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$(2,3,2)$–CONSTRAINED TOTAL LABELING OF GRAPHS

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Abstract. The $(2,3,2)$–constrained total labeled graph $G(V,E)$ is a bijective mapping $g : V \cup E \to \{1,2,\cdots,|V|+|E|\}$ with the constraints that $|g(v)-g(u)| \geq 2$ whenever $uv \in E$, $|g(uv)-g(u)| \geq 3$ and $|g(vu)-g(wv)| \geq 2$ whenever $w \neq u$. A graph G satisfies such labeling is known as a $(2,3,2)$-constrained total labeled graph, abbreviated as $(2,3,2)$-CTLG. The minimum number of isolated vertices that need to be added to a graph G to transform it into the resulting graph is $(2,3,2)$–constrained total labeled graph (CTLG) is called the $(2,3,2)$–constrained total number of G, denoted by $t_{(2,3,2)}(G)$. In this paper, we have obtained the $(2,3,2)$-constrained total number $t_{(2,3,2)}$ for path, cycle, and star graph. We also prove that wheel graph, double star graph, windmill graph, helm graph, fan graph, comb graph, sunlet graph, different types of ladder graph, and product graph are all $(2,3,2)$-constrained total labeled graph.

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1. Introduction

Graph theory, with its wide-ranging applications, highlighted by significant contributions from Rosa [1] in 1966 and Graham and Sloane [2] in 1980. Different kinds of labeling graphs can be recognized by using the field of discrete mathematics. Various applications in science, engineering and technology have been investigated through the varieties of graph labeling [3]. Hale [4] modeled the distance-constrained channel assignment problem by representing transmitters in a wireless communication network as vertices in a graph, with edges connecting transmitters

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that are in close proximity. The concept of distance-two labeling was proposed by Griggs and Yeh [5]. Yeh [6] provided a survey on graph labeling under distance two constraints. Chang *et al.* [7] derived exact formulas for $\lambda(G \cup H)$ and $\lambda(G + H)$, established that $\lambda(G) \leq \Delta^2 + \Delta$ for graphs with maximum degree Δ, and showed that $\lambda(G) \leq 2\Delta + 1$ for odd-sun-free chordal graphs and $\lambda(G) \leq \Delta + 2\chi(G) - 2$ for sun-free chordal graphs. They also proposed a polynomial-time algorithm to compute $\lambda(T)$ for trees. Ma-Lian Chia *et al.* [8] examined $L(3,2,1)$-labeling, providing upper bounds for general graphs and trees, and also studied cartesian products and powers of paths and cycles. Shil *et al.* [9] studied SVN-graph theory by introducing single-valued quadripartitioned neutrosophic graphs (SVQN-graphs), and defined their degree, order, and size, supported by illustrative examples. Das *et al.* [10] introduced single-valued pentapartitioned neutrosophic graphs (SVPN-graphs) by extending neutrosophic set theory, defining their degree, size, and order. They highlighted the ability to handle indeterminacy more finely and discussed potential applications in networks and hypergraphs.

For the whole graph labeling survey [11]. For graph theory terminology and notations, we consult Harary [12] textbook. The $\deg(G)$ denotes the maximum degree of a graph G, defined as the largest degree among its vertices. A compilation of several graph labeling and the underlying notion are available in [13]. Several authors have examined a range of graph labels. In recent times, Shreedhara Kunikullaya D and B. Sooryanarayana [14] presented a k-constrained total labeled graphs. In this study, we explore $(2,3,2)$–constrained total labeling, which is motivated to be considered.

Let G be an undirected, finite, and simple graph consisting of a vertex set V and edge set E. A $(2,3,2)$–constrained total labeling of G is a bijective mapping $g : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ if it meets the subsequent requirements:
For every $u, v, w \in V$,

(1) $|g(v) - g(u)| \geq 2$ whenever $uv \in E$,
(2) $|g(uv) - g(u)| \geq 3$ and
(3) $|g(vw) - g(uv)| \geq 2$ whenever $u \neq w$.

A graph G which satisfies such labeling is called $(2,3,2)$–constrained total labeled graph, denoted as $(2,3,2)$–CTLG. Here all graphs need not be $(2,3,2)$–CTLG. However, by introducing some isolated vertices, a graph G can be made $(2,3,2)$–CTLG. The least number n for which the graph $G \cup \bar{K_n}$ becomes a $(2,3,2)$–CTLG is called $(2,3,2)$–constrained total number of the graph G and is denoted by $t_{(2,3,2)}(G)$.

2. $(2,3,2)$– CONSTRAINED TOTAL LABELING OF SOME GRAPHS

We begin our results by proving that all paths with at-least four vertices is a $(2,3,2)$–CTLG.

Theorem 2.1. *For a path P_n with n vertices,* $t_{(2,3,2)}(P_n) = \begin{cases} 3, & \text{for } n=2 \\ 2, & \text{for } n=3 \\ 0, & \text{otherwise} \end{cases}$.

Proof. For $n = 2$ and $n = 3$, the results are trivial. Let $V(P_n) = \{w_1, w_2, \ldots, w_n\}$ where $n \geq 4$. A total labeling is defined as $g : V \cup E \rightarrow \{1, 2, 3, \ldots, 2n-1\}$ by $g(w_j w_{j+1}) = 2j$ for $1 \leq j \leq n-1$, $g(w_j) = 2j-5$ for $3 \leq j \leq n$, $g(w_1) = 2n-3$ and $g(w_2) = 2n-1$. Thus, the function g serves as $(2,3,2)$–constrained total labeling for P_n. □

Theorem 2.2. *For a cycle C_n with n vertices* $t_{(2,3,2)}(C_n) = \begin{cases} 2, & \text{for } n=3 \\ 0, & \text{otherwise} \end{cases}$.

Proof. For $n \geq 4$, the result follows by connecting the end vertices of P_n with the edge $w_1 w_n$ and applying Theorem 2.1, where the labeling is extended by setting $g(w_1 w_n) = 2n$.

For $n = 3$, suppose 3 is assigned to a vertex, then 4 is to be assigned to the edge not incident to the vertex labeled 3. However, the label 5 cannot be assigned to any remaining vertex or edge in C_3. Therefore, in every sequence of three consecutive integers, at least one must be left unused when labeling C_3. Hence, $(2,3,2)$–constrained total labeling of C_3 requires at least 8 distinct positive integers. So $t_{(2,3,2)}(C_3) = 2$. □

An illustrative example of the cycle graph C_3 is as follows.

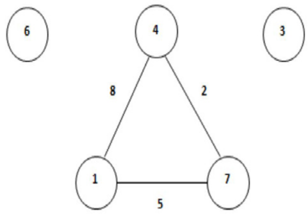

Figure 1. cycle graph

Theorem 2.3. *For star graph $K_{1,n}$, $n \geq 2$ $\quad t_{(2,3,2)}(K_{1,n}) = 2$.*

Proof. In $K_{1,n}$, the central vertex and each pendant vertex must be labeled with non-consecutive integers. Hence, $t_{(2,3,2)}(K_{1,n}) \geq 2$. Let w_0 be the central vertex, $w_1, w_2, \ldots, w_n$ be the pendent vertices of the star graph $K_{1,n}$. A total labeling is defined as $g : V \cup E \rightarrow \{1, 2, \ldots, 2n+3\}$ by $g\left(w_0 w_j\right) = 2j$ for $1 \leq j \leq n$, $g(w_0) = 2n+3$, $g(w_1) = 2n-1$, $g\left(w_j\right) = 2j-3$ for $2 \leq j \leq n$, $g(w_{n+1}) = 2n+1$ and $g(w_{n+2}) = 2(n+1)$. Hence, $t_{(2,3,2)}(K_{1,n}) = 2$. □

Theorem 2.4. *For integer $n \geq 3$, the wheel $W_{1,n}$ is a $(2,3,2)-$ constrained total labeled graph.*

Proof. Consider the vertex set of $W_{1,n}$ as $V = \{w_0, w_1, w_2, \ldots, w_n\}$ where w_0 denotes the central vertex. For $n > 3$, a total labeling is defined as $g : V \cup E \rightarrow \{1, 2, 3, \ldots, 3n+1\}$ by $g(w_0) = 3n+1$, $g\left(w_j\right) = 2j-1$ for $1 \leq j \leq n$. Now define g on edges by $g\left(w_0 w_j\right) = 2(j+1)$ for $1 \leq j \leq n-1$, $g(w_0 w_n) = 2$, $g(w_n w_1) = 2(n+1)$. Also, we define $g\left(w_j w_{j+1}\right) = 2(n+j+1)$ for $1 \leq j \leq \frac{n}{2} - 1$, $g\left(w_{\frac{n}{2}-1+j}\ w_{\frac{n}{2}+j}\right) = 2(n+j) - 1$ for $1 \leq j \leq \frac{n}{2}$ if n is even and $g\left(w_j w_{j+1}\right) = 2(n+j+1)$ for $1 \leq j \leq \frac{n-3}{2}$, $g\left(w_{\frac{n-3}{2}+j}\ w_{\frac{n-1}{2}+j}\right) = 2(n+j) - 1$ for $1 \leq j \leq \frac{n+1}{2}$ if n is odd. Hence, $W_{1,n}$ is $(2,3,2)$–constrained total labeled graph.

If $n = 3$, label the edges $w_0 w_{j+2}$ by $2(j+1)$ for $j = 0, 1$, $w_0 w_1$ by 10, $w_1 w_2$ by 8, $w_1 w_3$ by 6, $w_2 w_3$ by 1. Now label the vertices w_j by $2j+1$ for $j = 1, 2$, w_0 by 7, w_3 by 9 makes W_3 a $(2,3,2)$–constrained total labeled graph. □

We present an illustrative example of the wheel graph $W_{1,8}$ as follows.

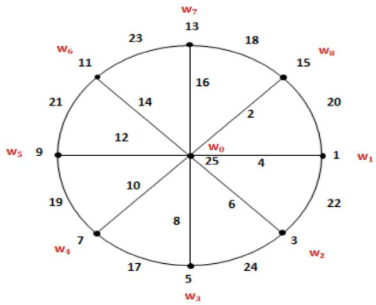

Figure 2. wheel graph

Theorem 2.5. *The double star graph $D_{m,n}$, $n \geq m$, is a $(2,3,2)-$ constrained total labeled graph.*

Proof. consider a vertex set of $D_{m,n}$ be
$V = \{w_0, w_1, w_2, \ldots, w_n, u_0, u_1, \ldots, u_m\}$ where $deg(w_0) = n$, $deg(u_0) = m$, w_i's and u_j's are pendant vertices and an edge set be $E = \left\{u_0u_i, w_0w_j, u_0w_0\right\}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

For $n = m$, a total labeling is defined as $g : V \cup E \rightarrow \{1, 2, 3, \ldots, 4n + 3\}$ by $g(w_iw_0) = 2i-1$, $g(u_iu_0) = 2(n+i)+1$ for $1 \leq i \leq n$ and $g(u_0w_0) = 2n+1$. Also, $g(u_i) = 2(i+1)$ for $0 \leq i \leq n$, $g(w_i) = 2(n+i+2)$ for $0 \leq i \leq n-1$ and $g(w_n) = 4n+3$ which provides $(2,3,2)-$constrained total labeling for $D_{n,n}$.

For $n > m$, label the edges as w_iw_0 by $2i-1$ for $1 \leq i \leq n$, u_ju_0 by $2(m+j)+1$ for $1 \leq j \leq m$, u_0w_0 by $2(n+m)+1$. Now label the vertices u_j by $2(j+1)$ for $0 \leq j \leq m$, w_i by $2(n+i+1)$ for $0 \leq i \leq n-1$ and $g(w_n)$ by $4n+1$. Hence, $D_{m,n}$ is a $(2,3,2)-$constrained total labeled graph. □

Theorem 2.6. *The wind mill graph $Wd(3,m)$ is a $(2,3,2)-$constrained total labeled graph.*

Proof. Let $G(V,E) = Wd(3,m)$ be a wind mill graph where m be the number of copies of K_3 with vertex set $V = \{w_0, w_1, w_2, \ldots, w_n\}$ where $n = 2m$ with $deg(w_0) = n$, $deg\left(w_j's\right) = 2$ and an edge set be $E = \left\{w_0w_j : 1 \leq j \leq n\right\} \cup \left\{w_{2j-1}w_{2j} : 1 \leq j \leq \frac{n}{2}\right\}$.

For $m > 2$, a total labeling is defined as $g : V \cup E \rightarrow \{1, 2, 3, \ldots, \frac{5n}{2} + 1\}$ by label the edges w_0w_j by $2j-1$ for $1 \leq j \leq n$, $w_{4j-1}w_{4j}$ by

$2(n+j)-1$ for $1 \leq j \leq \left\lfloor \frac{n-2}{4} \right\rfloor$, $w_{4j+1}w_{4j+2}$ by $2(n+j)$ for $1 \leq j \leq \left\lceil \frac{n-2}{4} \right\rceil$ and w_1w_2 by $\frac{5n}{2}$. Also, label the vertices w_j by $2j-4$ for $3 \leq j \leq n$, w_1 by $2(n-1)$, w_2 by $2n$ and w_0 by $\frac{5n+2}{2}$. Thus, $Wd(3,m)$ is a $(2,3,2)-$constrained total labeled graph.

If $m=2$, label the edges w_0w_j by $2j-1$ for $1 \leq j \leq n$, w_1w_2 by 11 and w_3w_4 by 9. Also label the vertices w_j by $2j-4$ for $3 \leq j \leq n$, w_1 by $2(n-1)$, w_2 by $2n$ and w_0 by 10. Therefore, $Wd(3,2)$ is a $(2,3,2)-$constrained total labeled graph. □

Theorem 2.7. *A ladder graph* (L_n) *is a* $(2,3,2)-$*constrained total labeled graph.*

Proof. Let $G(V,E)=L_n$ be a ladder graph with the vertex set
$V=\left\{u_j, w_j : 1 \leq j \leq n\right\}$ and edge set
$E=\left\{\left\{u_ju_{j+1}, w_jw_{j+1} : 1 \leq j \leq n-1\right\} \cup \left\{u_jw_j : 1 \leq j \leq n\right\}\right\}$.

For $n>3$, a total labeling is defined as $g: V \cup E \to \{1,2,3,\ldots,5n-2\}$ by label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$ for $j=1,2,3,\ldots,n$. Also, label edges u_ju_{j+1} by $2(2n+j-3)$ for $j=1,2$, $u_{j+2}w_{j+3}$ by $2j$ for $j=1,2,3,\ldots,n-3$, w_jw_{j+1} by $2(n+j-2)$, u_jw_j by $4n+j-1$ for $j=1,2,3,\ldots,n-1$ and u_nw_n by $2(n-2)$. Thus, L_n is a $(2,3,2)-$constrained total labeled graph.

If $n=3$, label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$ for $j=1,2,3,\ldots,n$. Also, label edges u_ju_{j+1} by $2(n+j)$, w_jw_{j+1} by $n+2j-1$, u_jw_j by $4n+j-1$ for $j=1,2$ and u_nw_n by 2. Therefore, L_3 is a $(2,3,2)-$constrained total labeled graph. □

Theorem 2.8. *An open ladder graph* (OL_n) *is a* $(2,3,2)-$*constrained total labeled graph.*

Proof. Let $G(V,E)=OL_n$ be an open ladder graph with vertex set
$V=\left\{u_j, w_j : 1 \leq j \leq n\right\}$ and an edge set
$E=\left\{\left\{u_{j+1}u_j, w_jw_{j+1} : 1 \leq j \leq n-1\right\} \cup \left\{u_jw_j : 1 \leq j \leq n\right\}\right\}$.

For $n>3$, a total labeling is defined as $g: V \cup E \to \{1,2,3,\ldots,5n-4\}$ by label vertices u_j by $2j-1$, w_j by $2(n+j)-1$ for $1 \leq j \leq n$. Also, label edges $u_{j+1}u_j$ by $2(2n+j-4)$ for $j=1,2$, $u_{j+2}u_{j+3}$ by $2j$, $u_{j+2}w_{j+2}$ by $4n+j-1$ for $j=1,2,3,\ldots,n-3$, w_jw_{j+1} by $2(n+j-3)$ for $1 \leq j \leq n-1$ and u_2w_2 by $2(2n-1)$. Thus, OL_n is a

$(2,3,2)$–constrained total labeled graph.

If $n = 3$, label the vertices u_j by $2j-1$, w_j by $2j+5$ for $1 \le j \le 3$. Also, label edges u_ju_{j+1} by $2(j+3)$, w_jw_{j+1} by $2j$ for $j = 1,2$ and u_2w_2 by 6. Therefore, OL_3 is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.9. *A slanting ladder graph* (SL_n) *is a* $(2,3,2)$*–constrained total labeled graph.*

Proof. Let $G(V,E) = SL_n$ be a slanting ladder graph with the vertex set $V = \left\{u_j, w_j : 1 \le j \le n\right\}$ and the edge set
$E = \left\{\left\{u_{j+1}u_j, w_{j+1}w_j : 1 \le j \le n-1\right\} \cup \left\{u_{j+1}w_j : 1 \le j \le n-1\right\}\right\}$.

For $n > 3$, a total labeling is defined as $g : V \cup E \to \{1,2,3,\ldots,5n-3\}$ by label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$ for $1 \le j \le n$. Also, label the edges u_ju_{j+1} by $2(2n+j-4)$ and w_ju_{j+1} by $2(2n+j-2)$ for $j = 1,2$, $u_{j+2}u_{j+3}$ by $2j$, $w_{j+2}u_{j+3}$ by $4n+j$ for $1 \le j \le n-3$, w_jw_{j+1} by $2(n+j-3)$ for $1 \le j \le n-1$.Thus, SL_n is a $(2,3,2)$–constrained total labeled graph.

If $n = 3$, label the vertices u_j by $2j-1$, w_j by $2j+5$ for $1 \le j \le 3$. Also, label the edges u_ju_{j+1} by $2(j+2)$, w_jw_{j+1} by $2j$, w_ju_{j+1} by $2(4+j)$ for $j = 1,2$. Therefore, SL_3 is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.10. *For a triangular ladder graph* (TL_n) *is a* $(2,3,2)$*–constrained total labeled graph.*

Proof. Let $G(V,E) = TL_n$ be a triangular ladder graph with vertex set $V = \left\{u_j, w_j : 1 \le j \le n\right\}$ and edge set
$E = \left\{\left\{u_ju_{j+1}, w_jw_{j+1}, w_ju_{j+1} : 1 \le j \le n-1\right\} \cup \left\{u_jw_j : 1 \le j \le n\right\}\right\}$.

For $n > 3$, a total labeling is defined as $g : V \cup E \to \{1,2,3,\ldots,6n-3\}$ by label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$ for $1 \le j \le n$. Also, label edges u_ju_{j+1} by $2(2n+j-4)$ for $j = 1,2$, $u_{j+2}u_{j+3}$ by $2j$ for $1 \le j \le n-3$, w_jw_{j+1} by $2(n+j-3)$, u_jw_j by $2(2n+j)-1$, w_ju_{j+1} by $2(2n+j-2)$ for $1 \le j \le n-1$, u_nw_n by $2(3n-2)$. Thus, TL_n is a $(2,3,2)$–constrained total labeled graph.

If $n = 3$, label the vertices u_j by $2j-1$, w_j by $2j+5$ for $1 \le j \le 3$. Also, label the edges u_ju_{j+1} by $2(j+2)$, w_jw_{j+1} by $2j$, w_ju_{j+1} by

$2(j+4)$ for $j=1,2$ and w_ju_j by $j+12$ for $j=1,2,3$. Therefore, TL_3 is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.11. *An open triangular ladder graph* (OTL_n) *is a* $(2,3,2)$–*constrained total labeled graph.*

Proof. Let $G(V,E)=OTL_n$ be an open triangular ladder graph with the vertex set $V=\left\{u_j,w_j : 1\le j\le n\right\}$ and edge set $E=\left\{u_ju_{j+1}, w_jw_{j+1}, w_ju_{j+1} : 1\le j\le n-1\right\}$.

For $n>3$, a total labeling is defined as $g: V\cup E\to\{1,2,3,\ldots,6n-5\}$ by label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$ for $1\le j\le n$. Also, label the edges u_ju_{j+1} by $2(2n+j-4)$, $w_{n+j-3}u_{n+j-2}$ by $6n+j-7$ for $j=1,2$, $u_{j+2}u_{j+3}$ by $2j$, w_ju_{j+1} by $2(2n+j)-1$ for $1\le j\le n-3$, $u_{j+1}w_{j+1}$ by $2(2n+j-2)$ for $1\le j\le n-2$ and w_jw_{j+1} by $2(n+j-3)$ for $1\le j\le n-1$. Thus, OTL_n is a $(2,3,2)$–constrained total labeled graph.

If $n=3$, label the vertices u_j by $2j-1$, w_j by $2j+5$ for $1\le j\le 3$. Also, label the edges u_1u_2 by 13, u_2u_3 by 8, w_2u_2 by 6, w_jw_{j+1} by $2j$, w_ju_{j+1} by $8+2j$ for $j=1,2$. Therefore, OTL_3 is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.12. *For integer* $n>2$, *a diagonal ladder graph* (DL_n) *is a* $(2,3,2)$– *constrained total labeled graph.*

Proof. Let $G(V,E)=DL_n$ be a diagonal ladder graph with the vertex set $V=\left\{u_j,w_j : 1\le j\le n\right\}$ and the edge set $E=\left\{\left\{u_ju_{j+1}, w_jw_{j+1}, u_jw_{j+1}, w_ju_{j+1} : 1\le j\le n-1\right\}\cup\left\{w_ju_j : 1\le j\le n\right\}\right\}$.

For $n>3$, a total labeling is defined as $g: V\cup E\to\{1,2,3,\ldots,7n-4\}$ by label the vertices u_j by $2(n+j)-3$, w_j by $2(2n+j-2)$ for $1\le j\le n$. Also, label the edges u_ju_{j+1} by $2(2n+j)-3$, w_jw_{j+1} by $2(n+j-1)$, u_jw_{j+1} by j, w_ju_{j+1} by $n+j-1$ for $1\le j\le n-1$, u_jw_j by $6n+j-4$ for $1\le j\le n$. Thus, DL_n is a $(2,3,2)$–constrained total labeled graph.

If $n=3$, label the vertices u_j by $2(j+4)$, w_j by $2j+1$ for $1\le j\le 3$. Also, label the edges u_ju_{j+1} by $2(j+1)$, w_jw_{j+1} by $2j+9$, $w_{j+1}u_j$ by j, w_ju_{j+1} by $7+j$ for $1\le j\le 2$ and w_ju_j by $14+j$ for $j=1,2,3$. Therefore, DL_3 is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.13. *For open diagonal ladder graph $O(DL_n$ is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = O(DL_n)$ be an open diagonal ladder graph with the vertex set $V = \left\{u_j, w_j : 1 \leq j \leq n\right\}$ and edge set
$E = \left\{u_j u_{j+1}, w_j w_{j+1}, u_j w_{j+1}, w_j u_{j+1}, u_{j+1} w_{j+1} : 1 \leq j \leq n-1\right\}$.

For $n > 3$, a total labeling is defined as $g : V \cup E \rightarrow \{1,2,3,\ldots,7n-6\}$ by label the vertices u_j by $2(n+j)-3$, w_j by $2(2n+j-2)$ for $1 \leq j \leq n$. Also, label the edges $u_j u_{j+1}$ by $2(2n+j)-3$, $w_j w_{j+1}$ by $2(n+j-1)$, $w_j u_{j+1}$ by $j+3$, $u_j w_{j+1}$ by j for $1 \leq j \leq n-1$ and $u_{j+1} w_{j+1}$ by $6n+j-4$ for $1 \leq j \leq n-2$. Thus, $O(DL_n)$ is a $(2,3,2)$–constrained total labeled graph.

If $n = 3$, label the vertices u_j by $2(j+4)$, w_j by $2j+1$ for $1 \leq j \leq 3$. Also, label the edges $u_j u_{j+1}$ by $2(j+1)$, $w_j w_{j+1}$ by $2j+9$, $w_{j+1} u_j$ by j, $w_j u_{j+1}$ by $7+j$ for $1 \leq j \leq 2$, $w_2 u_2$ by 15. Therefore, $O(DL_3)$ is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.14. *The helm graph H_n is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = H_n$ be the helm graph with vertex set
$V = \{w_0, w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_n\}$ and the edge set
$E = \left\{\left\{w_0 w_j, w_j u_j : 1 \leq j \leq n\right\} \cup \left\{w_j w_{j+1} : 1 \leq j \leq n-1\right\}\right\}$, Where w_0 is the center vertex, w_j's are the vertices on a cycle and u_j's are the pendent vertices.

For $n > 3$, a total labeling is defined as $g : V \cup E \rightarrow \{1,2,3,\ldots,5n+1\}$ by label the vertices w_0 by $4n+1$, w_j by $2j-1$, u_j by $4n+j+1$ for $1 \leq j \leq n$. Also, label the edges $w_0 w_n$ by 2, $w_0 w_j$ by $2(j+1)$, $w_j w_{j+1}$ by $2(n+j+1)$ for $1 \leq j \leq n-1$, $w_n w_j$ by $2(n+1)$, $w_1 u_1$ by $4n-1$, $u_j w_j$ by $2(n+j)-3$ for $2 \leq j \leq n$. Thus, H_n is a $(2,3,2)$–constrained total labeled graph.

If $n = 3$, label the vertices w_0 by $5n+1$, w_j by $2j-1$ and u_j by $2j+5$ for $1 \leq j \leq n$. Also, label the edges $w_0 w_j$ by $2(j+1)$, $w_j w_{j+1}$ by $2(j+4)$ for $j = 1, 2$, $w_3 w_1$ by 8 and $w_j u_j$ by $j+12$ for $1 \leq j \leq 3$. Therefore, H_3 is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.15. *A fan graph F_n is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = F_n$ be a fan graph with vertex set $V = \{w_0, w_1, w_2, \ldots, w_n\}$ and the edge set $E = \Big\{\Big\{w_0 w_j : 1 \le j \le n\Big\} \cup \{w_j w_{j+1} : 1 \le j \le n-1\}\Big\}$ Where w_0 is the center vertex, w_j's are vertices, a total labeling is defined as $g : V \cup E \to \{1, 2, \ldots, 4n+1\}$ by label the vertices w_j by $2j-4$ for $1 \le j \le n$, w_0 by $\frac{5n+2}{2}$ and w_1 by $2(n-1)$ and w_2 by $2n$. Also, label the edges $w_0 w_j$ by $2j-1$ for $1 \le j \le n$, $w_{4j-1} w_{4j}$ by $2(n+j)-1$ for $j = 1, 2, \ldots, \frac{n}{4}$ and $w_{4j+1} w_{4j+2}$ by $2(n+j)$ for $1 \le j \le \frac{n}{4} - 1$. Thus, F_n is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.16. *The product graph $C_n \times P_2$ for $n \ge 3$ is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = C_n \times P_2$ be a Product Graph, u_j's be the vertices of an outer cycle, w_j's be the vertices of an inner cycle. A vertex set $V = \Big\{w_j, u_j : 1 \le j \le n\Big\}$ and an edge set

$$E = \Big\{\Big\{u_j u_{j+1}, w_j w_{j+1} : 1 \le j \le n-1\Big\} \cup \Big\{u_j w_j : 1 \le j \le n\Big\}\Big\}.$$

For $n > 3$, a total labeling is defined as $g : V \cup E \to \{1, 2, 3, \ldots, 5n\}$ by label the vertices w_j by $2j-1$, u_j by $2(n+j)-1$ for $1 \le j \le n$. Also, label the edges $w_j w_{j+1}$ by $2(n+j-2)$, $u_j u_{j+1}$ by $2(2n+j-2)$ for $j = 1, 2$, $w_{j+2} w_{j+3}$ by $2j$, $u_{j+2} u_{j+3}$ by $2(n+j)$ for $1 \le j \le n-3$, $w_n w_1$ by $2(n-2)$, $u_n u_1$ by $4(n-1)$ and $w_n w_1$ by $2(n-2)$, $w_j u_j$ by $4n+j$ for $1 \le j \le n$. Thus, $C_n \times P_2$ is a $(2,3,2)$–constrained total labeled graph.

If $n = 3$, label the vertices w_j by $2j-1$, u_j by $2(n+j)-1$ for $1 \le j \le 3$. Also, label the edges $w_j w_{j+1}$ by $2(n+j+1)$, $u_j u_{j+1}$ by $2j$ for $j = 1, 2$, $u_j w_j$ by $4n+j$ for $1 \le j \le 3$, $w_3 w_1$ by 8, $u_3 u_1$ by 6. Therefore, $C_3 \times P_2$ is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.17. *The product graph $C_n \times P_3$ for $n \ge 3$ is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = C_n \times P_3$ be a product graph, w_j's be the vertices of an outer cycle, u_j's and v_j's are the vertices of an inner cycle. A vertex set V=$\Big\{u_j, v_j, w_j for 1 \le j \le n\Big\}$ and an edge set E=$\Big\{\Big\{u_j u_{j+1}, v_j v_{j+1}, w_j w_{j+1} : 1 \le j \le n-1\Big\} \cup \Big\{u_j v_j, u_j w_j : 1 \le j \le n\Big\}\Big\}$.

For $n > 3$, a total labeling is defined as $g : V \cup E \to \{1, 2, 3, \ldots, 8n\}$ by label the vertices u_j by $2j-1$, v_j by $2(n+j)-1$, w_j by $2(2n+j)-1$ for $1 \le j \le n$. Also, label the edges $u_j u_{j+1}$ by $2(n+j-2)$, $v_j v_{j+1}$

by $2(2n+j-2)$, w_jw_{j+1} by $2(3n+j-2)$ for $j=1,2$, $u_{j+2}u_{j+3}$ by $2j$, $v_{j+2}v_{j+3}$ by $2(n+j)$, $w_{j+2}w_{j+3}$ by $2(2n+j)$ for $1 \le j \le n-3$, u_nu_1 by $2(n-2)$, v_nv_1 by $4(n-1)$, w_nw_1 by $2(3n-2)$, u_jv_j by $6n+j$ and v_jw_j by $7n+j$ for $1 \le j \le n$. Thus, $C_n \times P_3$ is a $(2,3,2)$–constrained total labeled graph.

If $n=3$, label the vertices u_j by $2j-1$, v_j by $2(n+j)-1$, w_j by $2(2n+j)-1$ for $1 \le j \le 3$. Also, label the edges u_ju_{j+1} by $2(2n+j+1)$, v_jv_{j+1} by $2(j+1)$, w_jw_{j+1} by $2(n+j+1)$ for $j=1,2$, u_jv_j by $6n+j$, v_jw_j by $7n+j$ for $1 \le j \le 3$, v_3v_1 by 2, u_3u_1 by 14 and w_3w_1 by 8. Therefore, $C_3 \times P_3$ is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.18. *A mobius ladder graph M_n is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E)=M_n$ be a mobius ladder graph with vertex set $V=\left\{u_j,w_j : 1 \le j \le n\right\}$ and the edge set
$E=\left\{\left\{u_ju_{j+1},w_jw_{j+1} : 1 \le j \le n-1\right\} \cup \left\{u_jw_j : 1 \le j \le n\right\} \cup \{u_nw_1\} \cup \{w_nu_1\}\right\}$,
a total labeling is defined as $g : V \cup E \to \{1,2,3,\ldots,5n\}$ by label the vertices u_j by $2j-1$, w_j by $2n+2j-1$ for $1 \le j \le n$. Also, label the edges u_ju_{j+1} by $2(n+j+1)$, w_jw_{j+1} by $2(j+1)$ for $1 \le j \le n-1$, u_nw_1 by 2, u_1w_n by $2(n+1)$ and u_jw_j by $4n+j$ for $1 \le j \le n$. Thus, M_n is a $(2,3,2)$–constrained total labeled graph. □

The following is an illustrative example of the mobius ladder graph M_5.

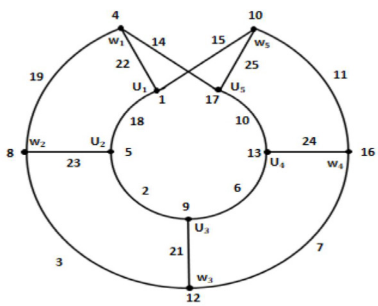

FIGURE 3. mobius ladder graph

Theorem 2.19. *A comb graph $(P_n \odot K_1)$ is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = P_n \odot K_1$ be a carona graph of a path P_n by joining each vertex to a pendent edge which becomes a comb graph with vertex set $V = \left\{u_j, w_j : 1 \le j \le n\right\}$ and edge set $E = \left\{\left\{u_j w_j : 1 \le j \le n\right\} \cup \left\{u_j u_{j+1} : 1 \le j \le n-1\right\}\right\}$, a total labeling is defined as $g : V \cup E \to \{1, 2, 3, \ldots, 4n-1\}$ by label the vertices w_j by $4j-3$, u_j by $4j-1$ for $1 \le j \le n$. Also, label the edges u_1u_2 by $2(2n-1)$, $u_{j+1}u_{j+2}$ by $4j$ for $1 \le j \le n-2$, w_1u_1 by $4(n-1)$ and $u_{j+1}w_{j+1}$ by $2(2j-1)$ for $1 \le j \le n-1$. Thus, $P_n \odot K_1$ is a $(2,3,2)-$constrained total labeled graph. □

Theorem 2.20. *A sunlet graph* $(C_n \odot K_1)$ *is a* $(2,3,2)-$*constrained total labeled graph.*

Proof. Let $G(V,E) = C_n \odot K_1$ be a carona graph of a cycle C_n by joining each vertex to a pendent edge which becomes a sunlet graph with the vertex set $V = \left\{u_j, w_j : 1 \le j \le n\right\}$ and the edge set $E = \left\{\left\{u_j w_j : 1 \le j \le n\right\} \cup \left\{u_j u_{j+1} : 1 \le j \le n-1\right\} \cup \{u_1 u_n\}\right\}$, a total labeling is defined as $g : V \cup E \to \{1, 2, 3, \ldots, 4n\}$ by label the vertices u_j by $4j-1$, w_j by $4j-3$ for $1 \le j \le n$. Also, label the edges $u_{j+1}u_{j+2}$ by $4j$ for $1 \le j \le n-2$, $u_{j+1}w_{j+1}$ by $2(2j-1)$ for $1 \le j \le n-1$, u_1u_2 by $4n$, u_1w_1 by $2(2n-1)$ and u_1u_n by $4(n-1)$. Thus, $C_n \odot K_1$ is a $(2,3,2)-$constrained total labeled graph. □

Theorem 2.21. *A carona product of ladder graph* $(L_n \odot K_1)$ *is a* $(2,3,2)-$*constrained total labeled graph.*

Proof. Let $G(V,E) = L_n \odot K_1$ be a carona graph of a ladder L_n by joining each vertex to a pendent edge which becomes a carona product of a ladder graph with the vertex set

$V = \left\{u_j, w_j, p_j, q_j : 1 \le j \le n\right\}$ and the edge set

$$E = \left\{\left\{u_j w_j, p_j w_j, u_j q_j : 1 \le j \le n\right\} \cup \left\{u_j u_{j+1}, v_j v_{j+1} : 1 \le j \le n-1\right\}\right\}.$$

For $n > 2$, a total labeling is defined as $g : V \cup E \to \{1, 2, 3, \ldots, 9n-2\}$ by label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$, q_j by $2(2n+j)-1$, p_j by $2(3n+j)-1$ for $1 \le j \le n$. Also, label an edges w_jw_{j+1} by $2j$, u_ju_{j+1} by $2(n+j-1)$ for $1 \le j \le n-1$, u_jw_j by $2(2n+j-2)$, v_jp_j by $2(3n+j-2)$ for $1 \le j \le n$, u_jq_j by $2(4n+j-2)$ for $j = 1, 2$ and $u_{j+2}q_{j+2}$ by $8n+j$ for $1 \le j \le n-2$. Thus, $L_n \odot K_1$ is a $(2,3,2)-$constrained total labeled graph.

If $n = 2$, label the vertices u_j by $2j-1$, w_j by $2j+3$, p_j by $2j+11$, q_j by $2j+7$ for $1 \le j \le 2$. Also, label the edges u_1u_2 by 8,

w_1w_2 by 2, u_jw_j by $2(j+6)$, u_jq_j by $2(j+1)$, w_jp_j by $2(j+4)$ for $j = 1,2$. Therefore, $L_2 \odot K_1$ is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.22. *For integer $n > 2$, a carona product of an open ladder graph $(OL_n \odot K_1)$ is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = OL_n \odot K_1$ be a carona graph of an open ladder OL_n by joining each vertex to a pendent edge which becomes a carona product of an open ladder graph with the vertex set $V = \left\{\left\{u_j, w_j : 1 \le j \le n\right\} \cup \left\{p_j, q_j : 1 \le j \le n-2\right\}\right\}$ and the edge set $E = \left\{\left\{u_ju_{j+1}, w_jw_{j+1} : 1 \le j \le n-1\right\} \cup \left\{u_{j+1}q_j, w_{j+1}p_j, u_{j+1}w_{j+1} : 1 \le j \le n-2\right\}\right\}$.

For $n > 3$, a total labeling is defined as $g : V \cup E \to \{1,2,3,\ldots,9n-12\}$ by label vertices u_j by $2j-1$, w_j by $2(n+j)-1$ for $1 \le j \le n$, q_j by $2(n+j)-5$ and p_j by $2(2n+j)-1$ for $1 \le j \le n-2$. Also, label edges u_ju_{j+1} by $2(n+j-1)$, w_jw_{j+1} by $2j$ for $1 \le j \le n-1$, $w_{j+1}p_j$ by $2(2n+j-2)$, $u_{j+1}q_j$ by $2(3n+j-4)$ for $1 \le j \le n-2$, $u_{j+1}w_{j+1}$ by $2(4n+j-6)$ for $j = 1,2$ and $u_{j+3}w_{j+3}$ by $8n+j-8$ for $1 \le j \le n-4$. Thus, $OL_n \odot K_1$ is a $(2,3,2)$–constrained total labeled graph.

If $n = 3$, label vertices u_j by $2j-1$, w_j by $2j+7$ for $1 \le j \le 3$, p_1 by 15 and q_1 by 7. Also, label edges u_ju_{j+1} by $2(j+4)$, w_jw_{j+1} by $2j$ for $j = 1,2$, u_2q_1 by 14, w_2p_1 by 6. Therefore, $OL_3 \odot K_1$ is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.23. *For integer $n \ge 2$, a carona product of a triangular ladder graph $(TL_n \odot K_1)$ is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = TL_n \odot K_1$ be a carona graph of a triangular ladder TL_n by joining each vertex to a pendent edge which becomes a carona product of a triangular ladder graph with vertex set $V = \left\{u_j, w_j, p_j, q_j : 1 \le j \le n\right\}$ and the edge set

$$E = \left\{\left\{u_ju_{j+1}, w_jw_{j+1} : 1 \le j \le n-1\right\} \cup \left\{u_jq_j, w_jp_j, u_jw_j : 1 \le j \le n\right\}\right\}.$$

For $n > 2$, a total labeling is defined as $g : V \cup E \to \{1,2,3,\ldots,10n-3\}$ by label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$, q_j by $4n+2j-1$, p_j by $2(3n+j)-1$ for$1 \le j \le n$. Also, label the edges u_ju_{j+1} by $2(n+j-1)$, w_jw_{j+1} by $2j$ for $1 \le j \le n-1$, u_jq_j by $2(2n+j-2)$, w_jp_j by $2(3n+j-2)$, u_jw_j by $9n+j-3$ for $1 \le j \le n$, $u_{j+1}w_j$ by $2(4n+j-2)$ for $1 \le j \le 2$ and $u_{j+3}w_{j+2}$ by $8n+j$ for

$1 \leq j \leq n-3$. Thus, $TL_n \odot K_1$ is a $(2,3,2)$–constrained total labeled graph.

If $n = 2$, label the vertices u_j by $2j-1$, w_j by $2j+3$, p_j by $2j+11$, q_j by $2j+7$ for $1 \leq j \leq 2$. Also, label an edges u_1u_2 by 12, w_1w_2 by 2, u_jw_j by $15+j$, w_1u_2 by 14, u_jq_j by $2(j+1)$, w_jp_j by $2(j+3)$ for $j = 1,2$. Therefore, $TL_2 \odot K_1$ is a $(2,3,2)$–constrained total labeled graph. □

Theorem 2.24. *For integer $n \geq 2$, a carona product of an open triangular ladder graph ($OTL_n \odot K_1$) is a $(2,3,2)$–constrained total labeled graph.*

Proof. Let $G(V,E) = OTL_n \odot K_1$ be a carona graph of an open triangular ladder OTL_n by joining each vertex to a pendent edge which becomes a carona product of an open triangular ladder graph with the vertex set $V = \left\{u_j, w_j, p_j, q_j : 1 \leq j \leq n\right\}$ and edge set $E = \left\{u_ju_{j+1}, w_jw_{j+1}, w_ju_{j+1} : 1 \leq j \leq n-1\right\} \cup \left\{u_jp_j, w_jq_j : 1 \leq j \leq n\right\} \cup \left\{u_{j+1}w_{j+1} : 1 \leq j \leq n-2\right\}$.

For $n > 2$, define a total labeling as $g : V \cup E \to \{1,2,3,\ldots,10n-5\}$ by label the vertices u_j by $2j-1$, w_j by $2(n+j)-1$, q_j by $3(n+j)-1$, p_j by $2(2n+j)-1$ for $1 \leq j \leq n$. Also, label the edges u_ju_{j+1} by $2(n+j-1)$, w_jw_{j+1} by $2j$ for $1 \leq j \leq n-1$, u_jp_j by $(2n+j-2)$, w_jq_j by $2(3n+j-2)$ for $1 \leq j \leq n$, $u_{j+1}w_{j+1}$ by $2(4n+j-2)$, $u_{j+1}w_j$ by $2(4n+j)-1$ for $1 \leq j \leq n-2$ and $w_{n-1}u_n$ by $2(5n-3)$. Thus, $OTL_n \odot K_1$ is a $(2,3,2)$–constrained total labeled graph.

If $n = 2$, label the vertices u_j by $2j-1$, w_j by $2j+3$, p_j by $2j+7$, q_j by $2j+11$ for $1 \leq j \leq 2$. Also, label the edges u_1u_2 by 6, w_1w_2 by 2, u_2w_1 by 14, u_jp_j by $4j$, w_jp_j by $2(j+4)$ for $j = 1,2$. Therefore, $OTL_2 \odot K_1$ is a $(2,3,2)$–constrained total labeled graph. □

Conclusion

We defined $(2,3,2)$–constrained total labeling of some graphs abbreviated as $(2,3,2)$–CTLG. The minimum number of isolated vertices that need to be added to a graph G to transform it into resulting graph is $(2,3,2)$-CTLG is called $(2,3,2)$-constrained total number of G, denoted by $t_{(2,3,2)}(G)$ and with the help of the ideas presented in this paper several types of the graphs can be studied. The basic concepts of a $(2,3,2)$−constrained total labeling of some graphs and the properties are investigated and can be extended in the future research with some applications. The proposed $(2,3,2)$−constrained total labeling of some graphs can be applied to more general and complex information systems for future research. Also, there are a lot of research scopes in this area. Thus, it is advantageous to use constrained total labeling in real life situations.

References

[1] Alexander Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium*, *Rome*, (1966) 349–355.

[2] R. L. Graham and N. J. A. Sloane, On Additive Bases and Harmonious Graphs, *SIAM J. Alg. Discrete Math.*, **1** (4) (1980) 382–404.

[3] G. S. Bloom, S. W. Golomb, Applications of numbered undirected graphs, *Proceedings of the IEEE*, **65** (4) (1977) 562–570.

[4] W. K. Hale, Frequency assignment: theory and application, *Proc. IEEE*, **68** (12) (1980), 1497–1514.

[5] Jerrold R. Griggs and Roger K. Yeh, Labelling graphs with a condition at distance 2, *SIAM Journal on Discrete Mathematics*, **5** (4) (1992) 586–595.

[6] Roger K. Yeh, A survey on labeling graphs with a condition at distance two, *Discrete Mathematics*, **306** (12), (2006) 1217–1231.

[7] Gerard J. Chang and David Kuo, The L(2,1)-labeling problem on graphs, *SIAM Journal on Discrete Mathematics*, **9** (2), (1996) 309–316.

[8] Ma-Lian Chia, David Kuo, Hong-ya Liao, Cian-Hui Yang, Roger K. Yeh, $L(3,2,1)$−Labeling of graphs, *Taiwanese J. Math.*, **15** (6) (2011) 2439 – 2457.

[9] Bimal Shil, Rakhal Das, Suman Das, Binod Chandra Tripathy, and Carlos Granados, Degree, order and size of single-valued quadripartitioned neutrosophic graph, *BISTUA Revista de la Facultad de Ciencias Básicas*, **20** (1) (2022), 63–69.

[10] Suman Das, Rakhal Das, and Surapati Pramanik, Single-valued pentapartitioned neutrosophic graphs, *Neutrosophic Sets and Systems*, **50** (1) (2022), 225–238.

[11] Fred Buckley, Frank Harary, Distance in Graphs, *Addison-wesley*, **63** (1990).

[12] F. Harary, Graph Theory, *Boca Raton*, *CRC Press*(1969).

[13] Joseph A. Gallian, A dynamic survey of graph labelling, *Electronic Journal of Combinatorics*, **17** (DS6) (6) 4-623.

[14] K. Shreedhar, B. Sooryanarayana and P. Raghunath, On Samarandachely K−constrained labeling of Graphs, *International Journal of Mathematical Combinatorics*, **1** (April 2009) 50–60.

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