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SQUARE POWER GRAPH OF FINITE ABELIAN GROUP OF ODD ORDER

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Abstract. For a finite abelian group G with identity element e, the square power graph of G, $\Gamma_{sq}(G)$ is the undirected simple finite graph having G as vertex set in which two different vertices u, v have edge iff $u + v = 2w$, for any $w \in G$ with $2w \neq e$. $\Gamma_{sq}(G)$ of a finite abelian group G of odd order is always a connected graph, which can be obtained by deleting the non-adjacent edges from complete graph with odd number of vertices. This research paper explores various characteristics and topological indices of the $\Gamma_{sq}(G)$ of a finite abelian group of odd order. Some of these characteristics are connectedness, degree of vertex, size of graph, eccentricity, chromatic number, clique number, perfectness, hamiltonicity, matching number, and topological indices like Wiener, Hyper-Wiener, Gutman, Schultz, Eccentric Connectivity, first and second Zagreb, Harary, Harmonic, Geometric-arithmetic index, Atomic-bond connectivity, General Randic and Randic. We have also studied the Laplacian matrix and its spectrum. We have calculated chromatic polynomials for almost complete graphs obtained by deleting non-adjacent edges from complete graph and so for square power graph of finite abelian group of odd order.

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1. Introduction

We can define many graphs for finite groups by using different group properties [1–8]. Edge-based topological indices for zero divisor graphs of commutative rings are studied in [1], for co-prime order graphs of finite abelian p-groups in [3] and wiener indices for connected graphs in [4]. Various spectral properties and characteristic polynomials of power graph are studied in [9, 10] R.R. Prathap and T.T. Chelvam [5] introduced and characterized many structural properties of the square power graph's complement, $\overline{\Gamma_{sq}(G)}$ for a abelian group G of finite order with 0 as its identity element. The square power graph $\Gamma_{sq}(G)$ is a undirected simple finite graph with vertex set group G itself and two different vertices v and u are adjacent iff $u + v = 2w$ for some $w \in G$ and $2w \neq 0$. Structure of square power graph of $\mathbb{Z}_n$ and $\mathbb{Z}_2^m \times \mathbb{Z}_{2^n}$ is given in [6] and for finite abelian group in [11] where as cubic power graph for dihedral group is studied in [12]. The degree of vertices of k^{th}-power graph is calculated in [7]. For $k \geq 2$, k^{th}-power

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graph $\Gamma_k(G)$ is an undirected simple finite graph, with vertex set group G itself and two different vertices v and u have edge iff $u+v=kw$ for some $w \in G$ with $kw \neq 0$.
In this paper, we have characterized various structural properties such as connectedness, degree of a vertex, size, clique number, chromatic number, perfectness, hamiltonicity, matching number etc in Section 3. Chromatic polynomial of almost complete graphs $K_n - \cup_{i=1}^{\frac{n-1}{2}} e_i$ obtained by deleting $\frac{n-1}{2}$ non-adjacent edges from complete graph K_n with n number of odd vertices and thus for $\Gamma_{sq}(G)$ of finite abelian group of odd order is also calculated in Section 3. Topological indices such as Randic, Wiener, Hyper-Wiener, Geometric-arithmetic, Gutman, Schultz, First and second Zagreb, Harary, eccentric connectivity, Harmonic and Atomic-bond connectivity are calculated in Section 4, where as matching, hamiltonicity and laplacian specteum in Section 5 for the square power graph $\Gamma_{sq}(G)$ of abelian group of finite odd order.

2. Preliminaries

To begin, let us refresh our recollections on a few fundamental graph theory terms, which are necessary for this research paper. Let $\Gamma_{sq}(G)$ is square power graph with finite abelian group G as its vertex set denoted as $V(\Gamma_{sq}(G))$ with corresponding set of edges denoted as $E(\Gamma_{sq}(G))$, where $uv \in E(\Gamma_{sq}(G))$ iff u and v forms edge in $\Gamma_{sq}(G)$. $|G|$, $|E(\Gamma_{sq}(G))|$ and $|V(\Gamma_{sq}(G))|$ denotes cardinality of G group, edge set & vertex set respectively. Degree of any node $w \in V(\Gamma_{sq}(G))$, $deg_{\Gamma_{sq(G)}}(w)$ is number of vertices having edge with w in $\Gamma_{sq}(G)$. For any vertices pair $w, u \in V(\Gamma_{sq}(G))$, distance between them is the shortest $u-w$ path in $\Gamma_{sq}(G)$ denoted as $d(u,w)$ whereas distance from u to any other vertex which is largest in $\Gamma_{sq}(G)$ is known by eccentricity of u and denoted by $ecc(u)$. If for every pair of vertices in $\Gamma_{sq}(G)$, we have path between them then graph is called connected graph otherwise graph is said to be disconnected. In $\Gamma_{sq}(G)$ if every pair of vertices have edge then graph is known as complete graph. By clique we mean the subset of vertex set of graph, such that subgraph having vertex set as that subset is complete. Number of elements or we can simply say vertices in maximal clique is called clique number, $\omega(\Gamma)$. A square power graph's vertex coloring is a mapping m : $V(\Gamma_{sq}(G)) \to K$. K elements are known as colors. If $|K| = k$, then c is known by k-coloring. If the colors of the vertices having edge are different, then coloring is said to be proper. A graph with proper k-coloring is said to be k-colorable. Chromatic number [13, 14] is the least value of k for which Γ_{sq} is k–colorable. For any graph, if chromatic number is equal to clique number then graph is said to be weakly perfect.
Topological indices are numerical values that characterize the structure of graph. Various topological indices for non-commuting graphs are studied by Fawad Ali *et al.* for finite non-abelian group in [15]. Let us discuss some topological properties which we have investigated in this research paper. For a graph $\Gamma_{sq}(G)$, $W(\Gamma_{sq}) = \sum_{\{u,w\} \subset V(\Gamma_{sq})} d(u,w)$ [16] and $WW(\Gamma_{sq}) = \frac{1}{2}W(\Gamma_{sq}) + \frac{1}{2}\sum_{\{u,w\} \subset V(\Gamma_{sq})} d(u,w)^2$ [17] are Wiener and Hyper-Wiener index respectively. First and second Zagreb indices for square power graph $\Gamma_{sq}(G)$

are $M_1(\Gamma_{sq}) = \sum_{w\in V(\Gamma_{sq})}(deg(w))^2$ and $M_2(\Gamma_{sq}) = \sum_{uw\in E(\Gamma_{sq})}[deg(u)\times deg(w)]$ respectively [18]. Harary index for $\Gamma_{sq}(G)$ is given as $\mathcal{H}(\Gamma_{sq}) = \sum_{\{u,w\}\subseteq V(\Gamma_{sq})}\frac{1}{d(u,w)}$ [19]. Schultz index for $\Gamma_{sq}(G)$ is given as $MTI(\Gamma_{sq}) = \sum_{\{w,v\}\subset V(\Gamma_{sq})} d(w,v)[deg(w)+deg(v)]$ [20]. $\xi(\Gamma_{sq}) = \sum_{w\in V(\Gamma_{sq})} ecc(w)deg(w)$ is defined as the eccentric connectivity index of $\Gamma_{sq}(G)$ [21].

Harmonic index of $\Gamma_{sq}(G)$ is $\mathcal{H}_r(\Gamma_{sq}) = \sum_{uw\in E(\Gamma_{sq})}\frac{2}{deg(u)+deg(w)}$ [22]. General Randic index [23] of $\Gamma_{sq}(G)$ is given as $R_\alpha(\Gamma_{sq}) = \sum_{wu\in E(\Gamma_{sq})}(deg(w)\times deg(u))^\alpha$ and Randic index [24] by $R_{-\frac{1}{2}}(\Gamma_{sq}) = \sum_{uw\in E(\Gamma_{sq})}\frac{1}{\sqrt{deg(u)deg(w)}}$. Gutman index for $\Gamma_{sq}(G)$ is given by $Gut(\Gamma_{sq}) = \sum_{\{u,w\}\subset V(\Gamma_{sq})} d(w,u)[deg(w)\times deg(u)]$ [25]. $ABC(\Gamma_{sq}) = \sum_{uw\in E(\Gamma_{sq})}\sqrt{\frac{deg(u)+deg(w)-2}{deg(u)deg(w)}}$ is Atomic-bond connectivity index [26], for $\Gamma_{sq}(G)$. Geometric-arithmetic index [27] of $\Gamma_{sq}(G)$ is given by $GA(\Gamma_{sq}) = \sum_{uw\in E(\Gamma_{sq})}\frac{2\sqrt{deg(u)\times deg(w)}}{deg(u)+deg(w)}$.

A set E of edges from graph Γ, s.t no edges pair in E have common node is called independent edge set or matching. Maximum matching is the matching which have maximum possible edges numbering and order of maximum matching is known as matching number denoted as $\mu(\Gamma_{sq}(G))$. If all the vertices of the graph get saturated by matching then it is known as perfect matching. So for a odd order graph we can not have perfect matching. A cycle which passes through every vertex in graph exactly once is known as Hamiltonian cycle and graph containing Hamiltonian cycle is said to be Hamiltonian graph.

For a finite abelain group G, $U = \{2u : u \in G\} \subseteq G$. When $|G|$ is odd then we have $U = G$. It is also clear that $y, x \in V(\Gamma_{sq}(G))$ have edge iff $y + x \in U \setminus \{0\}$, here 0 is identity of G. We have $\Gamma_{sq}(G)$ of finite abelian group of even order disconnected and of odd order connected. So we are more interested in studying square power graph of finite abelian group of odd order.

3. Structural Properties of $\Gamma_{sq}(G)$

Theorem 3.1. *Let G be an finite odd order abelian group, then $\Gamma_{sq}(G)$ is connected and $\Gamma_{sq}(G) = \overline{K_1 \cup \frac{|G|-1}{2}K_2}$.*

Proof. When G is abelian group with finite odd order with e identity element then we have $G = U$. Now for every $w \in V(\Gamma_{sq}(G)) \setminus \{e\}$ we have $e + w \in U$, so e is adjacent with every node in $\Gamma_{sq}(G)$. Thus $\Gamma_{sq}(G)$ is connected. Also, we have every pair of nodes $u, w \in V(\Gamma_{sq}(G))$ have edge iff $u \neq w^{-1}$. Hence, we have $\Gamma_{sq}(G) = \overline{K_1 \cup \frac{|G|-1}{2}K_2}$.

Square power graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is $\Gamma_{sq}(\mathbb{Z}_3 \times \mathbb{Z}_3) = \overline{K_1 \cup 4K_2}$, shown in figure 1.

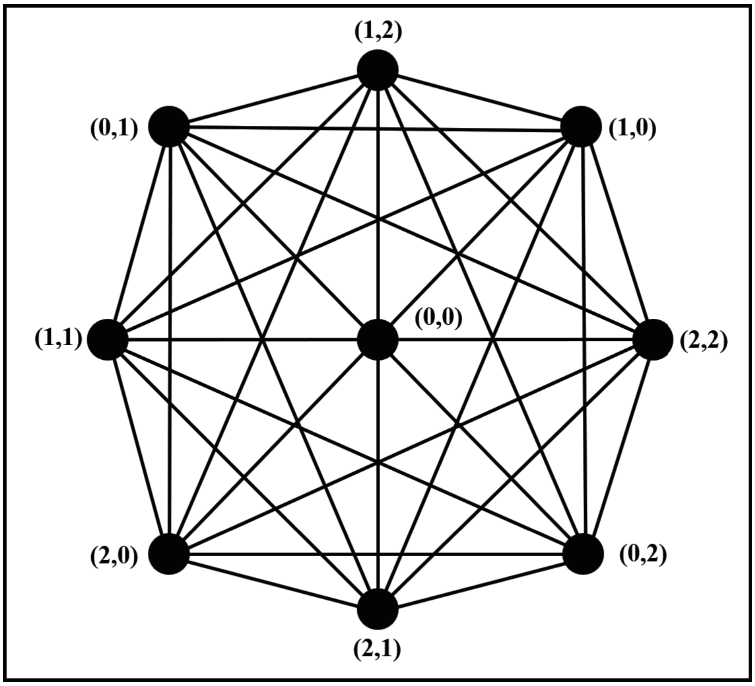

FIGURE 1. $\Gamma_{sq}(\mathbb{Z}_3 \times \mathbb{Z}_3)$

□

Theorem 3.2. *In a graph $\Gamma_{sq}(G)$ of abelian group G of finite odd order n,* $deg(w) = \begin{cases} |G|-1 \text{ when } w = e, \\ |G|-2 \text{ otherwise.} \end{cases}$ *where $w \in G$ and e is identity of G.*

Proof. From theorem 3.1, an identity element e adjacent with every other vertex in $\Gamma_{sq}(G)$, so $deg(e) = n-1$. When $w \neq e$ then we have $w + v \in U$ for every $v \in G \setminus \{w^{-1}\}$, so w is adjacent with every vertex other than its inverse vertex in square power graph. Hence $deg(w) = n-2$ when $w \neq e$. □

Corollary 3.3. *In a graph $\Gamma_{sq}(G)$ of G, abelian group of finite odd order n,* $|E(\Gamma_{sq}(G))| = \frac{(|G|-1)^2}{2}$.

Proof. From theorem 3.2, there is only a pair of vertices which are not adjacent with each other and are of type u, u^{-1}. We have total $\frac{n(n-1)}{2}$ distinct pairs of vertices and $\frac{n-1}{2}$ pairs of vertices of type u, u^{-1}. Hence $|E(\Gamma_{sq}(G))| = \frac{n(n-1)}{2} - \frac{n-1}{2} = \frac{(n-1)^2}{2} = \frac{(|G|-1)^2}{2}$. □

Theorem 3.4. *In a graph $\Gamma_{sq}(G)$ of G, abelian group of finite odd order n we have* $d(u,w) = \begin{cases} 2 \text{ if } u^{-1} = w, \\ 1 \text{ if } u^{-1} \neq w. \end{cases}$

Proof. For every pair of $w, u \in V(\Gamma_{sq}(G))$ vertices, $w + u \in U \setminus \{e\}$ whenever $u^{-1} \neq w$ and $w + u \notin U \setminus \{e\}$ whenever $u^{-1} = w$. Thus $d(u,w) = 1$ if $u^{-1} \neq w$ and 2 if $u^{-1} = w$. □

Theorem 3.5. *Let G be an finite odd order abelain group with identity element e then in* $\Gamma_{sq}(G)$ *we have* $ecc(u) = \begin{cases} 1 \text{ if } u = e, \\ 2 \text{ if } u \neq e. \end{cases}$

Proof. From theorem 3.2, vertex e is adjacent with every other vertex in square power graph and so maximum distance of vertex e to any other in $\Gamma_{sq}(G)$ is 1. Thus $ecc(u) = 1$ if $u = e$. When $u \neq e$ then we have u vertex adjacent with every vertex other than u^{-1}, and so maximum distance of u vertex to any other vertex in $\Gamma_{sq}(G)$ is 2. Thus $ecc(u) = 2$ if $u \neq e$. □

Theorem 3.6. *Let G be an finite odd order n abelian group then*
(i) Clique number, $\omega(\Gamma_{sq}(G)) = \frac{n+1}{2}$.
(ii) Chromatic number, $\chi(\Gamma_{sq}(G)) = \frac{n+1}{2}$.

Proof. **(i)** From theorem 3.2 in $\Gamma_{sq}(G)$, there is only identity element vertex e having edge with all other vertices of $\Gamma_{sq}(G)$ and $a \in G$ is adjacent with every $b \in G \setminus \{a^{-1}\}$ when $a \neq e$. Hence we have no edge between a, a^{-1} and we have $\frac{n-1}{2}$ such a, a^{-1} pairs in $\Gamma_{sq}(G)$. So we have maximal vertices set forming complete graph includes identity element and one vertex out of every $\frac{n-1}{2}$ pairs of a, a^{-1} in which $a \neq a^{-1}$. Hence we have $\omega(\gamma_{sq}(G)) = 1 + \frac{n-1}{2} = \frac{n+1}{2}$
(ii) From above discussion we have e adjacent with every other vertex in $\Gamma_{sq}(G)$ and $\frac{n-1}{2}$ pairs of a, a^{-1} in which a not adjacent to a^{-1} and $a \neq a^{-1}$. Assign different $n-1$ colours to each of $n-1$ pairs of a, a^{-1} in such a way that a and a^{-1} are assigned same colour and two different such pairs assigned different colours and different colour from already assigned colours to e vertex. This assignment gives proper colouring for $\Gamma_{sq}(G)$ and so $\omega(\Gamma_{sq}(G)) \leq \chi(\Gamma_{sq}(G)) \leq 1 + \frac{n-1}{2}$. Hence $\chi(\Gamma_{sq}(G)) = \frac{n+1}{2}$. □

Theorem 3.7. *Let G be finite odd order n abelian group then square power graph of G,* $\Gamma_{sq}(G)$ *is weakly perfect.*

Proof. From theorem 3.6 we have $\omega(\Gamma_{sq}(G)) = \chi(\Gamma_{sq}(G))$.Hence $\Gamma_{sq}(G)$ is weakly perfect. □

Theorem 3.8. *For odd number n, let* K_n *be a complete graph with n vertices. Then chromatic polynomial of almost complete graph* $K_n - \cup_{j=1}^{k} e_j$*, obtained by deleting k non-adjacent edges is* $P(K_n - \cup_{j=1}^{k} e_j, \lambda) = \sum_{i=0}^{k} {}^{k}C_i P(K_{n-i}, \lambda)$*, where* $k \leq \frac{n-1}{2}$.

Proof. For odd number n, let K_n be complete graph having n nodes & $K_n - \cup_{j=1}^{k} e_j$ be almost complete graph obtained by deleting k non-adjacent edges, where $k \leq \frac{n-1}{2}$.
For $k = 1$, $P(K_n - \cup_{j=1}^{1} e_j, \lambda) = P(K_n - e_1, \lambda)$. Using [28, Theorem 8.6], $P(K_n - e_1, \lambda) = P(K_n, \lambda) + P(K_{n-1}, \lambda)$. Thus $P(K_n - \cup_{j=1}^{k} e_j, \lambda) = \sum_{i=0}^{k} {}^{k}C_i P(K_{n-i}, \lambda)$ is true for $k = 1$.

Now let for $k = k'$, $P(K_n - \cup_{j=1}^{k'} e_j, \lambda) = \sum_{i=0}^{k'} {}^{k'}C_i P(K_{n-i}, \lambda)$ is true.
Now for $k = k'+1$, $P(K_n - \cup_{j=1}^{k'+1} e_j, \lambda) = P(K_n - \cup_{j=1}^{k'} e_j - e_{k'+1}, \lambda)$. Using [28, Theorem 8.6], $P(K_n - \cup_{j=1}^{k'} e_j - e_{k'+1}, \lambda) = P(K_n - \cup_{j=1}^{k'} e_j, \lambda) + P(K_{n-1} - \cup_{j=1}^{k'} e_j, \lambda)$.
Now by using above result (at $k = k'$), we get
$P(K_n - \cup_{j=1}^{k'+1} e_j, \lambda) = \sum_{i=0}^{k'} {}^{k'}C_i P(K_{n-i}, \lambda) + \sum_{i=0}^{k'} {}^{k'}C_i P(K_{n-1-i}, \lambda)$
$= \sum_{i=0}^{k'} {}^{k'}C_i P(K_{n-i}, \lambda) + \sum_{i=1}^{k'+1} {}^{k'}C_{i-1} P(K_{n-i}, \lambda)$
$= {}^{k'}C_0 P(K_n, \lambda) + \sum_{i=1}^{k'} [{}^{k'}C_i + {}^{k'}C_{i-1}] P(K_{n-i}, \lambda) + {}^{k'}C_{k'} P(K_{n-k'-1}, \lambda)$
$= {}^{k'+1}C_0 P(K_n, \lambda) + \sum_{i=1}^{k'} {}^{k'+1}C_i P(K_{n-i}, \lambda) + {}^{k'+1}C_{k'+1} P(K_{n-k'-1})$
$= \sum_{i=0}^{k'+1} {}^{k'+1}C_i P(K_{n-i}, \lambda)$. Thus $P(K_n - \cup_{j=1}^{k'+1} e_j, \lambda) = \sum_{i=0}^{k'+1} {}^{k'+1}C_i P(K_{n-i}, \lambda)$.
Hence $P(K_n - \cup_{j=1}^{k} e_j, \lambda) = \sum_{i=0}^{k} {}^{k}C_i P(K_{n-i}, \lambda)$. ◻

Theorem 3.9. *Let $\Gamma_{sq}(G)$ be a square power graph of abelian group G of finite odd order then chromatic polynomial, $P(\Gamma_{sq}(G), \lambda) = \sum_{i=0}^{\frac{n-1}{2}} {}^{\frac{n-1}{2}}C_i P(K_{n-i}, \lambda)$.*

Proof. Let $\Gamma_{sq}(G)$ be a square power graph of abelian group G of finite odd order n. Using theorem 3.1, $\Gamma_{sq}(G) = \overline{K_1 \cup \frac{n-1}{2} K_2}$. Now from theorem 3.8, we have $P(K_n - \cup_{j=1}^{\frac{n-1}{2}} e_j, \lambda) = \sum_{i=0}^{\frac{n-1}{2}} {}^{\frac{n-1}{2}}C_i P(K_{n-i}, \lambda)$. Almost complete graph $K_n - \cup_{j=1}^{\frac{n-1}{2}} e_j = \overline{K_1 \cup \frac{n-1}{2} K_2}$ obtained by deleting $\frac{n-1}{2}$ non-adjacent edges from the complete graph K_n having n odd number of vertices have the same structure as of the square power graph of finite abelian group of odd order n, $\Gamma_{sq}(G) = \overline{K_1 \cup \frac{n-1}{2} K_2}$. Hence $P(\Gamma_{sq}(G), \lambda) = \sum_{i=0}^{\frac{n-1}{2}} {}^{\frac{n-1}{2}}C_i P(K_{n-i}, \lambda)$. ◻

4. Topological Indices of $\Gamma_{sq}(G)$

Theorem 4.1. *Let $\Gamma_{sq}(G)$ be a square power graph of abelian group G of finite odd order then Wiener index is $W(\Gamma_{sq}) = \frac{|G|^2-1}{2}$.*

Proof. Let G be an abelian group of finite odd order n, then we have $\frac{n-1}{2}$ pairs of w, w^{-1} in G such that $w \neq w^{-1}$. Now from theorem 3.4, there exist $\frac{n-1}{2}$ pairs of vertices with $d(w,u) = 2$ and so pairs of vertices with $d(w,u) = 1$ is $\frac{n(n-1)}{2} - \frac{n-1}{2} = \frac{(n-1)^2}{2}$. Hence, $W(\Gamma_{sq}) = 2 \times \frac{n-1}{2} + \frac{(n-1)^2}{2} = \frac{n^2-1}{2} = \frac{|G|^2-1}{2}$. ◻

Theorem 4.2. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Hyper-Wiener index is $WW(\Gamma_{sq}) = \frac{|G|^2+|G|-2}{2}$.*

Proof. From theorem 3.4 and 4.1, $WW(\Gamma_{sq}) = \frac{1}{2}(\frac{n^2-1}{2}) + \frac{1}{2}(\frac{(n-1)^2}{2} + \frac{4(n-1)}{2}) = \frac{n^2+n-2}{2} = \frac{|G|^2+|G|-2}{2}$. ◻

Theorem 4.3. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then first Zagreb index is $M_1(\Gamma_{sq}) = (|G|-1)(|G|^2 - 3|G| + 3)$.*

Proof. From theorem 3.2, there is one node with degree $(n-1)$ and $n-1$ nodes with degree $n-2$ in $\Gamma_{sq}(G)$. Hence $M_1(\Gamma_{sq}) = (n-1)^2 + (n-1)(n-2)^2 = (n^2 - 3n + 3)(n-1) = (|G|^2 - 3|G| + 3)(|G| - 1)$. ◻

Theorem 4.4. *Let $\Gamma_{sq}(G)$ be a square power graph of abelian group of finite odd order n then second Zagreb index is* $M_2(\Gamma_{sq}) = \frac{(|G|-2)(|G|-1)(|G|^2-3|G|+4)}{2}$.

Proof. Using thm 3.2 & corollary 3.3, we get $n-1$ number of edges with one end having degree $n-1$ of vertex and another end degree $n-2$ of vertex, and $\frac{(n-3)(n-1)}{2}$ edges having vertices on both ends with degree $n-2$. Thus, $M_2(\Gamma_{sq}) = (n-1)^2(n-2) + (n-2)^2 \times \frac{(n-1)(n-3)}{2} = \frac{(n-2)(n-1)(n^2-3n+4)}{2} = \frac{(|G|-2)(|G|-1)(|G|^2-3|G|+4)}{2}$. □

Theorem 4.5. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Schultz index is* $MTI(\Gamma_{sq}) = (|G|-1)(|G|^2-|G|-1)$.

Proof. From thm 3.2 and 3.4, we get $\frac{n^2-4n+3}{2}$ pairs of vertices u, w in square power graph with $d(u,w) = 1$ and $deg(w) = deg(u) = n-2$, and $\frac{n-1}{2}$ pairs of vertices u, w in square power graph with $d(u,w) = 2$ and $deg(u) = deg(w) = n-2$, and $n-1$ pairs of vertices u, w in square power graph with $d(u,w) = 1$ having one with degree $n-1$ vertex and another with $n-2$ degree vertex. Thus $MTI(\Gamma_{sq}) = [(n-1)+(n-2)](n-1) + \frac{n^2-4n+3}{2}[(n-2)+(n-2)] + \frac{(n-1)}{2} \times 2 \times [(n-2)+(n-2)] = (n^2-n-1)(n-1) = (|G|-1)(|G|^2-|G|-1)$. □

Theorem 4.6. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Gutman index is* $Gut(\Gamma_{sq}) = \frac{|G|(|G|-2)(|G|-1)^2}{2}$.

Proof. Using theorem 3.2 and 3.4 with the same reasoning as given in theorem 4.5, we have $Gut(\Gamma_{sq}) = \frac{n^2-4n+3}{2}[(n-2)(n-2)] + \frac{n-1}{2}2[(n-2)(n-2)] + (n-1)[(n-1)(n-2)] = \frac{n(n-1)^2(n-2)}{2} = \frac{|G|(|G|-2)(|G|-1)^2}{2}$. □

Theorem 4.7. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then eccentric connectivity index is* $\xi(\Gamma_{sq}) = (|G|-1)(2|G|-3)$.

Proof. From theorem 3.2 and 3.5 we have only identity vertex e with $deg(e) = n-1$ and $ecc(e) = 1$, and all other $n-1$ nodes having degree $n-2$ and eccentricity 2. Thus $\xi(\Gamma_{sq}) = (n-1) \times 1 \times 1 + (n-2) \times 2 \times (n-1) = (n-1)(2n-3) = (|G|-1)(2|G|-3)$. □

Theorem 4.8. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Harary index is given as* $\mathcal{H}(\Gamma_{sq}) = \frac{(2|G|-1)(|G|-1)}{4}$.

Proof. From theorem 3.4 we have two possibilities for $d(w,u)$ which are $d(w,u) = 1$ if $w^{-1} \neq u$ and $d(w,u) = 2$ if $w^{-1} = u$. As discussed in theorem 4.1 we have $\frac{n-1}{2}$ possibilities of $d(u,v) = 2$ and number of pairs of edges with $d(w,u) = 1$ is $\frac{n(n-1)}{2} - \frac{n-1}{2} = \frac{(n-1)^2}{2}$. Hence $\mathbb{H}(\Gamma_{sq}) = \frac{(n-1)^2}{2} + \frac{1}{2} \times \frac{(n-1)}{2} = \frac{(n-1)(2n-1)}{4} = \frac{(2|G|-1)(|G|-1)}{4}$. □

Theorem 4.9. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Harmonic index is given by* $\mathcal{H}_r(\Gamma_{sq}) = \frac{(|G|-1)(2|G|^2-5|G|+1)}{2(|G|-2)(2|G|-3)}$.

Proof. From theorem 3.2, 3.4 & corollary 3.3, we have $n-1$ edges in $\Gamma_{sq}(G)$ with one end having degree $n-1$ vertex and another end having degree $n-2$

vertex and; $\frac{(n-1)(n-3)}{2}$ edges with both end with degree $n-2$ vertices. Thus we have $\mathcal{H}_r(\Gamma_{sq}) = \frac{2}{(n-1)+(n-2)} \times (n-1) + \frac{(n-1)(n-3)}{2} \times \frac{2}{(n-2)+(n-2)} = \frac{(n-1)(2n^2-5n+1)}{2(n-2)(2n-3)} = \frac{(|G|-1)(2|G|^2-5|G|+1)}{2(|G|-2)(2|G|-3)}$. □

Theorem 4.10. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then General Randic index,*
$R_\alpha(\Gamma_{sq}) = \frac{(|G|-1)(|G|-3)(|G|-2)^{2\alpha}}{2} + (|G|-1)[(|G|-1)(|G|-2)]^\alpha$.

Proof. Using theorem 3.2, 3.4 & corollary 3.3, we have $n-1$ edges in $\Gamma_{sq}(G)$ with one end having degree $n-1$ vertex and another end having degree $n-2$ vertex and; $\frac{(n-1)(n-3)}{2}$ edges with both end with degree $n-2$ vertices. Hence we have $R_\alpha(\Gamma_{sq}) = (n-1)[(n-1)(n-2)]^\alpha + \frac{(n-1)(n-3)[(n-2)(n-2)]^\alpha}{2} = (n-1)[(n-1)(n-2)]^\alpha + \frac{(n-1)(n-3)(n-2)^{2\alpha}}{2} = (|G|-1)[(|G|-1)(|G|-2)]^\alpha + \frac{(|G|-1)(|G|-3)(|G|-2)^{2\alpha}}{2}$. □

Theorem 4.11. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Randic index,*
$R_{-\frac{1}{2}}(\Gamma_{sq}) = \frac{\sqrt{(|G|-2)(|G|-1)}+(|G|-3)(|G|-1)}{2(|G|-2)}$.

Proof. Using theorem 4.10 for $\alpha = -\frac{1}{2}$, we get $R_{-\frac{1}{2}}(\Gamma_{sq}) = [(n-1)(n-2)]^{-\frac{1}{2}}(n-1) + \frac{(n-2)^{-1}(n-1)(n-3)}{2} = \frac{\sqrt{(n-2)(n-1)}+(n-3)(n-1)}{2(n-2)} = \frac{\sqrt{(|G|-2)(|G|-1)}+(|G|-3)(|G|-1)}{2(|G|-2)}$. □

Theorem 4.12. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Atomic-bond connectivity index,*
$ABC(\Gamma_{sq}) = \frac{2\sqrt{(|G|-2)(|G|-1)(2|G|-5)}+\sqrt{2}(|G|-3)^{\frac{3}{2}}(|G|-1)}{2(|G|-2)}$.

Proof. Using theorem 3.2, 3.4 & corollary 3.3, we have $n-1$ edges in $\Gamma_{sq}(G)$ with one end having degree $n-1$ vertex and another end having degree $n-2$ vertex and; $\frac{(n-1)(n-3)}{2}$ edges with both end with degree $n-2$ vertices. Thus we have $ABC(\Gamma_{sq}) = (n-1)\sqrt{\frac{(n-1)+(n-2)-2}{(n-1)(n-2)}} + \frac{(n-1)(n-3)}{2}\sqrt{\frac{(n-2)+(n-2)-2}{(n-2)(n-2)}} = \frac{2\sqrt{(n-1)(n-2)(2n-5)}+\sqrt{2}(n-1)(n-3)^{\frac{3}{2}}}{2(n-2)} = \frac{2\sqrt{(|G|-1)(|G|-2)(2|G|-5)}+\sqrt{2}(|G|-1)(|G|-3)^{\frac{3}{2}}}{2(|G|-2)}$. □

Theorem 4.13. *Let $\Gamma_{sq}(G)$ be a square power graph of finite abelian group G of odd order n then Geometric-arithmetic index,*
$GA(\Gamma_{sq}) = \frac{(|G|-1)[4\sqrt{(|G|-1)(|G|-2)}+|G|-3]}{2(2|G|-3)}$.

Proof. Using theorem 3.2, 3.4& corollary 3.3, we have $n-1$ edges in $\Gamma_{sq}(G)$ with one end having degree $n-1$ vertex and another end having degree $n-2$ vertex and; $\frac{(n-1)(n-3)}{2}$ edges with both end having degree $n-2$ vertices. Thus we have $GA(\Gamma_{sq}) = (n-1) \times \frac{2\sqrt{(n-1)(n-2)}}{(n-1)+(n-2)} + \frac{(n-1)(n-3)}{2} \times \frac{2\sqrt{(n-2)(n-2)}}{(n-2)+(n-2)} = \frac{(n-1)[4\sqrt{(n-1)(n-2)}+n-3]}{2(2n-3)} = GA(\Gamma_{sq}) = \frac{(|G|-1)[4\sqrt{(|G|-1)(|G|-2)}+|G|-3]}{2(2|G|-3)}$. □

5. Matching, Hamiltonicity and Laplacian spectrum of $\Gamma_{sq}(G)$

Theorem 5.1. *Square power graph $\Gamma_{sq}(G)$, of finite abelian group G of odd order n is Hamiltonian iff $n \geq 5$.*

Proof. Let G be a finite odd order n abelian group and $\Gamma_{sq}(G)$ be square power graph of G. Let $\Gamma_{sq}(G)$ is Hamiltonian then we have hamiltonian cycle in $\Gamma_{sq}(G)$ and so $\Gamma_{sq}(G)$ is connected. But for even values of n, $\Gamma_{sq}(G)$ is disconnected. Thus even value of n is not possible. Also for $n \in \{1,3\}$ we have no cycle in $\Gamma_{sq}(G)$ and so no hamiltonian cycle in $\Gamma_{sq}(G)$. Thus $n \notin \{1,3\}$. For odd value of $n \geq 5$, we have hamiltonian cycle $e-a_1-a_2-a_3-\cdots-a_{n-1}-e$ (where $a_i \in G$, e is identity element of G and $a_i \neq a_{i+1}$) in $\Gamma_{sq}(G)$. Hence the required result.
Conversely, For finite groups of odd order ≥ 5, we have hamiltonian cycle $e-a_1-a_2-a_3-\cdots-a_{n-1}-e$ (where $a_1 \in G$, e is identity element of G and $a_i \neq a_{i+1}$) in $\Gamma_{sq}(G)$. Hence $\Gamma_{sq}(G)$ is Hamiltonian for $n \geq 5$. □

Theorem 5.2. *Square power graph $\Gamma_{sq}(G)$ of finite abelian group G of odd order n have matching number, $\mu(\Gamma_{sq}(G)) = \frac{n-1}{2}$.*

Proof. Let $G = \{e, a_1, a_2, a_3, \cdots, a_{n-1}\}$ be a odd order n finite abelian group with identity element e, $a_i^{-1} = a_{i+1}$ and $\Gamma_{sq}(G)$ be its square power graph. As discussed in theorem 3.1, e vertex is adjacent with every other vertex in $\Gamma_{sq}(G)$ and every pair of vertices $u, w \in V(\Gamma_{sq}(G))$ have edge iff $u \neq w^{-1}$. We have only identity element e in G which is self-inverse. Let for pair of adjacent vertices a_i, a_j we have h_{ij} edge between them. For $n \geq 5$ we have the set of edges $E = \{h_{13}, h_{24}, h_{57}, h_{68} \cdots h_{(n-3)(n-1)}\}$ with order $\frac{n-1}{2}$ in which no two edges have common vertex.
When $n = 1$, we have no edge in $\Gamma_{sq}(G)$ and for $n = 3$ we have $\Gamma_{sq}(G)$ as path P_3. Also we have that maximum possible number of edges in matching is $\frac{\text{number of vertices}}{2}$. As n is odd number so maximum possible edges in maximal matching are $\frac{n-1}{2}$. Also matching E have order $\frac{n-1}{2}$. Hence E is the maximal-matching in $\Gamma_{sq}(G)$.
Hence matching number, $\mu(\Gamma_{sq}(G)) = \frac{n-1}{2}$. □

Theorem 5.3. *Let G be a abelian group with finite odd order n and $\Gamma_{sq}(G)$ be square power graph then we have laplacian matrix $L = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix}_{n\times n}$ and laplacian spectrum of L is 0 with multiplicity 1, $n-2$ with multiplicity $\frac{n-1}{2}$ and n with multiplicity $\frac{n-1}{2}$*

where $$P_2 = \begin{bmatrix} n-2 & -1 & -1 & -1 & \cdots & -1 & 0 \\ -1 & n-2 & -1 & -1 & \cdots & 0 & -1 \\ -1 & -1 & n-2 & -1 & \cdots & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & -1 & -1 & \cdots & n-2 & -1 \\ 0 & -1 & -1 & -1 & \cdots & -1 & n-2 \end{bmatrix}_{(n-1)\times(n-1)}$$

P_3 is $1 \times (n-1)$ matrix whose all entries are -1 and P_1 is 1×1 matrix with entry $n-1$.

Proof. Using theorem 3.1 & 3.2, $L = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix}_{n\times n}$ as laplacian matrix of square power graph of finite abelian group of odd order n.
We will now move on to determining the spectrum of L. Characteristic

polynomial of L is $C(x) = det(L - xI) = det\begin{bmatrix} P_1 - xI & P_3 \\ P_3^T & P_2 - xI \end{bmatrix}_{n\times n}$
Applying row operation $R_1 \to \sum_{i=1}^{n} R_i$ and taking $-x$ common from $1st$ row,
$C(x) = det(L - xI) = -xdet\begin{bmatrix} P_4 & -P_3 \\ P_3^T & P_2 - xI \end{bmatrix}_{n\times n}$ where P_4 is 1×1 matrix with 1 as its entry.
Now applying $R_i \to R_i + R_1 \ \forall\ 2 \le i \le n$.
$C(x) = det(L - xI) = -xdet\begin{bmatrix} P_4 & -P_3 \\ P_5 & P_6 \end{bmatrix}$ where P_5 is $(n-1)\times 1$ matrix with all entries 0, and

$$P_6 = \begin{bmatrix} n-1-x & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & n-1-x & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & n-1-x & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & n-1-x & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & n-1-x \end{bmatrix}_{(n-1)\times(n-1)}$$

$C(x) = det(L - xI) = -xdet\begin{bmatrix} P_4 & -P_3 \\ P_5 & P_6 \end{bmatrix} = -xdet(P_6)$
Applying row operation $R_1 \to \sum_{i=1}^{n} R_i$ to $det(P_6)$ and then taking $n-x$ common from $1st$ row, we get
$C(x) = det(L - xI) = -xdet(P_6) = -x(n-x)det(P_7)$

$$\text{where } P_7 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & n-1-x & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & n-1-x & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & n-1-x & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & n-1-x \end{bmatrix}_{(n-1)(n-1)}$$

Now applying row operation $R_n \to R_n - R_1$ to $det(P_7)$, and then solving $det(P_7)$ along $1st$ column we get
$C(x) = det(L - xI) = -x(n-x)det(P_7) = -x(n-x)det(P_8)$

$$\text{where } P_8 = \begin{bmatrix} n-1-x & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & n-1-x & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & n-1-x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 1 & 0 & \cdots & n-1-x & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & n-1-x & 0 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n-2-x \end{bmatrix}_{(n-2)(n-2)}$$

Now solving $det(Y)$ using $n-th$ column, we get
$C(x) = det(L - xI) = -x(n-x)det(P_8) = -x(n-x)(n-2-x)det(P_9)$

$$\text{where } P_9 = \begin{bmatrix} n-1-x & 0 & 0 & \cdots & 0 & 1 \\ 0 & n-1-x & 0 & \cdots & 1 & 0 \\ 0 & 0 & n-1-x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 1 & 0 & \cdots & n-1-x & 0 \\ 1 & 0 & 0 & \cdots & 0 & n-1-x \end{bmatrix}_{(n-3)(n-3)}$$

Now P_9 is $(n-3)\times(n-3)$ matrix of same kind as P_6, $(n-1)\times(n-1)$ matrix so applying same operation on $det(P_9)$ as applied on $det(P_6)$ and then we again get matrix of P_8 kind of order $(n-4)\times(n-4)$. By repeating above operations, in last we get $C(x) = det(L - xI) = -x(n-x)^{\frac{n-1}{2}}(n-2-x)^{\frac{n-1}{2}}$. Hence laplacian spectrum of L is 0 with multiplicity 1, $n-2$ with multiplicity $\frac{n-1}{2}$ and n with multiplicity $\frac{n-1}{2}$. □

Conclusion

In this research paper, representation of a Square power graph of finite abelian group of odd order is given and studied various properties such as connectedness, degree of vertex, size of graph, eccentricity, chromatic number, clique number, perfectness, hamiltonicity, matching number. We have also studied Laplacian spectrum and various topological indices such as Wiener, Hyper-Wiener, Gutman, Schultz, Eccentric Connectivity, first and second Zagreb, Harary, Harmonic, Geometric-arithmetic index, Atomic-bond connectivity, General Randic and Randic. We have also calculated chromatic polynomial of almost complete graph obtained by deleting non-adjacent edges from complete graph with odd number of vertices.

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