we first color the edge wx_2 with a color c_3 such that $c_3 \notin \varphi(v) \cup \varphi(z_2)$. Then, we color wx_1 with a color c_2 such that $c_2 \notin \varphi(v) \cup \varphi(z_1) \cup \{c_3\}$. There is at least one such color for wx_1.
(iii) When $|\varphi(v) \cap \varphi(z_1)| = 1$ and $|\varphi(v) \cap \varphi(z_2)| = 1$, (there is exactly one color unused at v and z_1 and exactly one at v and z_2) we color the edges wx_1 and wx_2 as follows.
(a) If $c_1 \neq \varphi(x_1z_1)$ and $c_1 \neq \varphi(x_2z_2)$, we first color the edge wx_1 with a color, say, c_2 such that $c_2 \notin \varphi(v) \cup \varphi(z_1)$. Then, we color the edge wx_2 with any available color, say, c_3. As $c_1 \notin \varphi(u) \cup \varphi(v^{'}) \cup \varphi(vv_l)$ for $l \in \{1, \ldots, k-3\}$, $F|(wx_2)| \leq |\varphi(z_2) \cup \{c_1, c_2\}| = k+2$. Since $k \geq 4$, there are at least $(2k-(k+2) = k-2 \geq 2)$ two colors available for wx_2.
(b) If $c_1 = \varphi(x_1z_1)$ and $c_1 \neq \varphi(x_2z_2)$, we color the edges wx_1 and wx_2 in order in similar way as given in (a) above.
(c) If $c_1 \neq \varphi(x_1z_1)$ and $c_1 = \varphi(x_2z_2)$, we first color the edge wx_2 with a color, say, c_3 such that $c_3 \notin \varphi(v) \cup \varphi(z_2)$. Then, we color the edge wx_1 with any available color, say, c_2. As $c_1 \notin \varphi(u) \cup \varphi(v^{'}) \cup \varphi(vv_l)$ for $l \in \{1, \ldots, k-3\}$, $F|(wx_1)| \leq |\varphi(z_1) \cup \{c_1, c_3\}| = k+2$. Since $k \geq 4$, there are at least $(2k-(k+2) = k-2 \geq 2)$ two colors available for wx_1.
(d) If $c_1 = \varphi(x_1z_1) = \varphi(x_2z_2)$, then first we recolor the edge vw with a color other than c_1. If there is a color, say, $c \notin \varphi(u) \cup \varphi(v^{'}) \cup \{\varphi(vv_l), c_1\}$ for $l \in \{1, \ldots, k-3\}$, we set $\varphi(vw) = c$ and remove the color of one of the pendant edge incident to v and color it with color c_1, so that the condition $|\varphi(v) \cap \varphi(z_1)| = 1$ and $|\varphi(v) \cap \varphi(z_2)| = 1$ is not disturbed. Then, we proceed as in (a) above. Otherwise, we swap the colors of one of the pendant edge incident to v and

c_1. So, $\varphi(vw) \neq \varphi(x_1z_1) = \varphi(x_2z_2)$ and appears on at most one of the vertices u and $v^{'}$. Then, we color the edge wx_1 as it is colored in (a) and the edge wx_2 is colored with an available color. After recoloring, let the colors of vw, wx_1 and wx_2 be called as c_1, c_2 and c_3 respectively.
(iv) If $|\varphi(v) \cap \varphi(z_1)| \geq 1$ and $|\varphi(v) \cap \varphi(z_2)| = 0$, (there is at least one color unused at v and z_1 and no color unused at v and z_2) we extend the coloring φ to the edges wx_1 and wx_2 in order in similar way as they are colored in (iii)(a).
(v) If $|\varphi(v) \cap \varphi(z_2)| \geq 1$ and $|\varphi(v) \cap \varphi(z_1)| = 0$, (there is at least one color unused at v and z_2 and no color unused at v and z_1) we color the edges wx_2 and wx_1 in order in similar way as they are colored in (iii)(c).

Now, we color the pendant edges incident to the vertices x_1 and x_2.

- In cases when the edge wx_1 is colored first, we choose two sets of colors, say, A_1 and A_2 such that each color in A_1 is not in $\varphi(z_1) \cup \{c_2, c_3\}$ and each color in A_2 is not in $\varphi(z_2) \cup \{c_1, c_3\}$. As $c_1 \notin \varphi(u) \cup \varphi(v^{'})$ and $c_1 \notin \varphi(z_1)$, c_1 can be in A_1 and as $c_3 \notin A_1$, c_2 can be in A_2. Since $|\varphi(z_1) \cup \{c_2, c_3\}| \leq k+2$, $|A_1| \geq k-2$ and $|\varphi(z_2) \cup \{c_1, c_3\}| \leq k+2$, $|A_2| \geq k-2$. So, we color the edges $x_1y_n^1$ with colors from A_1 and the edges $x_2y_n^2$, $n \in \{1, ...k-2\}$ with colors from A_2.
- In cases when the edge wx_2 is colored first, we choose two sets of colors, say, A_1 and A_2 such that each color in A_2 is not in $\varphi(z_2) \cup \{c_2, c_3\}$ and each color in A_1 is not in $\varphi(z_1) \cup \{c_1, c_3\}$. As $c_1 \notin \varphi(u) \cup \varphi(v^{'})$ and $c_1 \notin \varphi(z_2)$, c_1 can be in A_2 and as $c_3 \notin A_2$, c_3 can be in A_1. Since $|\varphi(z_2) \cup \{c_2, c_3\}| \leq k+2$, $|A_1| \geq k-2$ and $|\varphi(z_1) \cup \{c_1, c_3\}| \leq k+2$, $|A_2| \geq k-2$. So, we color the edges $x_1y_n^1$ with colors from A_1 and the edges $x_2y_n^2$, $n \in \{1, ...k-2\}$ with colors from A_2.

In all the above cases, any bi-colored P_5 or C_4 is not created. Hence, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H with $2k$ colors.
Case 2: When $|\varphi(v) \cap \varphi(z_1)| = 0$ and $|\varphi(v) \cap \varphi(z_2)| = 0$.

In this case, we extend the coloring to all the edges of H, by deriving key observations and recoloring certain edges, ensuring that the edges incident to v satisfy the condition in Case 1. We consider the same graph $H^{'}$ and proceed from the Situation (*).
Observation 1: $d_H(z_1) = d_H(z_2) = d_H(v) = k$.
If the degree of any one of the vertices v, z_1 and z_2 is less than k,

then we get one of the following three situations.
(i) If $d(v) < k$, then there is at least one color unused at v and z_1 and there is at least one color unused at v and z_2. **(ii)** If $d(z_1) < k$, then there is at least one color unused at v and z_1 and no color unused at v and z_2. **(iii)** If $d(z_2) < k$, then there is at least one color unused at v and z_2 and no color unused at v and z_1. Since here, $|\varphi(v) \cap \varphi(z_1)| = 0$ and $|\varphi(v) \cap \varphi(z_2)| = 0$, we have $\varphi(vw) \neq \varphi(x_1z_1)$ and $\varphi(vw) \neq \varphi(x_2z_2)$. So, in all the three cases above, we can easily extend φ from $H^{'}$ to H by using the coloring as given in Case 1 (iii)(a), (iv) and (v) respectively. Therefore, we assume that $d_H(z_1) = d_H(z_2) = d_H(v) = k$.

Now, suppose that φ cannot be extended to H. As $|\varphi(v) \cap \varphi(z_1)| = 0$ and $|\varphi(v) \cap \varphi(z_2)| = 0$, without loss of generality, we can assume the following coloring. Let $\varphi(v) = \{1, ..., k\}$, $\varphi(z_1) = \varphi(z_2) = \{k+1, ..., 2k\}$. Let $\varphi(vw) = c_1 = 1$, $\varphi(vv^{'}) = 2$ and $\varphi(uv) = 3$. In order to extend this coloring, we try to recolor the edge vw with a color other than the colors $1, ..., k$. If we get this desired color for vw, we can extend the coloring using Case 1. Otherwise, all the $2k-1$ colors are forbidden for vw.
Note: On any path, say, $v_1v_2v_3v_4$, if v_1v_2 is colored with color 1, v_2v_3 is colored with color 2 and v_3v_4 is colored with color 3, then for our convenience we write that the path $v_1v_2v_3v_4$ is colored with colors 1-2-3. In the figures, while recoloring an edge, we strike out the existing color on the edge and show new color on it.
Observation 2: $d_H(v^{'}) = k$.
Suppose $d_H(v^{'}) = k-1$. Since there is no desired color for vw in $H^{'}$, it means all the colors that appear at $v^{'}$, v and u are forbidden for vw. Therefore, $\varphi(uv) = 3 \in \varphi(x_3) \cap \varphi(x_4)$. Let $\varphi(ux_3) = c_4$ and $\varphi(ux_4) = c_5$. Note that, $c_4, c_5 \in \{k+1, ..., 2k\}$. Now, we uncolor the edge vw. If we can recolor the edge uv with color 1 and then recolor the edge vw with color c_4 or c_5, we are done. Otherwise, we can assume that $1 \in \varphi(x_3)$ or $1 \in \varphi(x_4)$. If we can recolor the edge uv with any of the color present on one of the pendant edges vv_l, $l \in \{1, ...k-2\}$, say, $\varphi(vv_1)$ and then recolor the edge vv_1 with color c_4 or c_5, we are done. Otherwise, we can assume that all the colors on vv_l, $l \in \{1, ..., k-3\}$ are present in $\varphi(x_3)$ or in $\varphi(x_4)$. It can be observed that, out of all the colors $1, 3, 4, ..., k$ at most two colors that appear on the edges x_3z_3 and x_4z_4 may not be available for

uv and the remaining colors are available for uv that appear on the pendant edges incident to both x_3 and x_4.

In this situation, if color 3 appears on one of the pendant edges incident to the vertex x_3 and also on one of the pendant edges incident to the vertex x_4, let $\varphi(x_3y_1^3) = \varphi(x_4y_1^4) = 3$. We choose the path $y_1^3x_3uv$ colored with colors 3-c_4-3 and recolor it with the colors c_4-3-c_4. If this recoloring of path does not create any bi-colored P_5 or C_4, we are done. Else, choose another path $y_1^4x_4uv$ colored with colors 3-c_5-3 and recolor it with colors c_5-3-c_5. Finally, retain $\varphi(vw) = 1$. One possible coloring is depicted in Figure 5.

If color 1 appears on one of the pendant edges incident to the vertex x_3 and also on one of the pendant edges incident to the vertex x_4, let $\varphi(x_3y_1^3) = \varphi(x_4y_1^4) = 1$. Then, we choose the path $y_1^3x_3uvw$ colored with colors 1-c_4-3-1 and recolor it with colors c_4-1-c_4-3. If this recoloring of path does not create any bi-colored P_5 or C_4, we are done. Else, we choose another path $y_1^4x_4uvw$ colored with colors 1-c_5-3-1 and recolor it with the colors c_5-1-c_5-3.

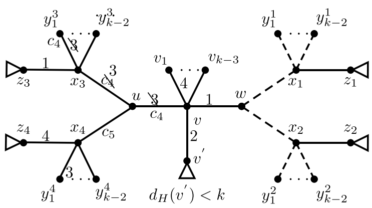

Figure 5. Illustration for Observation 2

Otherwise, choose a color, say, $c_6 \notin \{1, 3\}$ that appears on one of the pendant edges incident to all the vertices x_3, x_4 and v. We can always find such a color as there is no color available for uv such that vw can be colored either with c_4 or with c_5. Let $\varphi(x_3y_1^3) = \varphi(x_4y_1^4) = \varphi(vv_1) = c_6$. We choose the path $y_1^3x_3uvv_1$ colored with colors c_6-c_4-3-c_6 and recolor it with colors c_4-c_6-c_4-3. If this recoloring of path does not create any bi-colored P_5 or C_4, we are done. Else, choose another path $y_1^4x_4uvv_1$ colored with colors c_6-c_5-3-c_6 and recolor it with colors c_5-c_6-c_5-3. Finally, set $\varphi(vw) = 1$.

In all the above cases, we get either color c_4 or c_5 on the vertex v. Hence, we can extend the coloring φ of $H^{'}$ to a star edge coloring of H using the coloring as in Case 1(iii). This is possible because of the assumption that $d_H(v^{'}) < k$. Therefore, we conclude that $d_H(v^{'}) = k$.

Observation 3:

Consider the Situation (*). Suppose, we do not get any desired color for vw. This means all the colors that appear on the vertices $v^{'}$, u and v are forbidden for vw. The number of edges incident to the vertices $v^{'}$, v and u sum up to $k+(k-1)+2 = 2k+1$. Let $V^{'} = \{\varphi(vv^{'}), \varphi(v^{'}s_p)\}$, $p \in \{1,...,k-1\}$ be the set of colors used on the edges incident to $v^{'}$. Let $V = \{\varphi(vw), \varphi(uv), \varphi(vv_l)\}$, $l \in \{1,...,k-3\}$ be the set of colors used on the edges incident to v (except on $vv^{'}$). Let $U = \{\varphi(ux_3), \varphi(ux_4)\}$ be the set of colors used on the edges incident to u (except on uv). As all the $2k-1$ colors forbidden for vw appear on $2k$ edges, we can observe that there is exactly one color common among any two of the sets $V^{'}$, V and U.

Based on the observations above, we consider the following three subcases.

Subcase 2.1: When $|U \cap V| = 1$.

Let $U \cap V = \{\varphi(vv_l) = \varphi(ux_4)\}$, for some $l \in \{1,...,k-3\}$. As $\varphi(v) = \{1,...,k\}$, let $\varphi(vv_l) = \{4,5,...,k\}$, for $1 \le l \le k-3$. Let for $l = 1$, $\varphi(vv_1) = 4 = \varphi(ux_4)$ and $\varphi(ux_3) = c_4$. As there is no color available for vw other than 1, $\varphi(uv) = 3 \in \varphi(x_3)$. It can be noted that $\varphi(uv) = 3 \notin \varphi(x_4)$.

Subcase 2.1(A): When color 3 is on one of the pendant edges incident to x_3.

As $|U \cap V| = 1$, let $\varphi(vv_1) = 4 = \varphi(ux_4)$. Let $\varphi(x_3y_1^3) = 3$. We uncolor the edge $x_3y_1^3$. If we get an available color for the edge $x_3y_1^3$ other than 3, then we can uncolor vv_1 and recolor it with the color c_4 and we are done. Otherwise, all the colors that appear on the vertices z_3, x_3 and u are forbidden for $x_3y_1^3$. Therefore, we choose the path x_3uvv_1 colored with colors c_4-3-4 and recolor it with colors 3-c_4-3 as shown in Figure 6(**a**). Now, since $3 \notin \varphi(x_4)$, $|F(x_3y_1^3)| \le |\varphi(z_3) \cup \varphi(x_3) \cup \varphi(u)| = k+(k-2)+1 = 2k-1$. Therefore, there is at least one color available for the edge $x_3y_1^3$. So, we get the color c_4 on the vertex v.

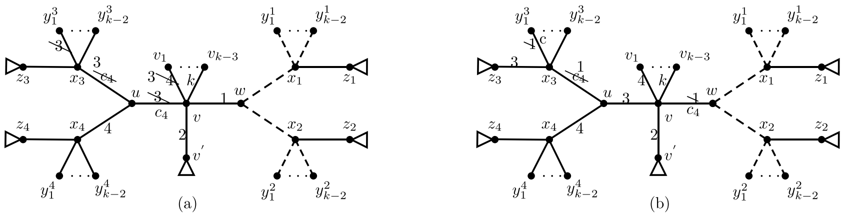

FIGURE 6. Illustration of Claim 3.5, (**a**) Subcase 2.1(A) (**b**) Subcase 2.1(B)(i)(a)

Subcase 2.1(B): When $\varphi(x_3z_3) = 3$.
Here, we uncolor the edge vw. If color 1 is available for the edge uv and c_4 for vw, we are done. Otherwise, we have one of the following three situations:
(i) $1 \in \varphi(x_3)$ but $1 \notin \varphi(x_4)$.
(a) If $\varphi(x_3y_1^3) = 1$ and $1 \notin \varphi(z_3)$, we uncolor the edges $x_3y_1^3$, ux_3 and vw. Then, set $\varphi(ux_3) = 1$ and $\varphi(vw) = c_4$. Since $1 \notin \varphi(x_4)$, $1 \notin \varphi(v)$ and color 3 is a common color on the vertices z_3 and u, $|F(x_3y_1^3)| \leq |\varphi(z_3) \cup \varphi(x_3) \cup \varphi(u)| = k + (k-2) + 0 = 2k-2$. So, there are at least two colors available for the edge $x_3y_1^3$. Hence, we get the color c_4 on the vertex v as shown in Figure 6(**b**).
(b) If $\varphi(x_3y_1^3) = 1$ and $1 \in \varphi(z_3)$, we uncolor the edges $x_3y_1^3$, uv and vw. As $1 \notin \varphi(x_4)$, we set $\varphi(uv) = 1$ and $\varphi(vw) = c_4$. Since color 1 is a common color on the vertices z_3 and u and color c_4 may be present on x_4, $|F(x_3y_1^3)| \leq |\varphi(z_3) \cup \varphi(x_3) \cup \varphi(u)| = k + (k-2) + 1 = 2k-1$. So, there is at least one color available for the edge $x_3y_1^3$. Hence, we get the color c_4 on the vertex v.
(ii) $1 \in \varphi(x_4)$ but $1 \notin \varphi(x_3)$.
(a) If color 1 is on one of the pendant edges incident to x_4, let $\varphi(x_4y_1^4) = 1$. Here, we uncolor the edges uv, vv_1 and vw. Since $1 \notin \varphi(x_3)$, we set $\varphi(uv) = 1$, $\varphi(vw) = 3$, and $\varphi(vv_1) = c_4$ and we are done.
(b) If $\varphi(x_4z_4) = 1$, we uncolor the edges ux_3, vv_1, vw and all the pendant edges incident to x_3. Now, if there is an available color for the edge ux_3 other than c_4, say, c_7, then we set $\varphi(ux_3) = c_7$ and $\varphi(vv_1) = c_4$. Since color 3 is a common color on the vertices z_3 and u, there is a set A of colors such that each color in $A \notin \varphi(z_3) \cup \{ux_3, ux_4\}$. As $|A| \geq 2k - |\varphi(z_3) \cup \{ux_3, ux_4\}| = 2k - (k+2) = k-2$, we

color all the pendant edges incident to the vertex x_3 with colors from the set A and we are done.

Otherwise, it can be observed that all the colors in $\varphi(z_3)\cup\varphi(v)\cup\{\varphi(ux_4),\varphi(x_4z_4)\}$ are forbidden for ux_3 and we can assume that color $4 \in \varphi(z_4)$ and $c_4 \notin \varphi(z_3)$. Also note that, c_4 do not appear at v'. So, we remove the color of the edge uv and then assign $\varphi(ux_3) = 2$, $\varphi(uv) = c_4$, $\varphi(vw) = 1$ and $\varphi(vv_1) = 3$. Now, we color the pendant edges incident to x_3. If $2 \notin \varphi(x_4)$, then we use all the colors $\notin \varphi(z_3)\cup\{ux_3, uv\}$ for coloring these edges. As $|\varphi(z_3)\cup\{ux_3, uv\}| \leq k+2$, there are $(2k-(k+2) \geq k-2)$ enough colors to color these pendant edges. So, we are done. Else, when $2 \in \varphi(x_4)$, it can only be on one of the pendant edges incident to x_4, as $\varphi(x_4z_4) = 1$. Let $\varphi(x_4y_1^4) = 2$. We uncolor the pendant edge $x_4y_1^4$ and color the pendant edges incident to x_3 with colors from the set A as above for case $2 \notin \varphi(x_4)$. Then, recolor the edge $x_4y_1^4$ with a color other than 2. Since color 4 is a common color at the vertices z_4 and u and after recoloring, $4 \notin \varphi(v)$, $|F(x_4y_1^4)| \leq |\varphi(z_4)\cup\varphi(x_4)\cup\varphi(u)| \leq k+(k-3)+1 = 2k-2$. So, there are at least two colors available for the edge $x_4y_1^4$. Therefore, we get the color c_4 on the vertex v.

(iii) $1 \in \varphi(x_3)$ and $1 \in \varphi(x_4)$.

(a) If $\varphi(x_4z_4) = 1$, then one of the pendant edges incident to x_3, say, $\varphi(x_3y_1^3) = 1$ as $\varphi(x_3z_3) = 3$. So, in this case, we can proceed in the similar way as given in Subcase 2.1(B)(ii)(b).

(b) If color 1 is on one of the pendant edges incident to x_4, let $\varphi(x_4y_1^4) = 1$. Here, since $\varphi(x_3z_3) = 3$, we have $\varphi(x_3y_1^3) = 1$. If there is an available color for the edge $x_3y_1^3$ other than 1, we color it with that color and then proceed in the similar way as given in Subcase 2.1(B)(ii)(a). Otherwise, color $1 \notin \varphi(z_3)$. So, we uncolor the edges $x_3y_3^1$, ux_3, vv_1 and vw and recolor these edges so that $\varphi(ux_3) = 1$, $\varphi(vv_1) = c_4$ and $\varphi(vw) = 4$. As $\varphi(x_4z_4) \neq 1$, $1 \notin \varphi(v)$ and $1 \notin \varphi(z_3)$ so, 1 is available for ux_3. As 3 is a common color at the vertices z_3 and u, $|F(x_3y_1^3)| \leq |\varphi(z_3)\cup\varphi(x_3)\cup\varphi(u)| \leq k+(k-2)+1 = 2k-1$. So, there is at least one color available for $x_3y_1^3$. Therefore, we get the color c_4 on the vertex v.

Note: In the above subcase, if $U\cap V = \{1\}$ and $\varphi(vw) = 1 = \varphi(ux_4)$, let $\varphi(vv_1) = 4$. First, we swap the colors of the edges vw and vv_1.

Then, we follow the above coloring except that everywhere we use color 1 for color 4 and color 4 for color 1.

Subcase 2.2: When $|V^{'} \cap U| = 1$.

Let $\varphi(v^{'}s_1) = \varphi(ux_4) = c_6 \in V^{'} \cap U$. Recall that, there is no color for vw other than 1. This means that $\varphi(uv) = 3 \in \varphi(x_3)$.

Subcase 2.2(A): When color 3 is on one of the pendant edges incident to x_3.

Let $\varphi(x_3y_1^3) = 3$. We uncolor the edge $x_3y_1^3$. If we get an available color for this edge other than 3, we color $x_3y_1^3$ with that color and then we uncolor the edge vv_1 and recolor it with color c_4. So, we are done. Otherwise, we can choose the path $y_1^3x_3uv$ colored with colors 3-c_4-3 and recolor it with colors c_4-3-c_4. So, we get the desired color on the vertex v as shown in Figure 7(**a**).

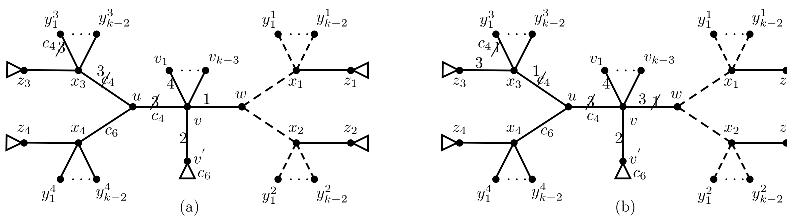

FIGURE 7. Illustration of Claim 3.5,(**a**) Subcase 2.2(A) (**b**) Subcase 2.2(B)(i)

Subcase 2.2(B): When $\varphi(x_3z_3) = 3$.

We uncolor the edges vw and uv. If 1 is an available color for uv and c_4 for vw, we are done. Otherwise, we have any one of the following situations.

(i) $1 \in \varphi(x_3)$ and $1 \notin \varphi(x_4)$.

Let $\varphi(x_3y_1^3) = 1$. We choose the path $y_1^3x_3uvw$ colored with colors 1-c_4-3-1 and recolor it with colors c_4-1-c_4-3. So, we get the desired color on v as shown in Figure 7(**b**).

(ii) $1 \notin \varphi(x_3)$ and $1 \in \varphi(x_4)$.

(a) If color 1 is on one of the pendant edges incident to x_4, let $\varphi(x_4y_1^4) = 1$. We uncolor the edges uv, vv_1 and vw and recolor these edges such that $\varphi(uv) = 1$, $\varphi(vv_1) = c_4$ and $\varphi(vw) = 4$.

(b) If $\varphi(x_4z_4) = 1$ and $c_6 \notin \varphi(z_4)$, we can easily extend the coloring as given in Subcase 2.2(B)(ii)(a). When $c_6 \in \varphi(z_4)$, if 4 is an available color for the edge uv and c_4 is an available color for the edge vv_1, we are done. Otherwise, we can assume that $4 \in \varphi(x_3)$. Let $x_3y_1^3 = 4$. We uncolor the edge $x_3y_1^3$ and set $\varphi(uv) = 4$ and $\varphi(vv_1) = c_4$. Now, if we get an available color for the edge $x_3y_1^3$, we are done. Else, we can choose the path $y_1^3x_3uvv_1$ and recolor it with the colors c_4-4-c_4-3. It can be easily checked that, this recoloring does not create any bi-colored P_5 or C_4 and we get the desired color on the vertex v.

(iii) $1 \in \varphi(x_3)$ and $1 \in \varphi(x_4)$.

(a) If color 1 is on one of the pendant edges incident to x_4, let $\varphi(x_4y_1^4) = 1$. As $\varphi(x_3y_1^3) = 1$, we choose the path $y_1^3x_3uvw$ colored with colors 1-c_4-3-4 and recolor it with the colors c_4-1-c_4-3. This recoloring does not create any bi-colored P_5 or C_4 and we get the desired color on the vertex v.

(b) If $\varphi(x_4z_4) = 1$ and $c_6 \notin \varphi(z_4)$, we can easily extend the coloring as given in Subcase 2.2(B)(iii)(a). Otherwise, when $c_6 \in \varphi(z_4)$, we follow the coloring as given in Subcase 2.2(B)(ii)(b).

SubCase 2.3: When $|V' \cap V| = 1$.

Let $V' \cap V = \{\varphi(vv_1) = \varphi(v's_1) = 4\}$. As there is no color available for vw other than 1, all $k-1$ colors at v', $k-2$ colors at v and 2 colors at u are forbidden for vw. Therefore, $\varphi(uv) = 3 \in \varphi(x_3) \cap \varphi(x_4)$. This case is similar to the assumption that $d_H(v') < k$. So, by using the coloring as given in Observation 2, we get the desired color on the vertex v.

In all the above subcases, we get the desired color on the vertex v and therefore the condition of Case 1(iii) is satisfied. So, we can extend the coloring φ of H' to H by using the coloring given in Case 1(iii).

Claim 3.6. *G' does not contain a 3-vertex v adjacent to **(i)** a 3_2-vertex w and a weak 3_1-vertex u or **(ii)** a 2-vertex w and a weak 3_1-vertex u.*

(i) Suppose G' contains a 3-vertex v adjacent to a 3_2-vertex w and a *weak* 3_1-vertex u. Let u be adjacent to a 3_2-vertex. In this case, the configuration is as given in Figure 4 except that the vertex x_4 is a 3_2-vertex. Let $N(x_4) = \{u, z_4, z_4'\}$, where z_4 and z_4' are 2-vertices. Let t_4 and t_4' be the neighbors of z_4 and z_4' respectively other than x_4. Let r_j and s_m be the 1-neighbors of z_4 and z_4' respectively for $j, m \in \{1, ..., k-2\}$. As shown in Claim 3.5, we can show that $d_H(w) = 3$. Using similar arguments, we have $d_H(x_4) = 3$. The vertex u may have 1- neighbors in H, let them be denoted by u_i, $i \in \{1, ..., k-3\}$.

Let the graph $H' = H \setminus \{ux_3, ux_4, uu_i, x_3y_n^3\}$ as shown in Figure 8. By minimality of H, the graph H' has a star edge coloring φ with $2k$ colors. First, we recolor the edge uv with a color, say, c_1 such that $c_1 \notin \varphi(w) \cup \varphi(v') \cup \varphi(vv_l)$ for $l \in \{1, ..., k-3\}$. This is done in a similar way as the edge vw is colored in Claim 3.5. Then, to color the edge ux_4 we choose a set of colors $U = \varphi(x_4) \cup \{\varphi(z_4t_4), \varphi(z_4't_4'), c_1\}$. If there is a color, say, $c_2 \notin U$ such that $c_2 \notin \varphi(z_4r_j) \cup \varphi(z_4's_m)$, for $j, m \in \{1, ..., k-2\}$, then we say c_2 is a required color for ux_4 so, we set $\varphi(ux_4) = c_2$. Now, let $U' = \varphi(z_3) \cup \{c_1, c_2\}$. If there is a color, say, $c_3 \notin U'$ such that either $c_3 \notin \varphi(v)$ or $c_3 \notin \varphi(x_4)$, then we say c_3 is a required color for ux_3 so, we set $\varphi(ux_3) = c_3$. Next, to color the pendant edges incident to x_3 we choose a set A of colors such that if $c_3 \notin \varphi(v)$, each color in A is not in $\varphi(z_3) \cup \{c_2, c_3\}$ and if $c_3 \notin \varphi(x_4)$, each color in A is not in $\varphi(z_3) \cup \{c_1, c_3\}$. As $|A| \geq k-2$, we color the edges $x_3y_n^3$, $n \in \{1, ..., k-2\}$ using the colors from A. Finally, it can be observed that the pendant edges incident to u can be easily colored as there are at most three forbidden colors for them. (*)

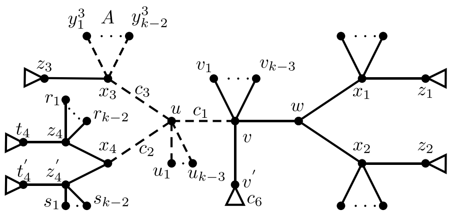

Figure 8. Illustration of Claim 3.6(i)

When $k > 4$ or $|U'| < k+2$, we can easily find the required color for the edge ux_3 which do not appear on the vertex x_4. When $k = 4$ and $|U'| = k+2$ and if we could not find any required color for ux_3, then the colors not in U' appear on x_4. So, here we recolor one of the edges x_4z_4 and x_4z_4' by swapping the color of the edge x_4z_4, say, a with a color on one of the pendant edges incident to z_4, say, a' or by swapping the color of the edge x_4z_4', say, b with a color on one of the pendant edges incident to $z_4^{'}$, say, $b^{'}$, whichever is possible. It can be observed that, if z_4 and $z_4^{'}$ are incident with pendant edges, then this swapping is possible for at least one of the edges x_4z_4 and $x_4z_4^{'}$. So, we set $\varphi(ux_3) = a$, if the color of x_4z_4 is swapped else, set $\varphi(ux_3) = b$. Then, we color the pendant edges incident to x_3 using the colors from a set A such that if the edge x_4z_4 is swapped, each color in A is not in $\varphi(z_3) \cup \{c_1, a\}$. Else, each color in A is not in $\varphi(z_3) \cup \{c_1, b\}$. Otherwise, we remove the color c_2 of the edge ux_4 and set $\varphi(ux_3) = c_2$. Then, we choose a required color other than c_2 for the edge ux_4. As z_4 and $z_4^{'}$ are not incident with pendant edges, we easily get such color, say, $c_2^{'}$ for ux_4. Then, we color the pendant edges incident to x_3 using the colors from a set A such that each color in A is not in $\varphi(z_3) \cup \{c_1, c_2\}$. Finally, the pendant edges incident to u can be easily colored as there are at most three forbidden colors for them.

If we could not find any required color for ux_4, then it means all the $2k$ colors are present in $\varphi(z_4) \cup \varphi(z_4^{'}) \cup \varphi(uv)$. As there are $2k + 1$ edges, we have at most one common color among any two of the vertices z_4, $z_4^{'}$ and u. So, we choose a color, say, $c_2^{'}$ which do not appear on the pendant edges incident to at least one of the vertices z_4 and $z_4^{'}$. Let $c_2^{'} \notin \varphi(z_4^{'})$ and $\varphi(z_4r_1) = c_2^{'}$. We uncolor the edge z_4r_1 and set $\varphi(ux_4) = c_2^{'}$. Then, we can easily find a color for z_4r_1 other than $c_2^{'}$ as $|F(z_4r_1)| \leq |\varphi(t_4) \cup \varphi(z_4) \cup \varphi(x_4)| \leq k+(k-2)+1 = 2k-1$. Next, we proceed the coloring as above in (*) to color the edge ux_3 and the remaining edges incident to x_3 and u except that we rename the color c_2 as $c_2^{'}$ in the given coloring. So, we can extend the coloring φ of $H^{'}$ to a star edge coloring of H.

Note that, this coloring can be extended in the similar way as above if x_2 is a 3_2-vertex. Therefore, we can also conclude that a 3_0-vertex v cannot be adjacent to two *weak* 3_1-vertices u and w.

(ii) Suppose $G^{'}$ contains a 3-vertex v adjacent to a 2-vertex w and a *weak* 3_1-vertex u. Let u be adjacent to a 3_2-vertex. In this case, the configuration is as given in Figure 8 except that the vertex w is a 2-vertex. Let x_1 be the other neighbor of w other than v. The vertex w may have 1-neighbors in H, let them be denoted by w_p for $p \in \{1, ..., k-2\}$.

Let the graph $H^{'} = H \setminus \{ux_3, ux_4, uu_i, x_3y_n^3\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. Here, if we get a color, say, c_1 such that $c_1 \notin \varphi(w) \cup \varphi(v^{'}) \cup \varphi(vv_l)$ for $l \in \{1, ..., k-3\}$, we say that c_1 is a required color for the edge uv and we set $\varphi(uv) = c_1$. Then, we can extend the coloring φ to $H^{'}$ by using the coloring as given in Claim 3.6(i).

If we could not get any required color for the edge uv, then we uncolor the pendant edges vv_l, $l \in \{1, ..., k-3\}$ and choose a set of colors $V = \varphi(v^{'}) \cup \{\varphi(vw), \varphi(wx_1)\}$. If $|V| < k+2$, then there are at least $k-1$ colors not in V. So, we easily get the required color for the edge uv and we use the remaining colors on the edges vv_l, $l \in \{1, ..., k-3\}$. Then, we proceed as in Claim 3.6(i). If $|V| = k+2$ and we could not get any required color for the edge uv, then w is incident with $k-2$ pendant edges and $|\varphi(w) \cap \varphi(v^{'})| = 0$. So, we uncolor one of the pendant edge, say, ww_1 colored with a color, say, c_1 and set $\varphi(uv) = c_1$. Now, we can easily color the edge ww_1 with a color other than c_1, as $\varphi(vw) \notin \varphi(v^{'})$ and $|F(ww_1)| \leq |\varphi(x_1) \cup \varphi(v) \cup \varphi(w)| \leq k + 1 + (k-3) = 2k-2$. Finally, we choose the colors not in $V \cup \{c_1\}$ to color the edges vv_l, $l \in \{1, ..., k-3\}$. As $|V \cup \{c_1\}| = k+3$, there are enough colors to color these edges. Then, we extend the coloring φ to $H^{'}$ by using the coloring as given in Claim 3.6(i).

Claim 3.7. *$G^{'}$ does not contain **(i)** a 4-vertex adjacent to two bad 2-vertices or **(ii)** a k-vertex, $k \geq 5$, adjacent to three bad 2-vertices.*

(i) Suppose that $G^{'}$ contains a 4-vertex u with $N(u) = \{u_i\}$ for $i \in \{1, ..., 4\}$, where u_1 and u_2 are two *bad* 2-vertices adjacent to the 2-neighbors v_1 and v_2 respectively. Let w_i be the other neighbors of v_i other than u_i, $i \in \{1, 2\}$. By Claim 3.3, we have $d_H(u_i) = d_H(v_i) = 2$ and $d_{G'}(u) = d_{G'}(w_i) = k = 4$.

Let $H^{'} = H \setminus \{u_1v_1\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. If we get an available color for the edge u_1v_1, we are done. Otherwise, all the colors used on the edges incident to the vertices u and w_1 are forbidden for u_1v_1. This means $|\varphi(u) \cap \varphi(w_1)| = 0$ and if $\varphi(uu_1) = c_1$, then $c_1 \in \varphi(u_2) \cap \varphi(u_3) \cap \varphi(u_4)$ forming bi-colored paths of length three and we have $\varphi(u_2v_2) = c_1$. So, we uncolor the edge u_2v_2. This makes the color of the edge uu_2, say, c_2 an available color for the edge u_1v_1. So, we set $\varphi(u_1v_1) = c_2$.

Now, if we get a color available for the edge u_2v_2, we are done. Otherwise, $|\varphi(w_2) \cap \varphi(u)| = 0$ and $c_2 \in \varphi(u_1) \cap \varphi(u_3) \cap \varphi(u_4)$. So, we uncolor the edge uu_2 and recolor it with a color other than c_2. As, $|F(uu_2)| \leq |\varphi(u) \cup \varphi(u_1) \cup \varphi(u_3) \cup \varphi(u_4)| \leq 3+1+1+1 = 6$, there are at least two colors available for uu_2 other than c_2. Then it is easy to see that there is at least one color available for the edge u_2v_2. So, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H with $2k$ colors.

(ii) Suppose $G^{'}$ contains a k-vertex u, $k \geq 5$, adjacent to three bad 2-vertices. Let $N(u) = \{u_i\}, 1 \leq i \leq k$. Let u_1, u_2 and u_3 be the *bad* 2-vertices adjacent to the 2-neighbors v_1, v_2 and v_3 respectively. Let w_i be the other neighbors of v_i other than u_i for $i \in \{1,2,3\}$. By Claim 3.3, we have $d_H(u_i) = d_H(v_i) = 2$ and $d_H(u) = d_H(w_i) = k$, for $i \in \{1,2,3\}$.

Let $H^{'} = H \setminus \{u_1v_1\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. If we get an available color for u_1v_1, we are done. Otherwise, all the colors used on the edges incident to the vertices u and w_1 are forbidden for u_1v_1. This means $|\varphi(u) \cap \varphi(w_1)| = 0$ and if $\varphi(uu_1) = c_1$, then $c_1 \in \bigcap_{i=2}^{k} \varphi(u_i)$ forming bi-colored paths of length three and we have $\varphi(u_2v_2) = \varphi(u_3v_3) = c_1$. So, we uncolor the edge u_2v_2. This makes the color of the edge uu_2, say, c_2 an available color for the edge u_1v_1. Now, we can observe that as $\varphi(u_3v_3) = c_1 \neq c_2$, $|F(u_2v_2)| \leq |\varphi(u) \cup \varphi(w_2)| \leq (k-1)+k = 2k-1$. There is at least one color available for the edge u_2v_2. Hence, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H with $2k$ colors.

Claim 3.8. *$G^{'}$ does not contain a 4-vertex u adjacent to **(i)** one bad 2-vertex and three weak 3_1-vertices or **(ii)** one bad 2-vertex and three 3_2-vertices or **(iii)** one bad 2-vertex and two 2-vertices.*

(i) Suppose $G^{'}$ contains a 4-vertex u with $N(u) = \{u_i\}$, $i \in \{1,...,4\}$, where u_1 is a *bad* 2-vertex adjacent to the 2-vertex v_1 and u_2, u_3 and u_4 are three *weak* 3_1-vertices which are adjacent to 3_2-vertices x_2, x_3 and x_4 respectively. Let w_1 be the other neighbor of v_1 other than u_1. By Claim 3.3, we have $d_H(u_1) = d_H(v_1) = 2$ and $d_{G'}(u) = d_{G'}(w_1) = k = 4$. For $i \in \{2,3,4\}$, let $x_i^{'}$ be the 2-neighbors of u_i and if these u_i's have 1-neighbors in H, let them be denoted by $u_i^{'}$. The vertices x_i and $x_i^{'}$ may also have 1-neighbors in H. As $d_G(u_i) = 3$, $i \in \{2,3,4\}$, all the two vertices $x_i^{'}$s are distinct from the two vertex v_1. If $x_2^{'} = x_3^{'}$, it is easy to see that $d_H(x_2^{'}) = d_H(x_3^{'}) = 2$ and in the graph $H^{'} = H \setminus \{u_2x_2^{'}\}$, which by minimality of H, has a star edge coloring φ with $2k$ colors we have, $|F(u_2x_2^{'})| \le |\varphi(u) \cup \varphi(x_2) \cup \varphi(u_3)| \le 7$. So, the coloring φ can be extended to H. Hence, $x_2^{'} \ne x_3^{'}$ and we conclude that for $i \in \{2,3,4\}$ all the 2-vertices $x_i^{'}$s are distinct.

Now, let $H^{'} = H \setminus \{u_1v_1\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. If there is no color available for u_1v_1, then $|\varphi(u) \cap \varphi(w_1)| = 0$ and $\varphi(uu_1) \in \varphi(u_2) \cap \varphi(u_3) \cap \varphi(u_4)$ forming bi-colored paths of length three. Therefore, wlog., we can assume this coloring as follows. Let $\varphi(w_1) = \{1,2,3,4\}$ and $\varphi(u) = \{5,6,7,8\}$. Let $\varphi(uu_2) = 6$, $\varphi(uu_3) = 7$ and $\varphi(uu_4) = 8$. If $\varphi(uu_1) = 5$, then $5 \in \bigcap_{i=2}^{4} \varphi(u_i)$.

If we can recolor the edge uu_1 with a color other than 5, then the color 5 becomes available for u_1v_1. So, we are done. Otherwise, the colors 1,2,3,4 appear on the edges u_ix_i and $u_ix_i^{'}$ ($i \in \{2,3,4\}$) such that if any color $c \in \{1,2,3,4\}$ is used on the edge uu_1, a bi-colored path of length four is created.

We can observe that, the color 5 must appear on one of the pendant edges incident to at least one of the vertices u_2, u_3 and u_4. Otherwise, when $5 \notin \varphi(u_iu_i^{'})$ for $i \in \{2,3,4\}$, we can recolor the edge uu_1 with a color other than 5. As $|F(uu_1)| \le |\bigcup_{i=2,3,4} \varphi(u_i)| \le 3+2+2 = 7$. The colors on the pendant edges incident to u_i may also be available for uu_1. This makes at least one color available for the edge u_1v_1. So, we are done. Therefore, we discuss the following cases when the color 5 appears on the pendant edges incident to the vertices u_i, $i \in \{2,3,4\}$.

Case 1: When color 5 appears on a pendant edge incident to exactly one of the u_i's, $i \in \{2,3,4\}$, say, u_2.

Let $\varphi(u_2u_2^{'}) = 5$. We uncolor the edge $u_2u_2^{'}$ and set $\varphi(u_1v_1) = \varphi(uu_2) = 6$. If there is an available color for the edge $u_2u_2^{'}$, we are done. Otherwise, all the colors incident to u, x_2 and $x_2^{'}$ are forbidden for $u_2u_2^{'}$ and the color 6 appears on at least one of the pendant edges $u_3u_3^{'}$ and $u_4u_4^{'}$. Let $u_3u_3^{'} = 6$. We uncolor $u_3u_3^{'}$ and set $\varphi(u_2u_2^{'}) = \varphi(uu_3) = 7$. Now, if there is an available color for the edge $u_3u_3^{'}$, we are done. Otherwise, all the colors incident to u, x_3 and $x_3^{'}$ are forbidden for $u_3u_3^{'}$ and the color 7 appears on the pendant edge $u_4u_4^{'}$. So, we uncolor the edge $u_4u_4^{'}$ and set $\varphi(u_3u_3^{'}) = \varphi(uu_4) = 8$. We can observe that as the color $8 \notin \varphi(u_2)$ and color 5 is a common color among two of the vertices x_4, $x_4^{'}$ and u, we have $|F(u_4u_4^{'})| \leq |\varphi(u) \cup \varphi(x_4) \cup \varphi(x_4^{'})| \leq 7$. So, we can easily color the edge $u_4u_4^{'}$. Hence, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H. One possible coloring for this situation is depicted in Figure 9(**a**).

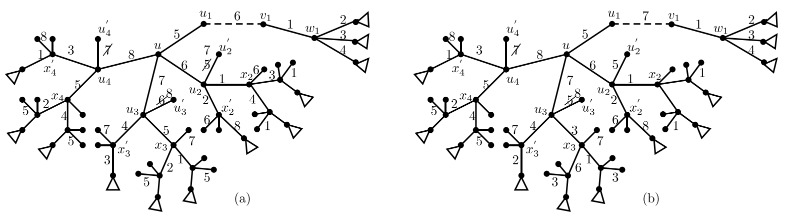

Figure 9. Configuration of Claim 3.8(i) (**a**)Case 1 (**b**)Case 2

Case 2: When color 5 appears on a pendant edge incident to exactly two of the u_i's, $i \in \{2,3,4\}$, say, u_2 and u_3.

Let $\varphi(u_2u_2^{'}) = \varphi(u_3u_3^{'}) = 5$. As there is no color for uu_1 other than 5, any two colors out of the four colors 1,2,3 and 4 must appear on at least one of the vertices u_2 and u_3. Let that be u_2. So, we uncolor the edge $u_3u_3^{'}$ (other than u_2) and set $\varphi(u_1v_1) = \varphi(uu_3) = 7$. If there is an available color for the edge $u_3u_3^{'}$, we are done. Otherwise, all the colors incident to u, x_3 and $x_3^{'}$ are forbidden for $u_3u_3^{'}$ and the color 7 appears on the vertex u_4. If $\varphi(u_4u_4^{'}) = 7$, we uncolor the edge $u_4u_4^{'}$ and set $\varphi(u_3u_3^{'}) = \varphi(uu_4) = 8$. As the color $8 \notin \varphi(u_2)$ and color 5 is a common color among two of the vertices x_4, $x_4^{'}$ and u, we have at least one color for $u_4u_4^{'}$. So, we are done. One possible coloring for this situation is depicted in Figure 9(**b**). Otherwise, retain the color 5 of the edge $u_3u_3^{'}$ and uncolor the edges u_1v_1 and $u_2u_2^{'}$. Then set $\varphi(u_1v_1) = 6$. Now if there is an available color for the edge $u_2u_2^{'}$, we are done. Otherwise, all the colors incident to u, x_2 and $x_2^{'}$ are forbidden for $u_2u_2^{'}$ and the color 6 appears on the pendant edge $u_4u_4^{'}$. We uncolor $u_4u_4^{'}$ and set $\varphi(u_2u_2^{'}) = 8$. Then, we recolor $u_4u_4^{'}$ with a color other than 6. As $8 \notin \varphi(u_3)$ and color 5 is a common color among two of the vertices x_4, $x_4^{'}$ and u, we have at least one color for $u_4u_4^{'}$.

Case 3: When color 5 appears on a pendant edge incident to all of the u_i's, $i \in \{2,3,4\}$.

Let $\varphi(u_2u_2^{'}) = \varphi(u_3u_3^{'}) = \varphi(u_4u_4^{'}) = 5$. If all the edges u_ix_i and $u_ix_i^{'}$ are colored with colors from $\{1,2,3,4\}$, then we uncolor the edge $u_2u_2^{'}$ and set $\varphi(u_1v_1) = 6$. As the color $6 \notin \varphi(u_3) \cup \varphi(u_4)$, there is at least one color available for $u_2u_2^{'}$. So, we are done. Therefore, we assume that at least one of the three vertices u_2, u_3 and u_4 are incident with at least one color other than the colors 1,2,3,4,5. Let that be u_3. We uncolor the edge $u_3u_3^{'}$ and set $\varphi(u_1v_1) = 7$. If there is an available color for the edge $u_3u_3^{'}$, we are done. Otherwise, all the colors incident to u, x_3 and $x_3^{'}$ are forbidden for $u_3u_3^{'}$ and the color 7 appears on one of the vertices u_2 and u_4. Let $7 \in \varphi(u_4)$. Here, we uncolor u_1v_1 and retain the color 5 of the edge $u_3u_3^{'}$. Then, we uncolor the edge $u_4u_4^{'}$ and set $\varphi(u_1v_1) = 8$. As the color $8 \notin \varphi(u_2) \cup \varphi(u_3)$, there is at least one color available for $u_4u_4^{'}$. Hence, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H.

(ii) This can be proved by similar arguments as in Claim 3.8(i).

(iii) Suppose $G^{'}$ contains a 4-vertex u with $N(u) = \{u_i\}$, $i \in \{1,...,4\}$, where u_1 is a *bad* 2-vertex adjacent to the 2-neighbor v_1 and u_2 and u_3 are 2-vertices. Let w_1 be the other neighbor of v_1 other than u_1. By Claim 3.3, we have $d_H(u_1) = d_H(v_1) = 2$ and $d_{G'}(u) = d_{G'}(w_1) = k = 4$. Let $u_2^{'}$ and $u_3^{'}$ be the neighbors of u_2 and u_3 other than u. If u_2 and u_3 have 1-neighbors in H, let them be denoted by w_j and y_j respectively, for $j \in \{1,2\}$.

Let $H^{'} = H \setminus \{u_1v_1\}$. By minimality of H, the graph $H^{'}$ has a star edge coloring φ with $2k$ colors. If there is an available color for u_1v_1, we are done. Otherwise, $|\varphi(u) \cap \varphi(w_1)| = 0$ and $\varphi(uu_1) \in \varphi(u_2) \cap \varphi(u_3) \cap \varphi(u_4)$ forming bi-colored paths of length three. Without loss of generality, we can assume this coloring as follows. Let $\varphi(w_1) = \{1,2,3,4\}$ and $\varphi(u) = \{5,6,7,8\}$. Let $\varphi(uu_1) =$

5 then $5 \in \bigcap_{i=2}^{4} \varphi(u_i)$ forming bi-colored paths of length three. Let $\varphi(uu_2) = 6$, $\varphi(uu_3) = 7$ and $\varphi(uu_4) = 8$.

If we can recolor the edge uu_1 with a color other than 5, we are done, as the color 5 becomes available for u_1v_1. Otherwise, as all the colors are forbidden for uu_1, the colors 1,2,3,4 and 5 appear on u_2, u_3 and u_4 such that if any color $c \in \{1, 2, 3, 4\}$ is used on uu_1, a bi-colored path of length four is created. This means that the color 5 must be on the pendant edges incident to u_2 and u_3. Let $\varphi(u_2w_1) = 5$. We remove the color of the edge uu_1 and then swap the color 5 of the edge u_2w_1 with the color 6 of the edge uu_2. This is possible because 5 is an available color for uu_2. Now, we set $\varphi(uu_1) = 6$, as $6 \notin \{\varphi(u_3u_3^{'})\} \cup \varphi(u_4)$, we have at least one color available for u_1v_1. Hence, the coloring φ of $H^{'}$ can be extended to a star edge coloring of H with $2k$ colors.

Discharging

Next, we show that a counterexample H does not exist. We set a weight function $w : V(G^{'}) \to \mathbb{R}$ such that $w(v) = d(v) - \frac{8}{3}$, $\forall$ $v \in V(G^{'})$. As $\text{Mad}(G^{'}) < \frac{8}{3}$, $\frac{2|E(G^{'})|}{|V(G^{'})|} < \frac{8}{3}$, that is $2|E(G^{'})| < \frac{8|V(G^{'})|}{3}$. Now, $\sum_{v \in V(G^{'})} w(v) = \sum_{v \in V(G^{'})}(d(v) - \frac{8}{3}) = 2|E(G^{'})| - \frac{8|V(G^{'})|}{3} < 0$. Therefore, the total sum of weights of all the vertices in the graph $G^{'}$ is strictly negative. Next, we redistribute the weights among the vertices according to the discharging rules described below, to obtain a weight function $w^{'}$. During the discharging process, the total sum of weights is kept fixed.

Discharging Rules:

R1: A 3-vertex sends $\frac{1}{3}$ to each of its adjacent 2-vertex.
R2: A 3-vertex sends $\frac{1}{3}$ to each of its adjacent *weak* 3_1-vertex.
R3: A 3-vertex sends $\frac{1}{3}$ to each of its adjacent 3_2-vertex.
R4: A k-vertex, $k \geq 4$ sends $\frac{1}{3}$ to each of its adjacent 2-vertex.
R5: A k-vertex, $k \geq 4$ sends $\frac{2}{3}$ to its adjacent *bad* 2-vertex.
R6: A k-vertex, $k \geq 4$ sends $\frac{1}{3}$ to each of its adjacent *weak* 3_1-vertex.

R7: A k-vertex, $k \geq 4$ sends $\frac{1}{3}$ to each of its adjacent 3_2-vertex.

As $G^{'}$ has no 1-vertices, $\delta(G^{'}) \geq 2$. Let $v \in V(G^{'})$ be a k-vertex.
Case 1: When $d(v) = 2$, $w(v) = -\frac{2}{3}$.
• If v is adjacent to 3^{+} vertices, then by R1 or R4, v receives $\frac{1}{3}$ units from each of them. Therefore, $w^{'}(v) = -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0$.
• If v is *bad*, then by Claim 3.3, it is adjacent to a k-vertex. As $k \geq 4$, by R5 it receives $\frac{2}{3}$ units from its k-neighbor. So, $w^{'}(v) = -\frac{2}{3} + \frac{2}{3} = 0$.
Case 2: When $d(v) = 3$, $w(v) = \frac{1}{3}$.
• If v is a 3_0 vertex, by Claim 3.5, it can be adjacent to at most one 3_2-vertex. If v is adjacent to one 3_2-vertex, say, u, then by Claim 3.6(i), such v is not adjacent to any *weak* 3_1-vertex. So, by R3, v sends $\frac{1}{3}$ units to u. Therefore, $w^{'}(v) = \frac{1}{3} - \frac{1}{3} = 0$.
If v is adjacent to a *weak* 3_1-vertex, then by Claim 3.6(i), v can be adjacent to at most one *weak* 3_1 vertex, say, u. So, by using R2, v sends $\frac{1}{3}$ units to u. Therefore, $w^{'}(v) = \frac{1}{3} - \frac{1}{3} = 0$. Otherwise, its weight remains unchanged.
• If v is not a *weak* 3_1 vertex, by R1, it sends $\frac{1}{3}$ units to its adjacent 2-neighbor. Therefore, $w^{'}(v) = \frac{1}{3} - \frac{1}{3} = 0$.
If v is a *weak* 3_1 vertex adjacent to a 3_2-vertex, say, u_1 and a 2-vertex, say, u_2, then by Claim 3.6(ii), the third neighbor of v can either be a 3_0-vertex, say, w which is not adjacent to a 3_2-vertex or a 4^{+}-vertex, say, w. So, by using R2 or R6, v receives $\frac{1}{3}$ units from w and sends $\frac{1}{3}$ units each to u_1 and u_2. Therefore, $w^{'}(v) = \frac{1}{3} + \frac{1}{3} - 2(\frac{1}{3}) = 0$.
• When v is a 3_2 vertex, by Claim 3.5, it is adjacent to a 3^{+}-vertex. So, by using R3 or R7, v receives $\frac{1}{3}$ units from its 3^{+}-neighbor and by using R1, v sends $\frac{1}{3}$ units to each of its 2-neighbors. Therefore, $w^{'}(v) = \frac{1}{3} + \frac{1}{3} - 2(\frac{1}{3}) = 0$.
Case 3: When $d(v) = 4$, $w(v) = \frac{4}{3}$.
• If v is adjacent to *weak* 3_1-vertices or 3_2-vertices or 2-vertices, then by R4, R6 or R7, v sends $\frac{1}{3}$ units to each of them. Therefore, $w^{'}(v) = \frac{4}{3} - p(\frac{1}{3}) \geq 0$, where p is the number of these vertices together and $0 \leq p \leq 4$.
• If v is adjacent to a *bad* 2-vertex then by Claim 3.7(i), it is adjacent to at most one *bad* 2-vertex and by Claim 3.8, such v can

be adjacent to at most two *weak* 3_1-vertices or at most two 3_2 vertices or at most one 2-vertex. Therefore, using R4, R5, R6 or R7, $w^{'}(v) = \frac{4}{3} - \frac{2}{3} - (p(\frac{1}{3}) + q(\frac{1}{3}) + r(\frac{1}{3})) \geq 0$, where p, q and r are the number of *weak* 3_1-vertices, 3_2-vertices and 2-vertices respectively adjacent to v such that $p, q \in \{0, 1, 2\}$, $r \in \{0, 1\}$ and $p + q + r \leq 2$.
Case 4: When $d(v) \geq 5$, $w(v) = k - \frac{8}{3}$.
By Claim 3.7(ii), v can be adjacent to at most two *bad* 2 vertices. Let v be adjacent to two *bad* two vertices and among other $k-2$ neighbors, let there be p *weak* 3_1-vertices, q 3_2-vertices and r 2-vertices, where $p, q, r \in \{0, 1, ..., k-2\}$ such that $p + q + r \leq k - 2$. Using R4, R5, R6 or R7, v sends $\frac{2}{3}$ units to each adjacent *bad* 2-vertex and $\frac{1}{3}$ units to each adjacent *weak* 3_1-vertex or 3_2-vertex or a 2-neighbor. Therefore, $w^{'}(v) = k - \frac{8}{3} - 2(\frac{2}{3}) - (p(\frac{1}{3}) + q(\frac{1}{3}) + r(\frac{1}{3})) \leq k - \frac{8}{3} - \frac{4}{3} - (k-2)(\frac{1}{3}) = \frac{2k-10}{3} \geq 0$ as $k \geq 5$.

Therefore, after discharging, in all the above cases, $w^{'}(v) \geq 0$ for every $v \in V(G^{'})$, a contradiction. So, the subgraph $G^{'}$ does not exist. Hence, the minimal counterexample H cannot exist. This completes the proof.

As every planar graph with girth g (length of the shortest cycle) satisfies (Folklore) $\text{Mad}(G) < \frac{2g}{g-2}$, the following corollary, can be easily derived from Theorem 1.1.

Corollary 3.9. *Let G be a planar graph with girth $g \geq 7$. Then $\chi_s^{'}(G) \leq 2\Delta$.*

References

[1] B. Lužar, M. Mockovčiaková, R. Soták, Note on list star edge coloring of subcubic graphs, *Journal of Graph Theory* **81** (1) (2016), 73–82.

[2] H. Lei, Y. Shi, Z. X. Song, Star chromatic index of subcubic multigraphs, *Journal of Graph Theory* **88** (4) (2018), 566–576.

[3] H. Xuling, L. Lingxi, T. Wang, Star edge coloring of some special graphs, arXiv:2010.14349, (2020).

[4] K. Pradeep, V. Vijayalakshmi, Star edge coloring of subcubic graphs, *Applied Mathematics and Information Sciences* **13** (2) (2019), 279–284.

[5] K. Pradeep, Star edge coloring of graphs with $\text{Mad}(G) < \frac{14}{5}$, *Turkish Journal of Mathematics* **45** (1) (2021), 54–65.

[6] L. Bezegová, B. Lužar, M. Mockovčiaková, R. Soták, R. Škrekovski, Star edge coloring of some classes of graphs, *J. of Graph Theory* **81** (1) (2016), 73–82.

[7] L. Xin-sheng, D. Kai, An upper bound on the star chromatic index of graphs with $\delta \geq 7$, *J. of Lanzhou University (Natural Sciences)* **44** (2008), 94–95.
[8] S. Kerdjoudj, A. V. Kostochka, A. Raspaud, List star edge coloring of subcubic graphs, *Discusscussions Math. Graph Theory*, **38** (2018), 1037–1054.
[9] S. Kerdjoudj, Kavita Pradeep, A. Raspaud, List star chromatic index of sparse graphs, *Discrete Math.*, **341** (2018), 1835–1849.
[10] S. Kerdjoudj, A. Raspaud, List star edge coloring of sparse graphs, *Discrete Applied Math.* **238** (2018), 115–125.
[11] Y. Wang, W. Wang, Y. Wang, Edge partition and star chromatic index, *Applied Mathematics and Computation* **333** (2018), 480–489.
[12] Y. Wang, Y. Wang, W. Wang, Star edge coloring of graphs with maximum degree four, *Applied Math. and Comput.* **340** (1) (2019), 268–275.
[13] Z. Dvořák, B Mohar, R. Šámal, Star chromatic index, *J. of Graph Theory*, **72** (3) (2013), 313–326.

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