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DISTANCE MAGIC LABELING OF GENERALIZED MYCIELSKIAN GRAPHS

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Abstract. A graph $G = (V, E)$ is said to be a distance magic graph if there is a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that the vertex weight $w(u) = \sum_{v \in N(u)} f(v) = k$ is constant and independent of u, where $N(u)$ is the open neighborhood of the vertex u. The constant k is called a *magic constant*, the function f is called a *distance magic labeling of the graph* G and the graph which admits such a labeling is called a *distance magic graph*. In this paper, we present some results on distance magic labeling of generalized Mycielskian graphs.

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1. Introduction

Throughout this paper, by a graph $G = (V, E)$, we mean a finite connected undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. For graph-theoretic terminology

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and notation, we refer to [1].

A *labeling* of a graph is any function that assigns elements of a graph (vertices or edges or both) to the set of numbers (positive integers or elements of groups, etc). In particular, if we have a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ then f is called a *vertex labeling*. The *neighborhood* of a vertex x in G is the set of all vertices adjacent to it and is denoted by $N_G(x)$. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the size of its neighborhood set. When a graph G is clear from the context, we will simply write $N(x)$ and $\deg(x)$ for the neighborhood and degree of a vertex x, respectively. The *weight* of a vertex v, denoted by $w(v)$, is defined as $w(v) = \sum_{u \in N(v)} f(u)$. If f is a vertex labeling such that $w(v) = k$, for all $v \in V(G)$, then k is called a *magic constant* and the labeling f is called a *distance magic labeling*. A graph that admits such a labeling is called a *distance magic graph*. For more details (see [2–7]).

Thus, it is natural to inquire into both the characterization of distance magic graphs and the determination of magic constants. In fact, it has been shown that for every natural number $k \in \mathbb{N} \setminus \{1, 2, 4, 6, 8, 12, 16\}$, there exists a distance magic graph with a magic constant k [8]. However, a complete characterization of the class of distance magic graphs is not yet known. Moreover, this class is believed to be very small and difficult to describe [8, 9]. Consequently, several attempts are made to characterize the distance magic graphs within a particular graph families–such as path, cycle, complete multipartite graphs, circulant graphs, etc [3]. In [10], the authors studied this problem for the graphs obtained by

Mycielskian construction for several graph classes. The Mycielskian $\mu(G)$ of a graph $G = (V, E)$ is the graph with $V(\mu(G)) = \{(x,0),(x,1) : x \in V(G)\} \cup \{u\}$ and $E(\mu(G)) = \{(x,0)(y,0),(x,0)(y,1) : xy \in E(G)\} \cup \{(x,1)u : x \in V(G)\}$ (see [11]). Using this construction Mycielski proved the existence of triangle-free graphs with arbitrarily large chromatic number.

The Mycielski's construction was generalized independently by Stiebitz [12] and Ngoc [13] in their dissertations in the following way: Given a graph $G = (V, E)$ and an integer $m \geq 0$, *Generalized Mycielskian graph* $\mu_m(G)$ as the graph with vertex set $\{(x,i) : 0 \leq i \leq m, x \in V(G)\} \cup \{u\}$, where there are edges $\{(x,0)(y,0),(x,i)(y,i+1) : 0 \leq i \leq m\}$ in $\mu_m(G)$ whenever there is an edge $xy \in E(G)$, and an edge $(x,m)u$ for all $x \in V(G)$. Observe that, for a connected graph G with $|V(G)| = n$, $\mu_m(G)$ is also connected with $|V(\mu_m(G))| = mn + n + 1$. Note that $\mu_0(G) \cong G$ and $\mu_1(G) \cong \mu(G)$. The construction is illustrated in Figure 1 for $\mu_2(P_3)$.

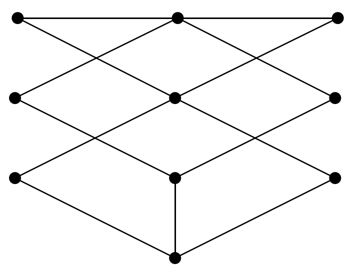

Figure 1. Generalized Mycielskian $\mu_2(P_3)$.

In this paper, we discuss the existence of distance magic labelings for some graph families obtained by applying the generalized Mycielskian construction. Throughout the paper, we assume $m \geq 2$.

We need the following results for our further investigation.

Theorem 1.1. *[4, 7] A graph G is not distance magic if there are vertices x and y in G such that* $|N(x) \triangle N(y)| = 1$ *or* 2.

Theorem 1.2. *[5–7] Let f be a distance magic labeling of a graph G. Then the sum of weights of all the vertices is given by:*

$$\sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} \deg(v) f(v) = kn,$$

where k is the magic constant and n is the number of vertices.

Corollary 1.3. *[4, 5, 7] No odd regular graph is distance magic.*

Theorem 1.4. *[5] Cycle* C_n *is distance magic if and only if* $n = 4$.

Theorem 1.5. *[5] A wheel* $W_n = C_n + K_1$ *is distance magic if and only if* $n = 4$.

Definition 1.6. *[14] A function* $g : V(G) \to [0, 1]$ *is called a total dominating function of the graph G if* $\sum_{u \in N(v)} g(u) \geq 1$ *for all* $v \in V$. *The fractional total domination number of a graph G is denoted by* γ_{ft} *and is given by*

$$\gamma_{ft} = \ min\{|g| : g \text{ is total dominating function of } G\},$$

where $|g| = \sum_{v \in V} g(v)$.

Theorem 1.7. *[15] For any graph G without isolated vertices,*

$$\gamma_{ft}(\mu_m(G)) = \begin{cases} \frac{m}{2}\gamma_{ft}(G) + 2 - \frac{1}{\gamma_{ft}(G)} & \text{if } m \text{ is even} \\ \frac{m+1}{2}\gamma_{ft}(G) + \frac{1}{\gamma_{ft}(G)} & \text{if } m \text{ is odd.} \end{cases}$$

Theorem 1.8. *[16,17] If G is a distance magic graph of order n, then the distance magic constant k is given by $k = \frac{n(n+1)}{2\gamma_{ft}(G)}$.*

2. Main Results

Let $G = (V, E)$ be a graph. The subgraph induced by the subset $S = \{(x, 0) \in V(\mu_m(G)) : x \in V(G)\}$ of $V(\mu_m(G))$ is isomorphic to G. First, we discuss some basic structural properties of the generalized Mycielskian of a graph such as regularity, degree conditions, etc. By construction, we have $\deg_{\mu_m(G)}(x_i, 0) = \deg_{\mu_m(G)}(x_i, j)$, for all $1 \le j \le (m-1)$.

Observation 1. *For $x \in V(G)$, $\deg_{\mu_m(G)}(x, i) = \deg_{\mu_m(G)}(x, 0) = 2\deg_G(x)$, for $1 \le i \le (m-1)$ and $\deg_{\mu_m(G)}(x, m) = \frac{\deg_{\mu_m(G)}(x,0)}{2} + 1$.*

Proposition 2.1. *$\mu_m(G)$ is r-regular if and only if $G \cong K_2$.*

Proof. Let G be a graph on n vertices such that $\mu_m(G)$ is r-regular. Therefore,

$$\deg(u) = \deg(x, i) = r \text{ for } i = 0, 1, 2, \ldots, m, \forall x \in V(G). \tag{1}$$

But, $\deg(u) = n$. Hence,

$$r = n. \tag{2}$$

By Observation 1, we have $\deg(x, 1) = \frac{\deg(x,0)}{2} + 1$. Using this in Equation (1) we get $r = 2$. From Equation (2), we get $n = r = 2$. This means G is a graph on 2 vertices such that $\mu(G)$ is 2-regular. Now on 2 vertices only two graphs are possible; K_2 or its complement $\overline{K_2}$. For $x \in V(\overline{K_2})$, $\deg_{\mu_m(\overline{K_2})}(x, 0) = 0$ and $\deg_{\mu_m(\overline{K_2})}(u) = 2$. Therefore, $\mu_m(\overline{K_2})$ is not regular. Hence, $G \ncong \overline{K_2}$. Therefore, G must be isomorphic to K_2. Conversely if $G \cong K_2$, then $\mu_m(K_2) \cong C_{2r+1}$ which is 2-regular. This completes the proof. □

Now we provide the sufficient conditions on a graph G, for the non-existence of distance magic labeling of its generalized Mycielskian graph $\mu_m(G)$ for $m \geq 2$.

Lemma 2.1. *For a graph G with $\delta(G) = 1$, then $\mu_m(G)$ is not distance magic.*

Proof. Let x_1 be a vertex in G such that $\deg_G(x_1) = 1$. Hence, there is a unique vertex $x_2 \in N_G(x_1)$. Then in $\mu_m(G)$ we have, $N_{\mu_m(G)}(x_1, 0) = \{(x_2, 0), (x_2, 1)\}$ and $N_{\mu_m(G)}(x_1, 1) = \{(x_2, 0), (x_2, 2)\}$. This gives $|N_{\mu(G)}(x_1, 0) \triangle N_{\mu(G)}(x_1, 1)| = |\{(x_2, 0), (x_2, 2)\}| = 2$. Hence, by Theorem 1.1, $\mu_m(G)$ is not distance magic. □

This lemma immediately gives non-existence of magic labeling of Mycielskian of a major family of graphs.

Corollary 2.2. *If T is a tree, then $\mu_m(T)$ is not distance magic.*

Lemma 2.3. *If a graph G contains two vertices x_i and x_j such that $|N_G(x_i) \triangle N_G(x_j)| = 2$, then $\mu_m(G)$ is not distance magic.*

Proof. Let x_i and x_j be two distinct vertices in G such that $|N_G(x_i) \triangle N_G(x_j)| = 2$. Then

$$N_{\mu_m(G)}(x_i, m) = \{(y, m-1) : y \in N_G(x_i)\} \cup \{u\}$$
$$N_{\mu_m(G)}(x_j, m) = \{(y, m-1) : y \in N_G(x_j)\} \cup \{u\}.$$

Therefore,

$$|N_{\mu_m(G)}(x_i, m) \triangle N_{\mu_m(G)}(x_j, m)| = |N_G(x_i) \triangle N_G(x_j)| = 2$$

and by Theorem 1.1, $\mu_m(G)$ is not distance magic. □

Corollary 2.4. *$\mu_m(K_n)$ is not distance magic graph.*

Proof. Let vertices of K_n be labeled as $x_1, x_2, \ldots, x_n$ in an anticlockwise manner. If $n = 1$, then $\mu(K_1) \cong K_1 \cup K_2$ is not distance magic. So, we assume $n \geq 2$. Then, $N_{K_n}(x_1) = \{x_2, x_3, \ldots, x_n\}$ and $N_{K_n}(x_2) = \{x_1, x_3, x_4, \ldots, x_n\}$. Hence, $|N_{K_n}(x_1) \triangle N_{K_n}(x_2)| = 2$ and by Lemma 2.3, $\mu_m(K_n)$ is not distance magic. □

Corollary 2.5. *If $n(\neq 4) \geq 3$ then $\mu_m(C_n)$ is not distance magic.*

Proof. Let $x_1, x_2, \ldots, x_n$ be the vertices of the cycle C_n taken in an anticlockwise manner. Now the $|N(x_1) \triangle N(x_{n-1})| = |\{x_2, x_{n-2}\}| = 2$ and hence by Lemma 2.3, $\mu_m(C_n)$ is not distance magic for $n(\neq 4) \geq 3$. □

We know that $\gamma_{ft}(C_n) = \frac{n}{2}$. By Theorem 1.7,

$$\gamma_{ft}(\mu_m(C_n)) = \begin{cases} \frac{mn^2+8n-8}{4n} & \text{if } m \text{ is even} \\ \frac{n^2(m+1)+8}{4n} & \text{if } m \text{ is odd} \end{cases}$$

and by Theorem 1.8, if $\mu_m(C_n)$ is distance magic then its magic constant is given by

$$k = \begin{cases} \frac{2n(mn+n+1)(mn+n+2)}{mn^2+8n-8} & \text{if } m \text{ is even} \\ \frac{2n(mn+n+1)(mn+n+2)}{n^2(m+1)+8} & \text{if } m \text{ is odd.} \end{cases}$$

Observe that $k = 8m + 10$ for $n = 4$ and for any $m \geq 1$ and it need not be always an integer for other values of m and n.

Theorem 2.6. *The generalized Mycielskian $\mu_m(C_n)$ is distance magic if and only if $n = 4$.*

Proof. Let $n \neq 4$, then by Corollary 2.5, $\mu_m(C_n)$ is not distance magic. For $n = 4$, let x_1, x_2, x_3, x_4 be the vertices of C_4 taken in an anticlockwise manner. We define the labeling f as follows:

$$f(x_i, j) = \begin{cases} 2j+1, \text{ if } i = 1, j = 0, 1, \ldots, m \\ 2j+2, \text{ if } i = 2, j = 0, 1, \ldots, m \\ 4(m+1) - 2j, \text{ if } i = 3, j = 0, 1, \ldots, m \\ 4(m+1) - 2j - 1, \text{ if } i = 4, j = 0, 1, \ldots, m \end{cases}$$

and $f(u) = 4m + 5$. It is easy to see that f is a distance magic labeling with magic constant $8m + 10$. A distance magic labeling of $\mu_3(C_4)$ is given in Figure 2. □

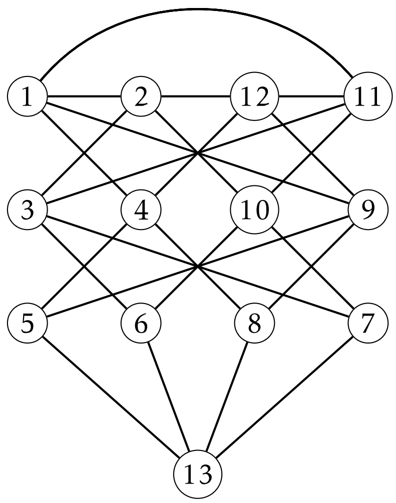

FIGURE 2. Distance magic labeling of $\mu_2(C_4)$.

Lemma 2.7. *$\mu_m(W_n)$, where $W_n = C_n + K_1$ is not distance magic for $n(\neq 4) \geq 3$.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the set of vertices that lie on the rim of $W_n = C_n + K_1$ and c_1 be the central vertex, where $n \geq 3$. If $n = 3$ then $W_3 \cong K_4$ and by Corollary 2.4, $\mu_m(W_3)$ is not distance magic. Next, we prove the case $n \geq 5$. On the contrary, suppose that the Mycielskian of the wheel $W_n = C_n + K_1$ is distance magic with a distance magic labeling f. Then, $N_G(x_1) = \{x_2, x_n, c_1\}$ and $N_G(x_3) = \{x_2, x_4, c_1\}$, where the subscripts are taken to be modulo n. Then, $|N_G(x_1) \triangle N_G(x_3)| = |\{x_4, x_n\}| = 2$. Therefore, by Lemma 2.3, the generalized Mycielskian of a wheel $W_n = C_n + K_1$ for $n(\neq 4) \geq 3$ is not distance magic. □

The fractional total domination number of W_4 is known to be $\gamma_{ft}(W_4) = \frac{3}{2}$. Referring to Theorem 1.7, we can determine that

$$\gamma_{ft}(\mu_m(W_4)) = \begin{cases} \frac{9m+16}{12} & \text{if } m \text{ is even} \\ \frac{9m+17}{12} & \text{if } m \text{ is odd} \end{cases}$$

Furthermore, according to Theorem 1.8, if $\mu_m(W_4)$ is a distance magic graph, its magic constant k is given by

$$k = \begin{cases} \frac{6(5m+6)(5m+7)}{9m+16} & \text{if } m \text{ is even} \\ \frac{6(5m+6)(5m+7)}{9m+17} & \text{if } m \text{ is odd} \end{cases}$$

It should be noted that k may not always be an integer. Specifically, for $m = 2$ and $m = 4$, we obtain integer values $k = 48$ and $k = 81$, respectively. Now we discuss the distance magic labeling of generalized Mycielskian of wheel graphs.

Lemma 2.8. *The generalized Mycielskian of wheel $\mu_m(W_4)$ is not distance magic.*

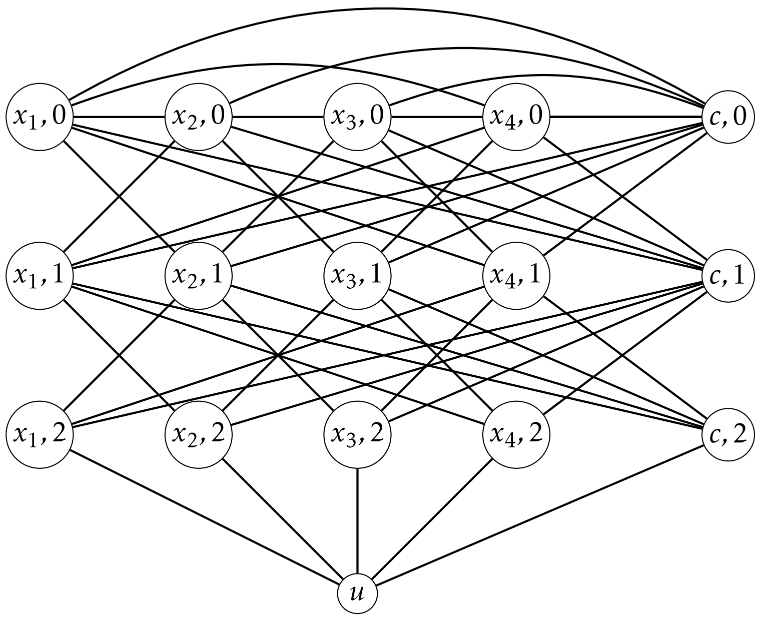

FIGURE 3. $\mu_2(W_4)$.

Proof. Suppose $\mu_m(W_4)$ is distance magic and having distance magic labeling f. Let $\{x_1, x_2, x_3, x_4\}$ be the set of vertices lying on the rim of W_4 and c_1 be the central vertex.

Then the neighborhood of the vertices are as follows:

$$
\begin{aligned}
N(x_1,0) &= N(x_3,0) = \{(x_2,0),(x_4,0),(x_2,1),(x_4,1),(c,0),(c,1)\}\\
N(x_2,0) &= N(x_4,0) = \{(x_1,0),(x_3,0),(x_1,1),(x_3,1),(c,0),(c,1)\}\\
N(x_1,i) &= N(x_3,i) = \{(x_2,i-1),(x_4,i-1),(x_2,i+1),(x_4,i+1),\\
&\qquad (c,i-1),(c,i+1)\}, \quad \text{for } 1 \le i \le m-1\\
N(x_2,i) &= N(x_4,i) = \{(x_1,i-1),(x_3,i-1),(x_1,i+1),(x_3,i+1),\\
&\qquad (c,i-1),(c,i+1)\}, \quad \text{for } 1 \le i \le m-1\\
N(x_1,m) &= N(x_3,m) = \{(x_2,m-1),(x_4,m-1),(c,m-1),u\}\\
N(x_2,m) &= N(x_4,m) = \{(x_1,m-1),(x_3,m-1),(c,m-1),u\}\\
N(u) &= \{(x_1,m),(x_2,m),(x_3,m),(x_4,m),(c,m)\}\\
N(c,0) &= \{(x_1,0),(x_2,0),(x_3,0),(x_4,0),(x_1,1),(x_2,1),(x_3,1),(x_4,1)\}\\
N(c,i) &= \{(x_1,i-1),(x_2,i-1),(x_3,i-1),(x_4,i-1),(x_1,i+1),\\
&\qquad (x_2,i+1),(x_3,i+1),(x_4,i+1)\}, \quad \text{for } 1 \le i \le m-1\\
N(c,m) &= \{(x_1,m-1),(x_2,m-1),(x_3,m-1),(x_4,m-1),u\}.
\end{aligned}
$$

We first prove the result for $m = 2$. The generalized Mycielskian of W_4 with $m = 2$ is shown in Figure 3. We assume that,

$$
\begin{aligned}
\alpha_i &= f(x_1,i) + f(x_3,i) \text{ for } i = 0,1,2\\
\beta_i &= f(x_2,i) + f(x_4,i) \text{ for } i = 0,1,2.
\end{aligned}
$$

With the above notations, we express the weights of vertices in $\mu_m(W_4)$ under f as follows:

(3) $$w(x_1,0) = \beta_0 + \beta_1 + f(c,0) + f(c,1)$$

(4) $$w(x_2,0) = \alpha_0 + \alpha_1 + f(c,0) + f(c,1)$$

(5) $$w(x_1,1) = \beta_0 + \beta_2 + f(c,0) + f(c,2)$$

(6) $$w(x_2,1) = \alpha_0 + \alpha_2 + f(c,0) + f(c,2)$$

(7) $$w(x_1,2) = \beta_1 + f(c,1) + f(u)$$

(8) $$w(x_2,2) = \alpha_1 + f(c,1) + f(u)$$

(9) $$w(c,0) = \alpha_0 + \beta_0 + \alpha_1 + \beta_1$$

(10) $$w(c,1) = \alpha_0 + \beta_0 + \alpha_2 + \beta_2$$

(11) $$w(c,2) = \alpha_1 + \beta_1 + f(u)$$

(12) $$w(u) = \alpha_2 + \beta_2 + f(c,2).$$

Now we equate: Equation (3) with Equation (7), Equation (4) with Equation (8), Equation (5) with Equation (12), Equation (6) with Equation (12), and Equation (9) with Equation (10) respectively, to obtain,

(13) $$f(u) = \beta_0 + f(c,0)$$

(14) $$f(u) = \alpha_0 + f(c,0)$$

(15) $$\alpha_2 = \beta_0 + f(c,0)$$

(16) $$\beta_2 = \alpha_0 + f(c,0)$$

(17) $$\alpha_1 + \beta_1 = \alpha_2 + \beta_2.$$

Again from Equations (14), (13), (15), (16) we obtain $f(u) = \alpha_2 = \beta_2$. Using this value of α_2 and β_2 in Equation (17) we get $\alpha_1 + \beta_1 = 2f(u)$. Therefore Equation (11) becomes $w(c,2) = 3f(u)$. Since f is assumed to be distance magic labeling we must have $w(u) = 3f(u)$. From Equation (12) we have $3f(u) = 2f(u) + f(c,2) \implies f(u) = f(c,2)$ which is a contradiction. Hence $\mu_m(W_4)$ is not distance magic for $m = 2$. Now, we suppose that $m \geq 3$. By equating the weights: $w(x_1, 0) = w(x_1, 2)$, $w(x_2, 0) = w(x_2, 2)$ and $w(c, 0) = w(c, 2)$, respectively, we get

(18)
$$f(x_2, 0) + f(x_4, 0) + f(c, 0) = f(x_2, 3) + f(x_4, 3) + f(c, 3)$$
(19)
$$f(x_1, 0) + f(x_3, 0) + f(c, 0) = f(x_1, 3) + f(x_3, 3) + f(c, 3)$$
(20)
$$f(x_1, 0) + f(x_2, 0) + f(x_3, 0) + f(x_4, 0) \\ = f(x_1, 3) + f(x_2, 3) + f(x_3, 3) + f(x_4, 3).$$

Adding Equations (18) and (19) we get

$$f(x_1, 0) + f(x_2, 0) + f(x_3, 0) + f(x_4, 0) + 2f(c, 0) \\ = f(x_1, 3) + f(x_2, 3) + f(x_3, 3) + f(x_4, 3) + 2f(c, 3).$$

Then using Equation (20), we get $f(c, 0) = f(c, 3)$, which is a contradiction. This completes the proof. □

From the Lemma 2.7 and the Lemma 2.8, the following result is evident.

Theorem 2.9. *$\mu_m(W_n)$ is not distance magic for $n \geq 3$.*

Let us calculate the fractional total domination number of generalized Mycielskian of complete bipartite graphs. We know that $\gamma_{ft}(K_{a,b}) = 2$. Therefore, by Theorem 1.7, $\gamma_{ft}(\mu_m(K_{a,b})) = \frac{2m+3}{2}$ for all $m \geq 1$. If $\mu_m(K_{a,b})$ is distance magic then by Theorem 1.8, its magic constant is given by $k = \frac{(m(a+b)+a+b+1)(m(a+b)+a+b+2)}{2m+3}$. When $a = b = 1$, then $k = 2m+4$, which is an integer but $K_{1,1} \cong K_2$ by Lemma 2.1, $\mu_m(K_2)$ is not distance magic. For the case where $a = b = 2$ and $m \geq 1$, the value of k evaluates to an integer $8m + 10$. However, for other values of a, b, and m, the calculated value of k may not necessarily be an integer. In the following theorem, we give a simple proof for the distance magic labeling of complete bipartite graphs.

Theorem 2.10. *$\mu_m(K_{a,b})$ is distance magic if and only if $a = b = 2$.*

Proof. Let $G \cong K_{a,b}$, where a and b both are at least 2. Otherwise, G will be a star and by Lemma 2.1, it is not distance magic. Let $V_1 = \{x_1, x_2, \ldots, x_a\}$ and $V_2 = \{y_1, y_2, \ldots, y_b\}$ be the partition of the vertex set of G. Suppose that $\mu_m(G)$ is distance magic with distance magic labeling f. We assume that

$$\sum_{i=1}^{a} f(x_i, j) = \alpha_j, \text{ for } j = 0, 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{b} f(y_i, j) = \beta_j, \text{ for } j = 0, 1, 2, \ldots, m.$$

Therefore, the vertex weights are as follows:

$$w(x_i, 0) = \beta_0 + \beta_1, \text{ for } i = 1, 2, \ldots, a$$
$$w(x_i, j) = \beta_{j-1} + \beta_{j+1}, \text{ for } i = 1, 2, \ldots, a; j = 1, 2, \ldots m-1$$
$$w(x_i, m) = \beta_{m-1} + f(u), \text{ for } i = 1, 2, \ldots, a$$
$$w(y_i, 0) = \alpha_0 + \alpha_1, \text{ for } i = 1, 2, \ldots, b$$
$$w(y_i, j) = \alpha_{j-1} + \alpha_{j+1}, \text{ for } i = 1, 2, \ldots, b;\ j = 1, 2, \ldots m-1$$
$$w(y_i, m) = \alpha_{m-1} + f(u), \text{ for } i = 1, 2, \ldots, b.$$

Since $\mu_m(G)$ is distance magic, the vertex weights are the same under f. Equating the weights, we get

$$\alpha_i = \beta_i = f(u). \tag{21}$$

The vertex u must receive the largest label i.e. $f(u) = (a+b)(m+1)+1$. Otherwise, if one of the vertices (x_i, j) or (y_i, j)

receives the label $(a+b)(m+1)+1$, for some i and j, then one of the equalities

$$\alpha_i = f(u),\ \beta_i = f(u)$$

is not possible. Therefore, from Equation (21) we have

$$\sum_{i=0}^{m} (\alpha_i + \beta_i) = 2(m+1)f(u). \tag{22}$$

Since $f(u) = (a+b)(m+1)+1$. Therefore, $\sum_{i=0}^{m}(\alpha_i + \beta_i)$ is the sum of the first $(a+b)(m+1)$ natural numbers. Therefore, Equation (22) becomes

$$\frac{(a+b)(m+1)[(a+b)(m+1)+1]}{2} = 2(m+1)[(a+b)(m+1)+1].$$

This implies $a+b=4$. Since a and b both are at least 2, we must have $a=b=2$. Conversely, suppose that $a=b=2$. In this case, $K_{2,2} \cong C_4$ and by Theorem 2.6, $\mu_m(C_4)$ is distance magic. This completes the proof. □

Theorem 2.11. *If G is an r-regular graph such that the* $\mu_m(G)$ *is distance magic then* $r < 2(m+1)$.

Proof. Let G be an r-regular graph on n vertices such that $\mu_m(G)$ admits distance magic labeling f. Then $\deg_{\mu_m(G)}(x,i) = 2r$, for $(1 \le i \le m-1)$, $\deg_{\mu_m(G)}(x,m) = r+1$

and $\deg_{\mu_m(G)}(u) = n$. Now by Theorem 1.2 we have

$$\sum_{x\in V(G)} w(x,0) = r\sum_{x\in V(G)} f(x,0) + r\sum_{x\in V(G)} f(x,1) = kn$$
$$\sum_{x\in V(G)} w(x,1) = r\sum_{x\in V(G)} f(x,0) + r\sum_{x\in V(G)} f(x,2) = kn$$
$$\ldots$$
$$\sum_{x\in V(G)} w(x,m-1) = r\sum_{x\in V(G)} f(x,m-2) + r\sum_{x\in V(G)} f(x,m) = kn$$
$$\sum_{x\in V(G)} w(x,m) = r\sum_{x\in V(G)} f(x,m-1) + nf(u) = kn.$$

Comparing the above set of equations we obtain,

$$nf(u) = r\sum_{x\in V(G)} f(x,m). \tag{23}$$

If we assign the smallest labels $1,2,3,\ldots,n$ to the n vertices (x,m) and label u with the largest label, that is, $f(u) = mn+n+1$, then using this in Equation (23), we get $n(mn+n+1) \geq \frac{rn(n+1)}{2}$ which on simplification yields $r < 2(m+1)$. □

Proposition 2.2. *Let G be a graph. If $\mu_m(G)$ is a regular graph, then $\mu_m(G)$ is not distance magic.*

Proof. Let G be a graph on n vertices such that $\mu_m(G)$ is r-regular. Then by Theorem 2.1, $G \cong K_2$. But $\mu_m(K_2) \cong C_{2m+3}$ and by Theorem 1.4, C_{2m+3} is not distance magic. This completes the proof. □

Observation 2. *The graph G and its generalized Mycielskian $\mu_m(G)$ do not share the property of being distance magic, that is, $\mu_m(G)$ is distance magic irrespective of G. For example, the path on 3 vertices P_3 is distance magic [5] but $\mu_m(P_3)$, $(m \geq 1)$ is not distance magic (see Corollary 2.2). Whereas,*

C_4 is distance magic [5] and $\mu_m(C_4)$, $(m \geq 1)$ is also distance magic (see Theorem 2.6).

Conclusion

In this paper, we have studied the distance magic labeling of graphs arising from generalized Mycielskian construction for several graph classes. A complete characterization of distance magic labeling of graphs obtained from generalized Mycielskian construction remains open.

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