Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 2, pp. 303 - 318
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.2.012

t-PARTIAL DOMINATION RATIO IN GRAPHS

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Abstract. Let $G(V, E)$ be a graph with n vertices. The cardinality of the smallest subset D of V such that every vertex in $V - D$ is adjacent to (or 'dominated by') at least one vertex in D is called the domination number of G, denoted by $\gamma(G)$. For a positive integer $t \leq \gamma(G)$, we introduce t-Partial dominated set which is a largest subset PD_t of V dominated by t vertices of G, and $\alpha_t(G)=\frac{|PD_t|}{n}$, the t-Partial domination ratio of G. Further, extending this to distance-2 domination, we define t-Partial distance-2 domination ratio $\alpha_t^{(2)}(G)$. Also, we establish some general bounds on these parameters, as well as bounds for product graphs like cartesian, corona and lexicographic. In situations where complete representation of the network or complete distribution of resources become infeasible, t-Partial domination is a parameter that can provide crucial insights.

2000 Mathematics Subject Classification 05C69, 05C12, 05C76
Keywords and phrases. Partial domination, Partial distance-2 domination, Product graphs
Submission Date: 30 January 2024

1. Introduction

Domination in graphs is a widely studied area with diverse applications. The domination number, denoted by $\gamma(G)$, represents the minimum number of vertices required to dominate all other vertices in a graph G. However, practical scenarios often impose limitations, such as budgetary constraints or high demand, making it infeasible to utilize $\gamma(G)$ vertices. In such cases, the objective shifts to maximizing the ratio of dominated vertices given a smaller, fixed number of vertices. This problem is addressed by partial domination, which is particularly relevant for large graphs.

This paper explores partial domination and its extension to distance domination. Distance domination is crucial for allocating finite resources in massively parallel architectures and for designing redundant parallel paths that can withstand node failures.

A significant application of partial distance domination can be seen in vaccine distribution during periods of limited supply. By strategically placing vaccination points in densely populated areas with high person-to-person contact rates, the available vaccine volumes can be efficiently utilized, maximizing the protected population ratio.

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Similarly, in urban planning, strategically adding a few nodes or allocating additional resources can substantially increase the proportion of the area that benefits, often at a relatively low cost. The key lies in optimizing the placement of these nodes to maximize the overall impact.

Traditional partial domination focuses on finding the minimum number of vertices needed to dominate a given fraction α of the total n vertices. In contrast, our research focuses on maximizing the ratio of vertices dominated by a fixed number t of vertices, where $t < \gamma(G)$.

2. Literature

Distance variants of domination have significant applications across numerous disciplines. In a graph G, a subset $D \subseteq V(G)$ is called a distance-k dominating set if every vertex in $V(G) \setminus D$ lies within distance k of some vertex in D. The smallest cardinality of such a set is called the distance-k domination number, denoted $\gamma_k(G)$.

Our particular interest is the case $k = 2$. Here, a subset $D \subseteq V(G)$ forms a distance-2 dominating set if each vertex not in D is at most two edges away from some vertex in D. The distance-2 domination number, $\gamma_2(G)$, is the minimum size of such a set. For more details on distance domination, refer to [6], [8] and [9].

This work mainly examines the notion of partial domination, introduced independently by [2] in 2017 and by [4]. In the context of partial domination, a set S is called a p-dominating set if it dominates at least a proportion p of the vertices of G. The p-domination number, $\gamma_p(G)$, denotes the smallest size of a p-dominating set. In [2], the authors focused on $\frac{1}{2}$-domination - identifying sets that dominate at least half of the vertices in G - and derived precise values for paths, cycles, and their cartesian products.

This concept was further extended by [4], introducing the α-partial domination number, and established several bounds in terms of graph order, maximum degree, and the classic domination number. Nordhaus-Gaddum-type results for α-partial domination were also derived. Formally, for $0 < \alpha \leq 1$, a subset $S \subseteq V(G)$ is an α-partial dominating set if $|N[S]| \geq \alpha|V(G)|$, where $N[S]$ represents the closed neighborhood of S. The minimum size of such a set is the α-partial domination number of G, denoted by $pd_\alpha(G)$. For more results on partial domination, refer [3].

Recent work by [1] provided an upper bound on pd_α for connected cubic graphs with $\alpha = \frac{13}{14}$, and classified those of order 14 and domination number 5.

In addition, [5] introduced the concept of domination defect. For a graph with domination number γ, the k-domination defect is defined as $\zeta_k(S) = |V(G)| - |N[S]|$, where $1 \leq k < \gamma$ and $S \subset V(G)$ with $|S| = \gamma - k$. The paper computed $\zeta_k(P_n)$ and $\zeta_k(C_n)$ for paths and cycles, as well as ζ_1 and ζ_2 for the Petersen graph. Relationships between domination defect and graph differential were established, along with various bounds and characterizations of graphs with domination defect 1.

The k-domination defect thus measures the minimum number of vertices left undominated by any $\gamma - k$ vertices.

Recent studies have extended domination concepts to hypergraphs; for instance, the work [7] introduces partial domination, defining the minimum set of vertices needed to dominate a given fraction of the hypergraph, and investigates structural conditions under which this parameter behaves predictably, establishing bounds, monotonicity, and sharper results for uniform hypergraphs.

In our study, we consider a concept closely related to k-domination defect, namely t-Partial domination. We also explore its distance-based version.

3. $t-$ Partial domination ratio

Unless stated otherwise, all graphs considered in this paper are simple and connected.

Definition 3.1. *Let $G(V,E)$ be a graph with n vertices and t be a positive integer $t \leq \gamma(G)$. Let PD_t be a set of maximum number of vertices dominated by t vertices and $\alpha_t(G)=\frac{|PD_t|}{n}$. Then PD_t is a t-Partial dominated set of G and $\alpha_t(G)$ the t-Partial domination ratio of G.*

Observation 3.2. *The following results are straightforward.*

(1) $0 < \alpha_t(G) \leq 1$.
(2) $\alpha_\gamma(G)=1$.
(3) *Let G be a path P_n or a cycle C_n of order $n > 3$, then $\alpha_t(G) = \frac{3t}{n}$ for $t < \gamma$.*
(4) $\alpha_1(G) = \frac{\Delta+1}{n}$, *$\Delta$ is the maximum degree of G.*
(5) *If $G= k_{m,n}$ $n > m$, $\alpha_1(G) = \frac{n+1}{m+n}$.*
(6) $\alpha_t(G) \leq \frac{t(\Delta+1)}{n}$.
(7) $\alpha_1(\overline{C}_n) = \frac{n-2}{n}$ *and* $\alpha_1(\overline{P}_n) = \frac{n-1}{n}$.
(8) $\alpha_1(\overline{G}) = \frac{n-k}{n}$ *if G is a k regular graph.*

Observation 3.3. *Let G and H be two graphs with same number of vertices and $\gamma(G) = \gamma(H)$. Then $\alpha_t(G)$ need not be equal to $\alpha_t(H)$ for $0 < t < \gamma(G)$.*

Lemma 3.4. *In a graph G, if $\alpha_t(G) \neq \alpha_{t+1}(G)$ then $\alpha_{t+1}(G) - \alpha_t(G) \geq \frac{1}{n}$.*

Proof. Since $\alpha_t(G) \neq \alpha_{t+1}(G)$, the set PD_{t+1} must include at least one additional vertex compared to PD_t. Hence, $|PD_{t+1} - PD_t| \geq 1$, which leads to $\alpha_{t+1}(G) - \alpha_t(G) \geq \frac{1}{n}$. □

Theorem 3.5. *For any graph G, $\alpha_1(G) < \alpha_2(G) < \alpha_3(G)...... < \alpha_\gamma(G) = 1$.*

Proof. If $\alpha_1(G) \neq 1$, then there exists at least one vertex not contained in $PD_1(G)$. Consequently, $\alpha_2(G) > \alpha_1(G)$. By lemma 3.4, $\alpha_2(G)$ is at least $\frac{1}{n}$ more than $\alpha_1(G)$. If $\alpha_2(G) \neq 1$, then there is at least one vertex which is not in $PD_2(G)$. So $\alpha_3(G) > \alpha_2(G)$ and $\alpha_3(G)$ is at least $\frac{1}{n}$ more than $\alpha_2(G)$ and so on. Continuing this way we obtain $\left|PD_\gamma\right| = n$. Hence the result. □

Theorem 3.6. *Let G be any graph and t be an integer $0 < t \leq \gamma(G)$. Then $\alpha_t(G) \leq t\alpha_1(G)$.*

Proof. Let $\alpha_1(G) = \frac{k}{n}$. Then the maximum number of vertices dominated by any one vertex is k. So the number vertices dominated by t vertices is tk or less. Thus, $\alpha_t(G) \le \frac{tk}{n}$. □

Corollary 3.7. *For any graph G,* $\alpha_1(G) \ge \frac{1}{\gamma}$.

Proof. By Theorem 3.6, for $0 < t \le \gamma$, $\alpha_t(G) \le t\alpha_1(G)$. So, $\alpha_\gamma(G) \le \gamma\alpha_1(G)$. By applying $\alpha_\gamma(G) = 1$ we get the result. □

Theorem 3.8. *For any graph G,*

$$\alpha_1(G) \ge \alpha_2(G) - \alpha_1(G) \ge \alpha_3(G) - \alpha_2(G) \ldots\ldots \ge \alpha_\gamma(G) - \alpha_{\gamma-1}(G).$$

Proof. By Theorem 3.6, $2\alpha_1(G) \ge \alpha_2(G)$. So $\alpha_1(G) \ge \alpha_2(G) - \alpha_1(G)$. Now we prove $\alpha_2(G) - \alpha_1(G) \ge \alpha_3(G) - \alpha_2(G)$. Let $|PD_1| = k$ then $|PD_2| \le 2k$. Let $|PD_2| = k + a$, $a \le k$ and $|PD_3| = k + a + b$. If $b > a$, $k + b > k + a$ which contradicts the maximality of $|PD_2|$. Hence, $b \le a \le k$. So $\alpha_2(G) - \alpha_1(G) \ge \alpha_3(G) - \alpha_2(G)$. Similarly the remaining inequalities follow. □

Theorem 3.9. *For any graph G,* $\alpha_{t_1+t_2}(G) \le \alpha_{t_1}(G) + \alpha_{t_2}(G)$*, where* $t_1 + t_2 \le \gamma(G)$ *and* $t_1 \le t_2$*. The equality occurs when* $\alpha_1(G) = \alpha_2(G) - \alpha_1(G) = \alpha_3(G) - \alpha_2(G) \ldots\ldots = \alpha_\gamma(G) - \alpha_{\gamma-1}(G)$.

Proof.

$$\begin{aligned}
\alpha_{t_1+t_2}(G) &= \alpha_{t_2}(G) + \alpha_{t_1+t_2}(G) - \alpha_{t_2}(G) \\
&= \alpha_{t_2}(G) + [\alpha_{t_1+t_2}(G) - \alpha_{t_1+t_2-1}(G)] \\
&\quad + [\alpha_{t_1+t_2-1}(G) - \alpha_{t_1+t_2-2}(G)] + \cdots \\
&\quad + [\alpha_{t_2+1}(G) - \alpha_{t_2}(G)] \\
&\le \alpha_{t_2}(G) + \alpha_1(G) + [\alpha_2(G) - \alpha_1(G)] \\
&\quad + [\alpha_3(G) - \alpha_2(G)] + \cdots \\
&\quad + [\alpha_{t_1}(G) - \alpha_{t_1-1}(G)] \text{ (by Theorem 3.8)} \\
&= \alpha_{t_1}(G) + \alpha_{t_2}(G).
\end{aligned}$$

Thus, $\alpha_{t_1+t_2}(G) \le \alpha_{t_1}(G) + \alpha_{t_2}(G)$. □

Corollary 3.10. *In any graph G,* $\alpha_t(G) + \alpha_{\gamma-t}(G) \ge 1$.

Proof. Since $\alpha_\gamma(G) = 1$, by using Theorem 3.9 the result follows. □

Theorem 3.11. *If G is an r regular bipartite graph with n vertices,* $\alpha_t(G) = \frac{(r+1)t}{n}$, $t \le \lceil \frac{n}{2r} \rceil$.

Proof. In any regular bipartite graph with n vertices, the vertices are partitioned into two sets V_1 and V_2 with $\frac{n}{2}$ vertices in each. A vertex u_1 in V_1 can dominate itself and r vertices in V_2. Similarly with t vertices in V_1, we can have $(r+1)t$ vertices in PD_t and thus $\alpha_t(G) = \frac{(r+1)t}{n}$. With V_2 containing $\frac{n}{2}$ vertices where each group of r vertices is dominated by a vertex in V_1, we have $t \le \lceil \frac{n}{2r} \rceil$. □

4. t-Partial distance-2 domination ratio

Definition 4.1. *Let $G(V,E)$ be a graph with n vertices and t be a positive integer $t \leq \gamma_2(G)$. Let PD_t^2 be a set of maximum number of vertices distance-2 dominated by t vertices and $\alpha_t^{(2)}(G)=\frac{|PD_t^2|}{n}$. Then PD_t^2 is a t-Partial distance-2 dominated set of G, and $\alpha_t^{(2)}(G)$, the t-Partial distance-2 domination ratio of G.*

Analogous to the results proved in the case of t-Partial domination, we have the following results for the distance-2 version.

Theorem 4.2. *If G is a path or cycle of order $n > 5$, $\alpha_t^{(2)}(G) = \frac{5t}{n}$, for $t < \gamma_2$.*

Proof. In a tree, a leaf is a vertex with degree one and the support vertex is a vertex adjacent to at least one leaf. Consider a vertex u which is not a leaf or support vertex in a path P_n, $n > 5$. Since degree of u is two and u is not a support vertex, u is adjacent to two vertices v and w which are not leaves. Here, v and w are adjacent to one more vertex each, other than u. Thus $PD_1^2(P_n)$ contains five vertices and $\alpha_t^{(2)}(P_n) = \frac{5t}{n}$, for t<$\gamma_2$. In a cycle, since degree of each vertex is two, any vertex u is adjacent to two other vertices and these two are adjacent to one more each. Hence $PD_1^2(C_n)$ contains five vertices and $\alpha_t^{(2)}(C_n) = \frac{5t}{n}$, for t<$\gamma_2$. □

Now we present the results from Theorem 3.5 to Corollary 3.10 from the previous section extended to distance 2-domination. Since the proofs follow on similar lines, they are omitted for brevity.

Theorem 4.3. *In a graph G, $\alpha_1^{(2)}(G) < \alpha_2^{(2)}(G) < \alpha_3^{(2)}(G)......\ \alpha_{\gamma_2}^{(2)}(G)=1$.*

Theorem 4.4. *Let G be any graph and t be an integer $0 < t \leq \gamma_2$. Then $\alpha_t^{(2)}(G) \leq t\alpha_1^{(2)}(G)$.*

Corollary 4.5. *In any graph G, $\alpha_1^{(2)}(G) \geq \frac{1}{\gamma_2}$.*

Theorem 4.6. *In any graph G, $\alpha_1^{(2)}(G) \geq \alpha_2^{(2)}(G) - \alpha_1^{(2)}(G) \geq \alpha_3^{(2)}(G) - \alpha_2^{(2)}(G)...... \geq \alpha_{\gamma_2}^{(2)}(G) - \alpha_{\gamma_2-1}^{(2)}(G)$.*

Theorem 4.7. *For a graph G, $\alpha_{t_1+t_2}^{(2)}(G) \leq \alpha_{t_1}^{(2)}(G)+\alpha_{t_2}^{(2)}(G)$, where $t_1+t_2 \leq \gamma_2$. The upper bound is reached when $\alpha_1^{(2)}(G) = \alpha_2^{(2)}(G) - \alpha_1^{(2)}(G) = \alpha_3^{(2)}(G) - \alpha_2^{(2)}(G)...... = \alpha_{\gamma_2}^{(2)}(G) - \alpha_{\gamma_2-1}^{(2)}(G)$.*

Corollary 4.8. *In any graph G, $\alpha_t^{(2)}(G) + \alpha_{\gamma_2-t}^{(2)}(G) \geq 1$.*

Theorem 4.9. *For any graph G, $\alpha_1^{(2)}(G) \leq \frac{\Delta^2+1}{n}$, where Δ is maximum degree of G.*

Proof. A vertex u in G can distance-2 dominate itself and at most Δ neighbors and $\Delta - 1$ neighbors of each of these Δ vertices. Thus maximum $|PD_1^2|$ is $\Delta^2 + 1$. □

Theorem 4.10. *In a graph G, $\alpha_{\gamma_2-1}^{(2)}(G) \leq \frac{n-1}{n}$. The bound is attained iff there exists a vertex v in G such that $\gamma_2(G) = \gamma_2(G - v) + 1$.*

Proof. We know that $\alpha^{(2)}_{\gamma-1}(G) \neq 1$. So, $\alpha^{(2)}_{\gamma_2-1}(G) \leq \frac{n-1}{n}$. If $\alpha^{(2)}_{\gamma_2-1}(G) = \frac{n-1}{n}$, except one vertex, say v all other vertices are in $PD^2_{\gamma_2-1}(G)$. Thus v should be in γ_2 set of G. So $\gamma_2(G)$ is one more than $\gamma_2(G-v)$. Conversely, if there exists a vertex v in G such that $\gamma_2(G)$=$\gamma_2(G-v)+1$, v is not in $PD^2_{\gamma-1}$ and hence $\alpha^{(2)}_{\gamma_2-1}(G)$=$\frac{n-1}{n}$. □

Corollary 4.11. *For any graph G,* $\alpha^{(2)}_{\gamma_2-1}(G)$=$\frac{n-k}{n}$ *iff there exists k vertices* $u_1, u_2, ...u_k$ *in G such that* $\gamma_2(G)$=$\gamma_2(G-\{u_1,u_2,...u_k\})+1$.

Theorem 4.12. *In any graph G,* $\frac{1}{2} \leq \alpha^{(2)}_{\lceil \frac{\gamma_2}{2} \rceil}(G) \leq \frac{n-1}{n}$.

Proof. The upper bound is attained when γ_2=2 and there exists a vertex v in G such that $\gamma_2(G-v) = 1$. By corollary 4.5, $\alpha^{(2)}_1(G) \geq \frac{1}{\gamma_2}$. Also by theorem 4.6, $\alpha^{(2)}_1(G) \geq \alpha^{(2)}_2(G) - \alpha^{(2)}_1(G) \geq \alpha^{(2)}_3(G) - \alpha^{(2)}_2(G)...... \geq \alpha^{(2)}_{\gamma_2}(G) - \alpha^{(2)}_{\gamma_2-1}(G)$. The equality in theorem 4.6 gives, $\alpha^{(2)}_2(G) \geq \frac{2}{\gamma_2}$, $\alpha^{(2)}_3(G) \geq \frac{3}{\gamma_2}$.....
Thus $\alpha^{(2)}_{\lceil \frac{\gamma_2}{2} \rceil}(G) \geq \alpha^{(2)}_{\frac{\gamma_2}{2}}(G) \geq \frac{1}{2}$. □

Theorem 4.13. *In any graph G,* $\frac{1}{n} \leq \alpha^{(2)}_{t+1}(G) - \alpha^{(2)}_t(G) \leq \frac{\Delta^2+1}{n}$.

Proof. The minimum possible difference in the cardinality of $PD^2_{t+1}(G)$ and $PD^2_t(G)$ is 1, as the newly added vertex to $PD^2_t(G)$ dominates at least itself. This gives the lower bound. Now, any given vertex can distance-2 dominate at most itself, it's Δ neighbors and $\Delta(\Delta-1)$ neighbors of these Δ vertices. Thus, $\alpha^{(2)}_{t+1}(G) - \alpha^{(2)}_t(G) \leq \frac{\Delta(\Delta-1)+\Delta+1}{n}$ which gives the upper bound. □

Theorem 4.14. *If a new edge e is added between two non-adjacent vertices u and v in a graph G, then* $\alpha^{(2)}_t(G+e) - \alpha^{(2)}_t(G) \leq \frac{\Delta+1}{n}$.

Proof. Let v be a vertex in $PD^2_t(G)$ and u be a vertex not in $PD^2_t(G)$ of degree $\Delta(G)$. Adding an edge e from v to u enables v to dominate u as well as distance-2 dominate the $\Delta(G)$ vertices that u was adjacent originally in G. Thus addition of one edge results in an increase of PD^2_t by at most $\Delta+1$ which gives the upper bound. □

Next, we consider the problem for a *total graph*. A total graph $T(G)$ of a graph G is a graph such that the vertex set of $T(G)$ corresponds to the vertices and edges of G and, two vertices are adjacent in $T(G)$ iff their corresponding elements are either adjacent or incident in G.

Theorem 4.15. *If* $G = P_n$, *in the total graph* $T(G)$, $\alpha^{(2)}_t(T(G)) = \frac{9t}{2n+1}$ *for* $t < \gamma_2(P_n)$ *and* $n \geq 5$.

Proof. Consider the path $u_1e_1u_2e_2u_3e_3u_4e_4u_5$ on five vertices. Let x_1, x_2, x_3, x_4 be the vertices of $T(G)$ corresponding to e_1, e_2, e_3, e_4, respectively. Then u_3 is within distance-2 of the nine vertices $u_1, x_1, u_2, x_2, u_3, x_3, u_4, x_4$, and u_5 in $T(G)$. This holds for each of the t chosen vertices; hence, PD^2_t of $T(P_n)$ has exactly $9t$ vertices. □

Theorem 4.16. *If the graph G is a linear hexagonal chain with N hexagons and n vertices,* $\alpha^{(2)}_t(G) = \frac{8t}{n}$ *if* $N \geq 2$ *and* $t \leq \gamma_2 - 2$.

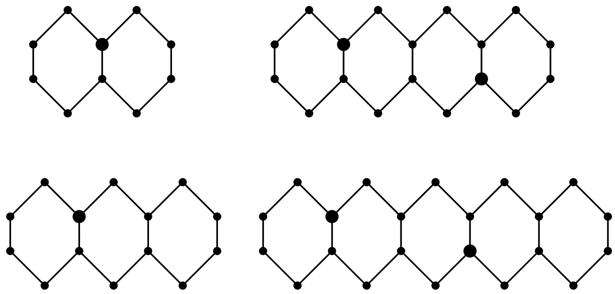

FIGURE 1. Hexagonal chain with $t \leq \gamma_2 - 1$.

Proof. There are eight vertices dominated by one vertex in linear hexagonal chain. So, t vertices, distance-2 dominates $8t$ vertices as shown in Figure 1.

□

Theorem 4.17. *If the graph G is a linear hexagonal chain with N hexogans,* $\alpha^{(2)}_{\gamma_2-1}(G) = \frac{n-2}{n}$ *if N is even and* $\alpha^{(2)}_{\gamma_2-1}(G) = \frac{n-1}{n}$ *if N is odd. Also,* $\alpha^{(2)}_{\frac{N}{2}}(G) = \frac{n-2}{n}$ *if N is even and* $\alpha^{(2)}_{(\frac{N+1}{2})}(G) = \frac{n-1}{n}$ *if N is odd.*

Proof. When N is even, place the $\gamma_2 - 1 = \frac{N}{2}$ vertices in the position marked as v_i's so that the number of vertices within distance two is maximum.

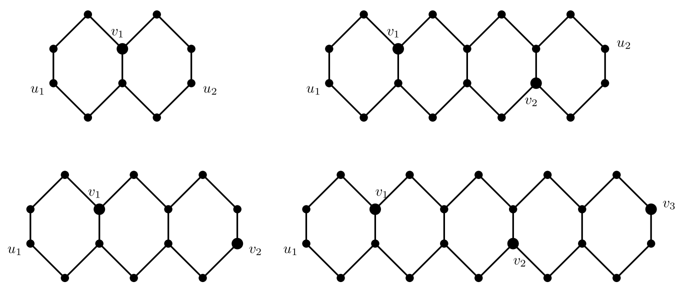

FIGURE 2. Hexagonal chain with $t = \gamma_2 - 1$.

We can see from the pattern in Figure 2, that except u_1 and u_2, all other vertices are within distance-2 from v_i's. Hence $\alpha^{(2)}_{\gamma_2-1}(G) = \frac{n-2}{n}$ if N is even. When N is odd, place the $\gamma_2 - 1 = \frac{N+1}{2}$ vertices in the position marked as v_i's, so that except u_1 all other vertices are within distance-2 from these vertices.

□

We know that the eccentricity $e(v)$ of a vertex v in a graph is the maximum distance from v to any other vertex in the graph.The maximum eccentricity in a graph G is called the diameter of G, $diam(G)$.

Theorem 4.18. *In any graph G with* $diam(G) \geq 4$, $\alpha^{(2)}_1(G) \geq \frac{5}{n}$.

Proof. Since $diam(G) \geq 4$, there exists at least one path in G containing minimum 5 vertices and hence at least 5 vertices in $PD_1^2(G)$ as well. Hence $\alpha_1^{(2)}(G) \geq \frac{5}{n}$. □

Theorem 4.19. *In a tree T, if there exist exactly k vertices with eccentricity 5 and all other vertices have eccentricity less than 5 then* $\dfrac{n - \lfloor \frac{k}{2} \rfloor}{n} \leq \alpha_1^{(2)}(T) \leq \frac{n-1}{n}$.

Proof. Suppose$\{u_1, u_2, ...u_k\}$ are the k vertices with eccentricity 5. Then there are paths consisting of six vertices with any two vertices say u_i, u_j from $\{u_1, u_2, ...u_k\}$ as end points. Then, either one of the vertices u_i, u_j will not be in $PD_1^2(T)$. As shown in the Figure 3, x distance-2 dominates all the vertices of the tree, except the leaf at u_1. This gives the upper bound.

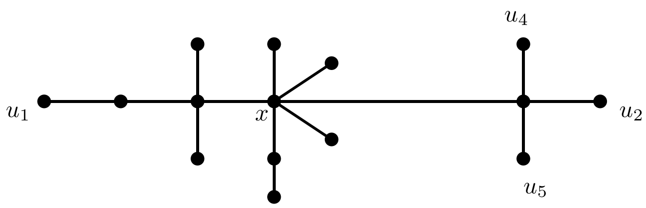

Figure 3. Example of a tree which attains the upper bound

Now, we illustrate the lower bound by Figure 4, which has 7 vertices with eccentricity 5 namely, $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$. The vertices x and y, each having eccentricity 3, can distance-2 dominate the maximum number of vertices in T. The vertex x distance-2 dominates all vertices of T except u_5, u_6, u_7, while the vertex y dominates all except u_1, u_2, u_3, u_4. The difference in the number of vertices which are not distance-2 dominated by x and y individually is at most 1. Thus at most $\lfloor \frac{k}{2} \rfloor$ vertices are not distance-2 dominated by either only by x or only by y. Thus, $PD_1^2(T)$ contains at least $n - \lfloor \frac{k}{2} \rfloor$ vertices.

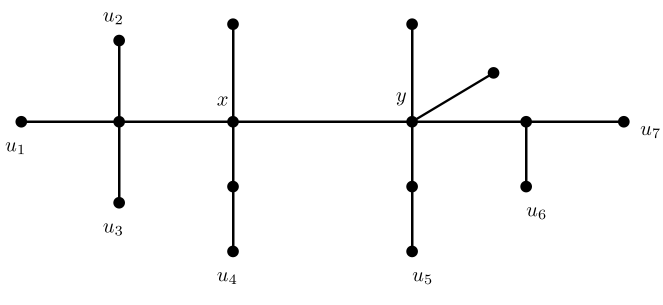

Figure 4. Example of a tree which attains the lower bound

□

Now we consider subdivision graph of a tree. The subdivision graph of a graph G is the graph obtained from G by replacing each of its edges by a path of length 2, or equivalently by inserting an additional vertex into each edge of G.

Theorem 4.20. *Let T be a tree with n vertices and Δ be the maximum degree of T. Let T' be subdivision graph of T. Then $\alpha_1^{(2)}(T') = \frac{2\Delta+1}{2n-1}$.*

Proof. We have seen that $\alpha_1(G) = \frac{\Delta+1}{n}$. Let u be a vertex of maximum degree in T. Since each edge is subdivided in the subdivision graph, u distance-2 dominates maximum number of vertices in T'. So including u, in $PD_1^2(T')$ there are $\Delta + 1$ vertices which are originally dominated by u in T and additionally Δ vertices obtained by subdividing the corresponding Δ edges of T. □

The splitting graph $S(G)$ of a graph G is that graph obtained from G by adding to G a new vertex u' for each vertex u of G and joining u' to the neighbors of u in G.

Theorem 4.21. *In the splitting graph of a path P_n or cycle C_n, for $n \geq 5$ $\alpha_t^{(2)}(S(P_n)) = \frac{8t}{2n}$ and $\alpha_t^{(2)}(S(C_n)) = \frac{10t}{2n}$, $t < \gamma_2(G)$.*

Proof. In Figure 5 we can see, $PD_1^2(S(P_n))$ has 8 vertices and $PD_1^2(S(C_n))$ has 10 vertices and thus $\left|PD_t^2(S(P_n))\right|$ is 8t and $\left|PD_t^2(S(C_n))\right|$ is 10t for $t < \gamma(G)$.

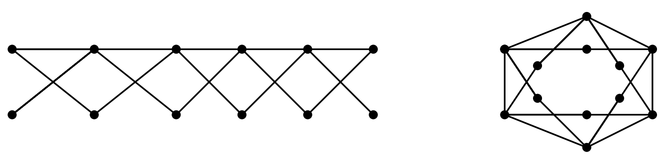

FIGURE 5. Splitting graph of P_6 and C_6

□

Theorem 4.22. *Let $S(T)$ is splitting graph of a tree T with n vertices, then*

$$\alpha_1^{(2)}(S(T)) \leq \frac{2\Delta^2 + 2}{2n}.$$

Proof. Let u be a vertex in T which can distance-2 dominate maximum number of vertices. Let $v_1, v_2, ...v_\Delta$ be the vertices adjacent to u in T. For maximality, other than u let $v_{i1}, v_{i2}, v_{i3}...v_{i\Delta-1}$ be the vertices adjacent to v_i, $i = 1, 2, 3...\Delta$. We have seen that for any graph, $|PD_1^2| \leq 1 + \Delta^2$. Let $u', v_1', v_2'...v_\Delta'$ and $v_{i1}', v_{i2}', v_{i3}'...v_{i\Delta-1}'$, $i = 1, 2, 3...\Delta_G$ be the corresponding vertices in $S(T)$. The newly added vertices are joined to the neighbours of the corresponding vertices of T in $S(T)$. Then u can be adjacent to utmost $v_1', v_2'...v_\Delta'$.. Moreover $v_1, v_2, ...v_\Delta$ are adjacent to $v_{i1}', v_{i2}', v_{i3}'...v_{i\Delta-1}'$, $i = 1, 2, 3...\Delta_G$ and u'. Thus $|PD_1^2(S(T))|$ is at most $2(1 + \Delta^2)$. □

5. RESULTS ON PRODUCT GRAPHS

The three different graph products that we consider here are :

1. Cartesian product: The cartesian product of graphs G and H denoted by $G \square H$ is the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent in $G \square H$ if and only if either i)$u = v$ and u' is adjacent to v' in H, or ii)$u' = v'$ and u is adjacent to v in G.

2. Lexicographic product: The lexicographic product of two graphs G and H denoted by $G[H]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ iff either $u_1 u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$.

3. Corona product: The corona of two graphs G and H denoted by GoH is the graph obtained by taking one copy of G of order n and n copies of H and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

5.1. Results for α_t on product graphs.

Theorem 5.1. *Let G and H be two graphs with number of vertices n and m respectively. If $\alpha_t(H) = t\alpha_1(H)$ for $t \leq \gamma_H$, then in the cartesian product $G \square H$,*

$$\alpha_t(G \square H) = \frac{t[\Delta_H + \Delta_G + 1]}{mn}.$$

Proof. Let the vertex set of G be $\{v_1, v_2, ...v_n\}$ and that of H be $\{u_1, u_2, ...u_m\}$. We have seen that, $\alpha_1(H) = \frac{\Delta_H + 1}{m}$. Since $\alpha_t(H) = t\alpha_1(H)$ for all $t \leq \gamma_H$, any PD_t set in H has $t(\Delta_H + 1)$ elements. Let (u_c, v_d) be a vertex in $G \square H$ which can dominate maximum number of vertices. $PD_1(G \square H)$ consists of (u_c, v_d) and vertices $(u_1, v_d), (u_2, v_d), ...(u_{\Delta_H}, v_d)$ where $\{u_1, u_2, ...u_{\Delta_H}\}$ are neighbors of u_c in H, along with the vertices $(u_c, v_1), (u_c, v_2), ...(u_c, v_{\Delta_G})$ where $\{v_1, v_2, ...v_{\Delta_G}\}$ are neighbors of v_d in G. Since $\alpha_t(H) = t\alpha_1(H)$ for all $t \leq \gamma_H$, $PD_t(G \square H)$ consists of $t[(\Delta_H + 1) + \Delta_G]$ elements. □

Theorem 5.2. *In the cartesian product of a complete graph K_m and a path P_n,*

$$\alpha_t(K_m \square P_n) = \frac{t(m+2)}{mn}$$

if $t \leq \gamma(P_n) - \{\lceil \frac{n}{3} \rceil - \lfloor \frac{n}{3} \rfloor\}$.

Proof. Let the vertex set of K_m and P_n be $\{u_1, u_2, ...u_m\}$ and $\{v_1, v_2, ...v_n\}$ respectively. A vertex (u_c, v_d) in $PD_1^2(K_m \square P_n)$ consists of the vertices (u_i, v_d) where $i = 1, 2, ..m$ along with the two vertices (u_c, v_{d-1}) and (u_c, v_{d+1}) where v_{d-1} and v_{d+1} are neighbors of v_d in P_n. Figure 6 shows an example where $m = 4$ and $n = 3$. For higher order paths P_n when $n = 3k$, the pattern in Figure 6 repeats and the bound for t follows. □

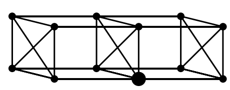

FIGURE 6. t-Partial domination in $K_4 \square P_3$ with $t = 1$

Theorem 5.3. *In cartesian product of cycle C_m and path P_n where $n, m \geq 3$,*

$$\alpha_t(C_m \square P_n) = \frac{5t}{mn}$$

for $t < \gamma(C_m)$.

Proof. Let the vertex set of C_m and P_n be $\{u_1, u_2, ...u_m\}$ and $\{v_1, v_2, ...v_n\}$ respectively. A vertex (u_c, v_d) in $PD_1^2(C_m \square P_n)$ dominates the vertices (u_{i+1}, v_d) and (u_{i-1}, v_d) where u_{i+1} and u_{i-1} are neighbors of u_c in C_m. Also the two vertices (u_c, v_{d-1}) and (u_c, v_{d+1}) where v_{d-1} and v_{d+1} are neighbors of v_d in P_n are dominated by (u_c, v_d). Thus (u_c, v_d) dominates 5 vertices in $C_m \square P_n$. Similarly for $t < \gamma(C_m)$, there are $5t$ vertices in PD_t^2. □

Theorem 5.4. *In lexicographic product of G and H with n and m vertices respectively,* $\alpha_t(G[H]) = \frac{m[n\alpha_t(G)-t]+t(\Delta_H+1)}{mn}$, $t \leq \gamma(G)$.

Proof. Let the vertex set of G and H be $\{u_1, u_2, ...u_n\}$ and $\{v_1, v_2, ...v_m\}$ respectively. Let $\{u_{i+1}, u_{i+2}, ...u_{i+t}\}$ be the t vertices in G which dominate maximum number of vertices in G and v_j is the vertex of maximum degree in H. Correspondingly the t vertices $(u_{i+1}, v_j), (u_{i+2}, v_j), (u_{i+3}, v_j)......(u_{i+t}, v_j)$ in $G[H]$ dominates maximum number of vertices in $G[H]$. Each of these (u_{i+k}, v_j), $k = 1, 2, ...t$, dominates all (u_i, v_c) where u_i's are adjacent to u_{i+k} and v_c is any vertex in H. Moreover each of these are adjacent to (u_i, v_d) also, where v_d is adjacent to v_j in H. ie, Corresponding to the $n\alpha_t(G)$ vertices in $PD_t^2(G)$, there are $mn\alpha_t(G) - mt$ vertices in $PD_t^2(G[H])$ (using the first condition of lexicographic product)and additionally $t(\Delta_H + 1)$ vertices using the second condition of lexicographic product. Figure 8 illustrates $PD_t^2(G[H])$ of graphs G and H shown in Figure 7.

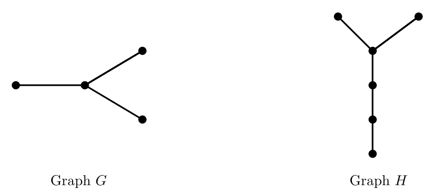

FIGURE 7. Graphs G and H

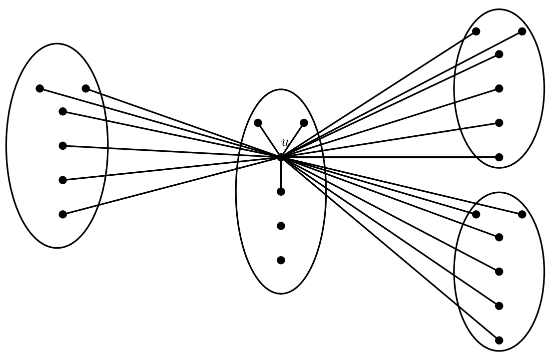

FIGURE 8. Lexicographic product of G and H.

□

Theorem 5.5. *In corona product of G and H with n and m vertices respectively,*

$$\alpha_t(GoH) = \frac{n\alpha_t(G) + tm}{mn},$$

for $t \leq \gamma(G)$.

Proof. Since GoH contains a copy of G and n copies of H, the t vertices which dominates maximum number of vertices in G and GoH will be same. The maximum number of vertices dominated by t vertices in G is same as the corresponding vertices dominated by t vertices in GoH with additional m vertices which are adjacent to each of these t vertices. Thus $PD_t(GoH)$ includes all the vertices in $PD_t(G)$ and additionally tm vertices. □

5.2. Results for $\alpha_t^{(2)}$ on product graphs.

Theorem 5.6. *In a graph* $G = P_2 \square P_n$, $n \geq 5$, $\alpha_t^{(2)}(G) = \frac{8t}{2n}$ *for* $t < \gamma_2(G)$ *if* $n \neq 4k$ *and for* $t < \gamma_2(G) - 1$ *if* $n = 4k$.

Proof. To find $\alpha_t^{(2)}$ when $n \neq 4k$, from Figure 9, it is easy to see that there is a trapezium shaped tiling pattern where each tile is made of 8 vertices which are distance-2 dominated by a single vertex in it. An extra vertex set $W = \{2, 4, 6\}$, where $|W| = 2n$ *modulo* 8, remains out of the t tiles. To establish the bound for t, it can be seen that by shifting all the t distance-2 dominating vertices one position to the left and including the neighbor of the right end vertex following the shorter side of the last trapezium tile, we get the distance-2 dominating set, i.e., $\gamma_2(P_2 \square P_n)$. In the case of $n = 4k$ similar argument follows except that now we must also include the vertex in the left end which is out of the tile.

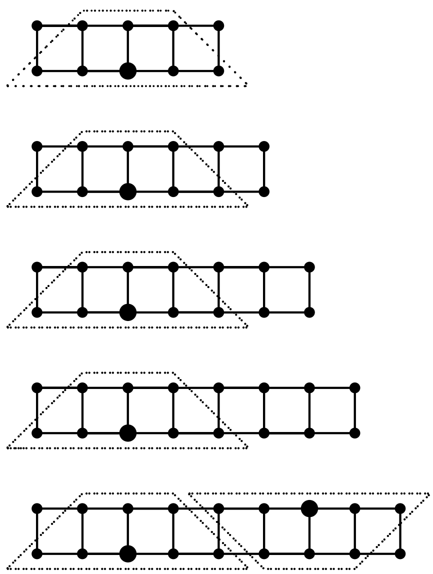

FIGURE 9. t-Partial domination in $P_2 \square P_n$

□

Theorem 5.7. *For a graph G of order n and path P_m, $\alpha_t^{(2)}(G) \le \alpha_{mt}^{(2)}(G \square P_m)$.*

Proof. Let the vertex set of G and P_m be $\{u_1, u_2, ...u_n\}$ and $\{v_1, v_2, ...v_m\}$ respectively. Let $u_1, u_2, ...u_t$ be the t vertices which distance-2 dominates maximum number of vertices of G. Let $\left|PD_t^2(G)\right| = k$. Then $\alpha_t^2(G) = \frac{k}{n}$. The mt vertices (u_i, v_j), $i = 1,2,3....t$, $j = 1,2,3....m$ can distance-2 domiminate at least mk vertices in $G \square P_m$. That is corresponding to the k vertices dominated by t vertices in G, at least mk vertices are dominated by mt vertices in $G \square P_m$. Since any $(u_i v_j)$ is also adjacent to (u_i, v_{j-1}) and (u_i, v_{j+1}), these two vertices along with their neighbours are also within distance-2 from (u_i, v_j). This increases $\left|PD_{mt}^2(G \square P_m)\right|$. Thus, $\alpha_t^{(2)}(G) \le \alpha_{mt}^{(2)}(G \square P_m)$. □

Theorem 5.8. *For the graph $G = K_m \square P_n$,*

$$\alpha_t^{(2)}(G) = \frac{(3m+2)t}{mn},$$

for $t \le \lfloor \frac{n}{5} \rfloor$ when $n \in \{6,7,8,11,12,16\}$, and for $t \le \lceil \frac{n}{5} \rceil$ otherwise.

Proof. Let the vertex set of K_m and P_n be $\{u_1, u_2, ...u_m\}$ and $\{v_1, v_2, ...v_n\}$ respectively. Let (u_i, v_j) be a vertex in $k_m \square P_n$ which can dominate maximum number of vertices and $v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}$ be 5 consecutive vertices in P_n. Then (u_i, v_j) can distance-2 dominate at most vertices of the type (u_i, v_j), (u_i, v_{j-1}) and (u_i, v_{j+1}) for $i = 1,2,3...m$ along with the two vertices, (u_i, v_{j-2}) and (u_i, v_{j+2}). So $PD_t^2(k_m \square P_n)$ contains $(3m+2)t$ vertices.

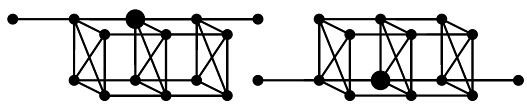

FIGURE 10. t-Partial domination in $k_4 \square P_9$ with $t = 2$

From Figure 10, it is clear that when $n = 5 + 5s + 4t$, for $s, t = 0,1,2,3...$ (which includes all n except 6,7,8,11,12,16), the maximum possible value of t for which $\left|PD_t^2(k_m \square P_n)\right| = (3m+2)t$ is $\gamma(P_n)$. □

Theorem 5.9. *For graphs G and H of order n and m respectively,*

$$\alpha_1^{(2)}(G \square H) \le \frac{\Delta_G^2 + \Delta_H^2 + \Delta_G \Delta_H + 1}{mn}.$$

Proof. Let the vertex set of G and H be $\{u_1, u_2, ...u_n\}$ and $\{v_1, v_2, ...v_m\}$ respectively. We have seen that, any PD_1^2 set in G can have maximum $\Delta_G^2 + 1$ elements. Let (u_i, v_j) be a vertex in $G \square H$ which can dominate maximum number of vertices. $PD_1^2(G \square H)$ consists of at most vertices of the type:

(1) $(u_1, v_j), (u_2, v_j), (u_3, v_j), ...(u_{\Delta_G^2+1}, v_j)$, where $u_1, u_2, u_3.......u_{\Delta_G^2+1}$ are in $PD_1^2(G)$, which counts to $\Delta_G^2 + 1$ vertices including (u_i, v_j).

(2) Each of Δ_G vertices in $G\square H$ corresponding to the vertices in $PD_1^2(G)$ are adjacent to Δ_H other vertices of the type (u_c, v_d), where u_c is adjacent to u_i in G and v_d is adjacent to v_j in H. So there are $\Delta_G\Delta_H$ such vertices.
(3) $(u_i, v_1), (u_i, v_2), (u_i, v_3)....(u_i, v_{\Delta_H^2+1})$ where $v_1, v_2, ...v_{\Delta_H^2+1}$ are vertices distance-2 dominated by v_j in H. There are Δ_H^2 vertices of this kind excluding (u_i, v_j).

So, $\alpha_1^{(2)}(G\square H) \leq \frac{\Delta_G^2+\Delta_H^2+\Delta_G\Delta_H+1}{mn}$, as shown in Figure 11 and Figure 12.

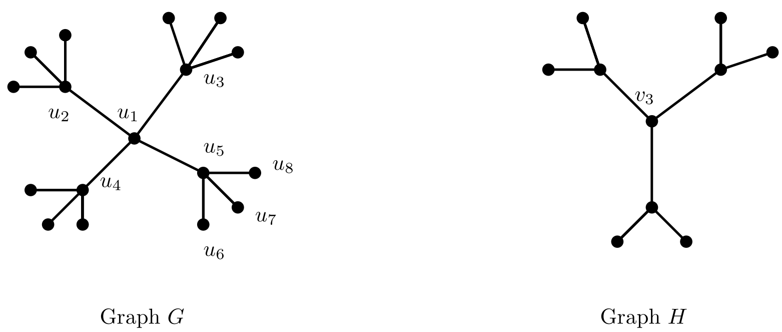

Figure 11. Graphs G and H

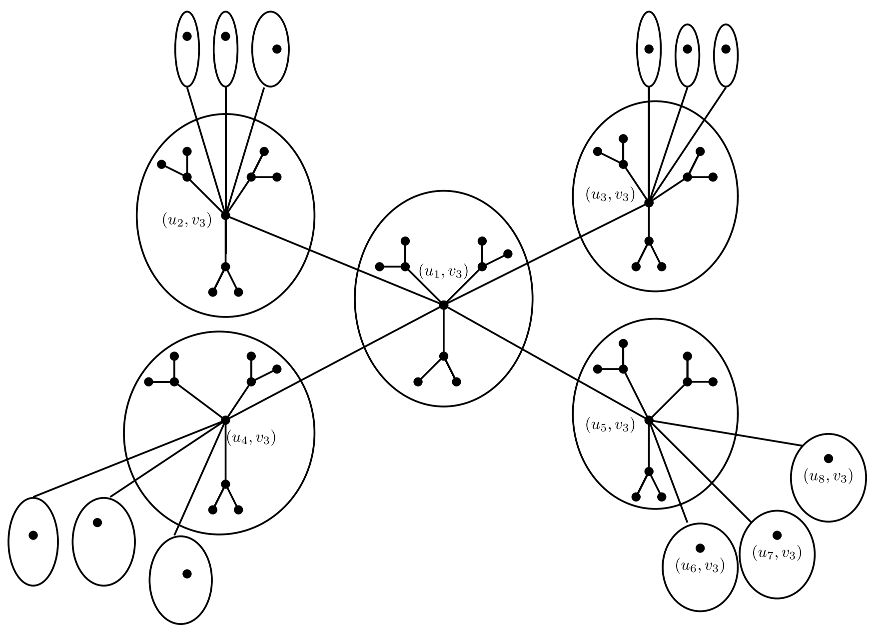

Figure 12. Cartesian product of G and H

□

Corollary 5.10. *If G is a graph of order n then $\alpha_1^{(2)}(G\square P_m) \leq \frac{(\Delta+1)^2+4}{mn}$ for $m \geq 5$, where Δ is maximum degree of G.*

Theorem 5.11. *In lexicographic product of G and H, $\alpha_1^{(2)}(G[H]) = \alpha_1^2(G)$.*

Proof. Let the vertex set of G and H be $\{u_1, u_2, ...u_n\}$ and $\{v_1, v_2, ...v_m\}$ respectively. $PD_1^2(G)$ contains $n\alpha_1^{(2)}(G)$ vertices. Consider a vertex (u_c, v_d) in $G[H]$ which can dominate maximum number of vertices in $G[H]$ where u_c is adjacent to $n\alpha_1^{(2)}(G)$ vertices in G, including itself. (u_c, v_d) is within distance-2 from all such (u_i, v_j) where u_i is in $PD_1^2(G)$ and v_j is any vertex in the vertex set of H. Hence, $PD_1^2(G[H])$ consists of $mn\alpha_1^{(2)}(G)$ elements. So $\alpha_1^{(2)}(G[H]) = \alpha_1^{(2)}(G)$. □

Theorem 5.12. *In corona product of G and H of order n and m respectively,*

$$\alpha_1^{(2)}(GoH) \leq \frac{(\Delta_G+1)m+\Delta_G^2+1}{mn}.$$

Proof. Let u be a vertex in G which distance-2 dominates maximum number of vertices. As GoH contains a copy of G and n copies of H with each i^{th} vertex of G joined to all the vertices of i^{th} copy of H, u distance-2 dominates maximum number of vertices in GoH also. We have seen that $PD_1^2(G)$ contains at most Δ^2+1 vertices. Among these, u and its Δ neighbours are also adjacent to all the m vertices of H in GoH. Then $PD_1^2(GoH)$ consists of at most $(\Delta_G+1)m + (\Delta_G^2+1)$ elements. □

Theorem 5.13. *In the corona product of C_n and P_m,*

$$\alpha_t^{(2)}(C_n o P_m) = \frac{(5+3m)t}{mn+n},$$

for $t < \gamma(C_n)$ and $n \geq 5$.

Proof. Let $u_1, u_2, u_3, u_4, u_5...u_n$ be the vertices of cycle C_n. There are n copies of P_m and each vertex of these $(P_m)_i$ is joined to u_i in the corona product $C_n o P_m$. One vertex say u_3 distance-2 dominates u_1, u_2, u_3, u_4, u_5 and also all the vertices of 3 copies of P_m, which are joined to u_2, u_3 and u_4. Thus, $|PD_t^2(C_n o P_m)| = (3m+5)t$. □

Conclusion

In this study, we presented bounds related to the t-partial domination ratio, investigated its distance-2 variant, and explored these concepts in various graph products, including cartesian, lexicographic, and corona products. The distance-2 version of partial domination provides the added advantage of an extended reach, making it more effective for identifying influential or strategic vertices. This extended range enhances decision-making and optimization by enabling more meaningful comparisons between cost and coverage. In short, the t-partial domination ratio has promising applications in all domains where there is a need to balance coverage efficiency with structural constraints such as network design, resource allocation, and other areas of operational planning.

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