

EQUAL-CUBIC GRAPH OF FINITE GROUPS

SHILPA AGGARWAL*, PANKAJ RANA, AMIT SEHGAL, AND POOJA BHATIA

ABSTRACT. Equal-cubic graph of a finite group G denoted as $EC(G)$ is finite, simple and undirected graph with the vertex set G in which pair of distinct vertices x and y are adjacent if and only if $x^3 = y^3$. In this paper graphical representation of equal-cubic graph of finite groups is done. Structural properties such as independent number, girth, chromatic number, chromatic polynomial, clique number, matching number, weakly perfectness, Laplacian polynomial, domination number of equal-cubic graph. Topological indices such as first Zagreb index, second Zagreb index, harmonic index, Randic and general Randic index, geometric-arithmetic index and atomic-bond connectivity index of equal-cubic graph of dihedral group are also studied.

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1. INTRODUCTION

In recent years various graphs associated with groups and algebraic structures such as equal-square graph [1, 2], square power graph [3–5], cubic power graph [6, 7], k^{th} -power graph [8], square element graph [9, 10] etc have been introduced and studied. In [11] Laplacian spectra of power graphs of certain finite groups is studied. Edge-based topological indices for zero divisor graphs of commutative rings are studied in [12], for coprime order graphs of finite abelian p -groups in [13] and Wiener indices for a connected graph in [14]. General Randic energy of the various graphs through graph operations is obtained in [15]. Various structural properties such as girth, diameter, completeness, traversability, bipartiteness and chromatic number of line graph are studied in [16].

In [1] authors have introduced and studied graphical representation of equal-square graphs associated with finite groups. Equal-square graph of a finite group is a simple and undirected graph with finite group G as its vertex set such that a pair of different vertices x and y are adjacent if and only if $x^2 = y^2$. In this paper we have introduced and studied the equal-cubic graph of a finite group. Equal-cubic graph of a finite group G , is simple undirected graph with G as its vertex set having pair of different vertices x and y adjacent if and only if $x^3 = y^3$.

In Section 2, basic definitions, notations and results related to group and

*Corresponding Author.

graph theory used in this paper are given. In Section 3, structure, connectedness, girth, independent number, clique number, chromatic number, chromatic polynomial, matching number, weakly perfectness, Laplacian polynomial and domination number of equal-cubic graph of finite groups are studied. In Section 4, topological indices of equal-cubic graph of dihedral group are studied.

2. PRELIMINARIES

$EC(G)$ denotes the equal-cubic graph of a finite group G with vertex set $V(EC(G)) = G$ and edge set $E(EC(G))$. Number of vertices in $EC(G)$ is denoted as $|V(EC(G))|$ or $|G|$ and number of edges by $|E(EC(G))|$. Complete graph is a graph in which every pair of vertices are adjacent. With n vertices, complete graph is denoted as K_n . The length of the smallest cycle in $EC(G)$ is known as girth of $EC(G)$, denoted as $gr(EC(G))$. A set of vertices in which no pair of vertices are adjacent then that set is independent set and the number of vertices in independent set having maximum number of vertices is known as the independent number of $EC(G)$, denoted as $\beta(EC(G))$. Complete subgraph of $EC(G)$ is known as clique of $EC(G)$ and number of vertices in maximum clique is known as the clique number denoted as $\omega(EC(G))$. Vertex coloring of $EC(G)$ in such a way that no pair of adjacent vertices get the same colors is known as a proper coloring and the least number of colors required for proper coloring of $EC(G)$ is known as the chromatic number denoted as $\chi(EC(G))$. Chromatic polynomial $P(EC(G), x)$ of graph $EC(G)$ gives the number of ways of properly coloring the graph. If K_n is complete graph with n vertices then its chromatic polynomial $P(K_n, x) = x(x-1)(x-2)\cdots(x-n+1)$. If $EC_1(G_1), EC_2(G_2), \dots, EC_n(G_n)$ are disjoint components of $EC(G)$ then chromatic polynomial of $EC(G)$, $P(EC(G), x) = P(EC_1(G_1), x) \times P(EC_2(G_2), x) \times \cdots \times P(EC_n(G_n), x)$. A set of edges is said to be matching if no pair of edges in set share a common vertex. A matching containing the largest possible number of edges is known as maximum matching and number of edges in maximum matching is known as the matching number.

$D(EC(G))$ and $A(EC(G))$ denote the vertex degree diagonal matrix and adjacency matrix respectively. Vertex degree diagonal matrix is a diagonal matrix having degree of vertices as diagonal entries. Whereas adjacency matrix is a square matrix having i_j^{th} entry 1 if $i \neq j$ & v_i, v_j vertices are adjacent and 0, otherwise. The difference of vertex degree diagonal matrix and adjacency matrix of $EC(G)$ is known as the Laplacian matrix of $EC(G)$, denoted as $L(EC(G))$. Characteristic polynomial of $L(EC(G))$ is known as Laplacian polynomial of $EC(G)$, denoted as $\Theta(EC(G), x)$. If K_n is complete graph with n vertices then $\Theta(K_n, x) = x(x-n)^{n-1}$.

If $EC_1(G_1), EC_2(G_2), \dots, EC_n(G_n)$ are disjoint components of $EC(G)$ then Laplacian polynomial of $EC(G)$, $\Theta(EC(G), x) = \Theta(EC_1(G_1), x) \times \Theta(EC_2(G_2), x) \times \cdots \times \Theta(EC_n(G_n), x)$ [17]. A subset D of $V(EC(G))$ is known as a dominating set if every vertex $EC(G)$ either lie in D or is adjacent to atleast one vertex in D . Number of vertices in minimum dominating set is known as the dominating number denoted as $\gamma(EC(G))$. \boxtimes is used for strong product of graphs. $D_n = \{a^i b^j : a^n = e, b^2 = e, a^i b = b a^{n-i}, 1 \leq i \leq n, 1 \leq j \leq 2\}$ be the

dihedral group of order $2n$ and $D_n^3 = \{x^3 : x \in D_n\}$. Let e is the identity element of the group G , then order of an element $x \in G$, $o(x)$ is the smallest positive integer n for which $x^n = e$.

3. STRUCTURAL PROPERTIES OF EQUAL-CUBIC GRAPH

Theorem 3.1. *Let G be a finite group of order n then $EC(G)$ is the empty graph if and only if $\gcd(3, n) = 1$.*

Proof. Let G be a finite group of order n and $EC(G)$ is the Equal-cubic graph of G .

Let $\gcd(3, n) = 1$. Then we have n of form $3k + 1$ or $3k + 2$. Let us assume that there exists a pair of vertices $x, y \in V(EC(G))$ are adjacent in $EC(G)$.

Case 1. When $n = 3k + 1$

As x is adjacent with y , so we have $x^3 = y^3 \implies (x^3)^{2k+1} = (y^3)^{2k+1} \implies x^{6k+3} = y^{6k+3} \implies x^{3k+1}x^{3k+1} = y^{3k+1}y^{3k+1} \implies x = y$. Thus we have no pair of adjacent vertices in $EC(G)$. Thus $EC(G)$ is empty.

Case 2. When $n = 3k + 2$

As x is adjacent with y , so we have $x^3 = y^3 \implies (x^3)^{k+1} = (y^3)^{k+1} \implies x^{3k+3} = y^{3k+3} \implies x^{3k+2}x = y^{3k+2}y \implies x = y$. Thus we have no pair of adjacent vertices in $EC(G)$. Thus $EC(G)$ is empty.

Thus if $\gcd(3, n) = 1$ then $EC(G)$ is the empty graph.

Conversely, let $EC(G)$ be the empty graph. If the order of G is n such that $\gcd(3, n) = 3$, then by using Cauchy's theorem we always have element x of order 3 in G adjacent with identity element e of G as $e^3 = e = x^3$.

Hence the required result. □

Theorem 3.2. *The following assertions are equivalent for a group G with order $o(G) \geq 2$.*

- (i) $EC(G)$ is connected.
- (ii) $EC(G)$ is complete.
- (iii) G is a 3-group.

Proof. (i) \implies (ii) Let $EC(G)$ is a connected graph. For any vertex $x \in G$ there exists a path in $EC(G)$ from x to e which can be represented as $x - x_1 - x_2 - \dots - e$. Thus we have $x^3 = x_1^3 = x_2^3 = \dots = e^3 = e$. So, $x^3 = e$ for all $x \in G$. Hence we have $x^3 = y^3$ for all $x, y \in G$ and so $EC(G)$ is complete graph.

(ii) \implies (iii) If $EC(G)$ is complete graph then we have $x^3 = y^3 \forall x, y \in G$. So we have $x^3 = e^3 = e$ for all $x \in G$. Thus we have every non-identity element of G of order 3. Hence G is a 3-group.

(iii) \implies (i) If G is a 3-group then we have every non-identity element of order three. So we have $x^3 = e$ for all $x \in G$. Thus we have every non-identity element vertex adjacent with identity element e in $EC(G)$. Hence $EC(G)$ is a connected graph. □

Corollary 3.3. *Let G be a elementary abelian 3-group then $EC(G) = K_{3^n}$.*

Theorem 3.4. *Let G be a finite cyclic group of order $3n$ then $EC(G) = nK_3$.*

Proof. Let G be a finite cyclic group of order $3n$ i.e

$G = \langle x \rangle = \{e, x, x^2, x^3, \dots, x^n, \dots, x^{2n}, \dots, x^{3n-1}\}$. Then $(x^i)^3 = (x^{n+i})^3 = (x^{2n+i})^3,$

where $0 \leq i \leq n-1$. Let x^i be adjacent with x^j where $0 \leq i \leq j \leq 3n-1$. So $(x^i)^3 = (x^j)^3 \implies x^{3i} = x^{3j} \implies x^{3(j-i)} = e \implies o(x)$ divides $3(j-i) \implies 3n|3(j-i) \implies n|(j-i), 0 \leq j-i \leq 3n-1 \implies j-i = 0$ or n or $2n$, then $j = i$ or $j = i+n$ or $j = i+2n$. Thus in $EC(G)$, x^i is adjacent to only two vertices i.e x^{n+i} and x^{2n+i} where $0 \leq i \leq n-1$. Hence we have $EC(G) = nK_3$. \square

Corollary 3.5. Let G be a finite cyclic group of order $3n$ then

- (i) Girth, $gr(EC(G)) = 3$.
- (ii) Independent number, $\beta(EC(G)) = n$.
- (iii) Clique number, $\omega(EC(G)) = 3$.
- (iv) Chromatic number, $\chi(EC(G)) = 3$.
- (v) Chromatic polynomial, $P(EC(G), x) = x^n(x-1)^n(x-2)^n$.
- (vi) Matching number, $\mu(EC(G)) = n$.
- (vii) $EC(G)$ is weakly perfect.
- (viii) Laplacian polynomial, $\Theta(EC(G), x) = x^n(x-3)^{2n}$.
- (ix) Domination number, $\gamma(EC(G)) = n$.

Theorem 3.6. Let D_n be dihedral group of order $2n$ then

$$EC(D_n) = \begin{cases} 2nK_1 & \text{if } \gcd(3, n) = 1, \\ \frac{n}{3}K_3 \cup nK_1 & \text{if } \gcd(3, n) = 3. \end{cases}$$

Proof. Let $D_n = \{a^i b^j : a^n = e, b^2 = e, a^i b = b a^{n-i}, 1 \leq i \leq n, 1 \leq j \leq 2\}$ be the dihedral group of order $2n$ and $EC(D_n)$ be the Equal-cubic graph of D_n . Let $D_n^3 = \{x^3 : x \in D_n\}$.

Case 1. When $\gcd(3, n) = 1$

$\gcd(3, n) = 1 \implies \gcd(3, 2n) = 1$, so by using Theorem 3.1 we have $EC(D_n)$ is the empty graph. Hence $EC(D_n) = 2nK_1$.

Case 2. When $\gcd(3, n) = 3$

For $1 \leq i \leq n, j = 1$ we have reflection elements $x = a^i b^j$ of D_n , such that $x^3 = x$. So every reflection element have degree 0 i.e forms no edge in $EC(D_n)$. Thus n reflection elements forms $n - K_1$ type components of $EC(D_n)$. For $1 \leq i \leq n, j = 2$ we have rotation elements $a^i b^j = a^i$ such that $(a^i)^3 = (a^{i+\frac{n}{3}})^3 = (a^{i+\frac{2n}{3}})^3$. So in $EC(D_n)$, a^i is adjacent to only two vertices i.e $a^{i+\frac{n}{3}}$ and $a^{i+\frac{2n}{3}}$ where $1 \leq i \leq n$. Thus we have $n - K_1$ type components and $\frac{n}{3} - K_3$ type components in $EC(D_n)$. Hence $EC(D_n) = \frac{n}{3}K_3 \cup nK_1$. \square

Corollary 3.7. Let D_n be dihedral group of order $2n$ then

- (i) Girth, $gr(EC(D_n)) = \begin{cases} \infty & \text{if } \gcd(3, n) = 1, \\ 3 & \text{if } \gcd(3, n) = 3. \end{cases}$
- (ii) Independent number, $\beta(EC(D_n)) = \begin{cases} 2n & \text{if } \gcd(3, n) = 1, \\ \frac{4n}{3} & \text{if } \gcd(3, n) = 3. \end{cases}$
- (iii) Clique number, $\omega(EC(D_n)) = \begin{cases} 1 & \text{if } \gcd(3, n) = 1, \\ 3 & \text{if } \gcd(3, n) = 3. \end{cases}$
- (iv) Chromatic number, $\chi(EC(D_n)) = \begin{cases} 1 & \text{if } \gcd(3, n) = 1, \\ 3 & \text{if } \gcd(3, n) = 3. \end{cases}$

- (v) Chromatic polynomial, $P(EC(D_n), x) = \begin{cases} x^{2n} & \text{if } \gcd(3, n) = 1, \\ x^{\frac{4n}{3}}(x-1)^{\frac{n}{3}}(x-2)^{\frac{n}{3}}. \end{cases}$
- (vi) Matching number, $\mu(EC(D_n)) = \begin{cases} 0 & \text{if } \gcd(3, n) = 1, \\ \frac{n}{3} & \text{if } \gcd(3, n) = 3. \end{cases}$
- (vii) $EC(G)$ is weakly perfect.
- (viii) Laplacian polynomial, $\Theta(EC(D_n), x) = \begin{cases} x^{2n} & \text{if } \gcd(3, n) = 1, \\ x^{\frac{4n}{3}}(x-3)^{\frac{2n}{3}} & \text{if } \gcd(3, n) = 3. \end{cases}$
- (ix) Domination number, $\gamma(EC(D_n)) = \begin{cases} 2n & \text{if } \gcd(3, n) = 1, \\ \frac{4n}{3} & \text{if } \gcd(3, n) = 3. \end{cases}$

Example 3.1. Equal-cubic graph of dihedral group of order 12.

$$D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$$

Using Theorem 3.6, $EC(D_6) = 2K_3 \cup 6K_1$.

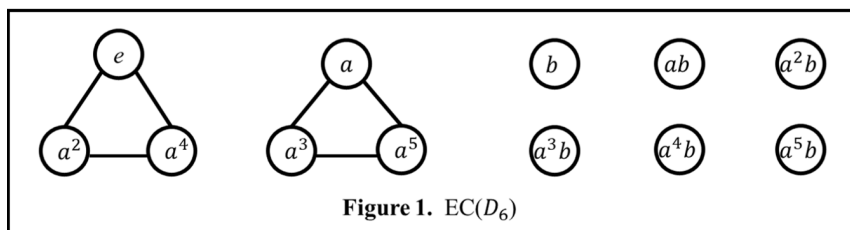


Figure 1. $EC(D_6)$

Theorem 3.8. For any two finite groups H_1 and H_2 , $EC(H_1 \times H_2) = EC(H_1) \boxtimes EC(H_2)$.

Proof. Clearly, $V(EC(H_1 \times H_2)) = V(EC(H_1)) \boxtimes V(EC(H_2)) = H_1 \times H_2$. Let $(h_1, h_2), (h'_1, h'_2) \in H_1 \times H_2$ such that $(h_1, h_2)(h'_1, h'_2) \in E(EC(H_1 \times H_2))$. Then $(h_1, h_2)^3 = (h'_1, h'_2)^3$ and so $h_1^3 = h'^3_1$ and $h_2^3 = h'^3_2$. From $h_1^3 = h'^3_1$ we get either $h_1 = h'_1$ or $h_1 h'_1 \in E(H_1)$. Also from $h_1^3 = h'^3_1$ we get either $h_2 = h'_2$ or $h_2 h'_2 \in E(H_2)$. Thus we obtain that $h_1 = h'_1$ and $h_2 h'_2 \in E(H_2)$, or $h_1 h'_1 \in E(H_1)$ and $h_2 = h'_2$, or $h_1 h'_1 \in E(H_1)$ and $h_2 h'_2 \in E(H_2)$. Hence, $(h_1, h_2)(h'_1, h'_2) \in E(EC(H_1) \boxtimes EC(H_2))$.

Conversely, let $(h_1, h_2), (h'_1, h'_2) \in H_1 \times H_2$ such that $(h_1, h_2)(h'_1, h'_2) \in E(EC(H_1) \boxtimes EC(H_2))$. This implies that $h_1 = h'_1$ and $h_2 h'_2 \in E(H_2)$, or $h_1 h'_1 \in E(H_1)$ and $h_2 = h'_2$, or $h_1 h'_1 \in E(H_1)$ and $h_2 h'_2 \in E(H_2)$. Thus either $h_1 = h'_1$ or $h_1^3 = h'^3_1$ and also either $h_2 = h'_2$ or $h_2^3 = h'^3_2$. Therefore $h_1^3 = h'^3_1$ and $h_2^3 = h'^3_2$. Hence $(h_1, h_2)(h'_1, h'_2) \in E(EC(H_1 \times H_2))$.

Hence the required result. \square

Theorem 3.9. Let $H_i = \langle a_i \rangle$ be a cyclic group of order m_i for every $1 \leq i \leq n$ and s number of integers in set $\{m_1, m_2, m_3, \dots, m_n\}$ such that $\gcd(m_i, 3) = 3$. Then $EC(\prod_{i=1}^n H_i) = \frac{m_1 m_2 \dots m_n}{3^s} K_{3^s}$.

Proof. Using Theorem 3.8, we obtain $EC(\prod_{i=1}^n G_i) = \boxtimes_{i=1}^n EC(H_i)$. and by using Theorem 3.1 and 3.4, we have for each $1 \leq i \leq n$

$$EC(H_i) = \begin{cases} m_i K_1, & \text{if } \gcd(3, m_i) = 1, \\ \frac{m_i}{3} K_3 & \text{if } \gcd(3, m_i) = 3. \end{cases}$$

Now using Lemma 1. from [1] (i.e $rK_m \boxtimes sK_n = rsK_{mn}$ for all $m, n, r, s \in \mathbb{Z}^+$), we get $EC(\prod_{i=1}^n H_i) = \frac{m_1 m_2 \dots m_n}{3^s} K_{3^s}$. □

Theorem 3.10. For distinct prime numbers $3, p_1, p_2, \dots, p_n$, let $G \cong \mathbb{Z}_{3^{b_1}} \times \mathbb{Z}_{3^{b_2}} \times \dots \times \mathbb{Z}_{3^{b_t}} \times \mathbb{Z}_{p_1^{b_{11}}} \times \dots \times \mathbb{Z}_{p_1^{b_{1t_1}}} \times \dots \times \mathbb{Z}_{p_n^{b_{n1}}} \times \dots \times \mathbb{Z}_{p_n^{b_{nt_n}}}$ be a finite abelian group of order $3^a p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$ where $a = b_1 + b_2 + \dots + b_t$, $a_i = b_{i1} + b_{i2} + \dots + b_{it_i}$ with $b_1 \geq b_2 \geq \dots \geq b_t \geq 1$ and $b_{i1} \geq b_{i2} \geq \dots \geq b_{it_i} \geq 1$ for each i ; $1 \leq i \leq n$. Then we have $EC(G) = \left(\frac{|G|}{3^t}\right) K_{3^t}$.

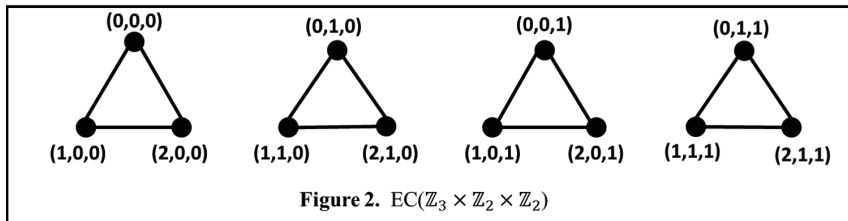
Proof. Using Theorem 3.8 we get $EC(G) = EC(\mathbb{Z}_{3^{b_1}}) \boxtimes EC(\mathbb{Z}_{3^{b_2}}) \boxtimes \dots \boxtimes EC(\mathbb{Z}_{3^{b_t}}) \boxtimes EC(\mathbb{Z}_{p_1^{b_{11}}}) \boxtimes \dots \boxtimes EC(\mathbb{Z}_{p_1^{b_{1t_1}}}) \boxtimes \dots \boxtimes EC(\mathbb{Z}_{p_n^{b_{n1}}}) \boxtimes \dots \boxtimes EC(\mathbb{Z}_{p_n^{b_{nt_n}}})$. Now applying Theorem 3.1 and 3.4, we get $EC(G) = 3^{b_1-1} K_3 \boxtimes \dots \boxtimes 3^{b_t-1} K_3 \boxtimes p_1^{b_{11}} K_1 \boxtimes \dots \boxtimes p_1^{b_{1t_1}} K_1 \boxtimes \dots \boxtimes p_n^{b_{n1}} \boxtimes \dots \boxtimes p_n^{b_{nt_n}}$. Using Lemma 1 of [1], we get $EC(G) = (3^{b_1+b_2+\dots+b_t-t} p_1^{a_1} \dots p_n^{a_n}) K_{3^t} = \frac{|G|}{3^t} K_{3^t}$. Hence $EC(G) = \frac{|G|}{3^t} K_{3^t}$. □

Corollary 3.11. For distinct prime numbers $3, p_1, p_2, \dots, p_n$, let $G \cong \mathbb{Z}_{3^{b_1}} \times \mathbb{Z}_{3^{b_2}} \times \dots \times \mathbb{Z}_{3^{b_t}} \times \mathbb{Z}_{p_1^{b_{11}}} \times \dots \times \mathbb{Z}_{p_1^{b_{1t_1}}} \times \dots \times \mathbb{Z}_{p_n^{b_{n1}}} \times \dots \times \mathbb{Z}_{p_n^{b_{nt_n}}}$ be a finite abelian group of order $3^a p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$ where $a = b_1 + b_2 + \dots + b_t$, $a_i = b_{i1} + b_{i2} + \dots + b_{it_i}$ with $b_1 \geq b_2 \geq \dots \geq b_t \geq 1$ and $b_{i1} \geq b_{i2} \geq \dots \geq b_{it_i} \geq 1$ for each i ; $1 \leq i \leq n$. Then we have

- (i) Girth, $gr(EC(G)) = 3$.
- (ii) Independent number, $\beta(EC(G)) = \frac{|G|}{3^t}$.
- (iii) Clique number, $\omega(EC(G)) = 3^t$.
- (iv) Chromatic number, $\chi(EC(G)) = 3^t$.
- (v) Chromatic polynomial, $P(EC(G), x) = \{x(x-1)(x-2) \dots (x-3^t+1)\}^{\frac{|G|}{3^t}}$.
- (vi) Matching number, $\mu(EC(G)) = \frac{(3^t-1) \times |G|}{2 \times 3^t}$.
- (vii) $EC(G)$ is weakly perfect.
- (viii) Laplacian polynomial, $\Theta(EC(G), x) = \{x(x-3^t)^{3^t-1}\}^{\frac{|G|}{3^t}}$.
- (ix) Dominating number, $\gamma(EC(G)) = \frac{|G|}{3^t}$.

Example 3.2. Equal-cubic graph of $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Using Theorem 3.10, $EC(\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 4K_3$.



4. TOPOLOGICAL INDICES OF EQUAL-CUBIC GRAPH OF DIHEDRAL GROUP

Theorem 4.1. *Let D_n be the dihedral group of order $2n$. Then*

$$(i) \text{ First Zagreb index, } M_1(EC(D_n)) = \begin{cases} 4n \text{ when } \gcd(3, n) = 3, \\ 0 \text{ when } \gcd(3, n) = 1. \end{cases}$$

$$(ii) \text{ Second Zagreb index, } M_2(EC(D_n)) = \begin{cases} 4n \text{ when } \gcd(3, n) = 3, \\ 0 \text{ when } \gcd(3, n) = 1. \end{cases}$$

Proof. Let D_n be the $2n$ order dihedral group.

(i) $M_1(EC(D_n)) = \sum_{u \in V(EC(D_n))} (deg(x))^2$ is the first Zagreb index [18] of $EC(D_n)$.

Using Theorem 3.6, $EC(D_n)$ is the empty graph when $\gcd(3, n) = 1$. So all the vertices have 0 degree. Thus $M_1(EC(D_n)) = 0$ when $\gcd(n, 3) = 1$.

Using Theorem 3.6, there are n vertices with 0 degree and n vertices with degree 2 in $EC(D_n)$ when $\gcd(3, n) = 3$. Thus $M_1(EC(D_n)) = n \times (0)^2 + n \times (2)^2 = 4n$ when $\gcd(3, n) = 3$.

(ii) $M_2(EC(D_n)) = \sum_{u_1 u_2 \in E(EC(D_n))} [deg(u_1) \times deg(u_2)]$ is Second Zagreb index [18] of $EC(D_n)$.

Using Theorem 3.6, $EC(D_n)$ is the empty graph when $\gcd(n, 3) = 1$. So $E(EC(D_n))$ is the empty set. Thus $M_2(EC(D_n)) = 0$ when $\gcd(3, n) = 1$.

Using Theorem 3.6, there are n edges with vertices on both ends having degree 2 when $\gcd(3, n) = 3$. Thus $M_2(EC(D_n)) = n \times (2 \times 2) = 4n$ when $\gcd(3, n) = 3$. □

Theorem 4.2. *Let D_n be the dihedral group of order $2n$. Then the harmonic*

$$\text{index, } H_r(EC(D_n)) = \begin{cases} \frac{n}{2} \text{ when } \gcd(3, n) = 3, \\ 0 \text{ when } \gcd(3, n) = 1. \end{cases}$$

Proof. Let D_n be the $2n$ order dihedral group.

$H_r(EC(D_n)) = \sum_{u_1 u_2 \in E(EC(D_n))} \frac{2}{deg(u_1) + deg(u_2)}$ is Harmonic index [19] of $EC(D_n)$.

Case 1. When $\gcd(n, 3) = 3$

Using Theorem 3.6, there are n edges with vertices on both ends having degree 2 when $\gcd(n, 3) = 3$. Thus $H_r(EC(D_n)) = n \times \frac{2}{2+2} = \frac{n}{2}$ when $\gcd(n, 3) = 3$.

Case 2. When $\gcd(n, 3) = 1$

Using Theorem 3.6, $EC(D_n)$ is the empty graph when $\gcd(n, 3) = 1$. So $E(EC(D_n))$ is the empty set. Thus $H_r(EC(D_n)) = 0$ when $\gcd(n, 3) = 1$. □

Theorem 4.3. *Let D_n be the dihedral group of order $2n$. Then*

$$(i) \text{ general Randic index, } R_\alpha(EC(D_n)) = \begin{cases} 4^\alpha n \text{ when } \gcd(3, n) = 3, \\ 0 \text{ when } \gcd(3, n) = 1. \end{cases}$$

$$(ii) \text{ Randic index, } R_{-\frac{1}{2}}(EC(D_n)) = \begin{cases} \frac{n}{2} \text{ when } \gcd(3, n) = 3, \\ 0 \text{ when } \gcd(3, n) = 1. \end{cases}$$

Proof. Let D_n be the dihedral group of order $2n$.

(i) $R_\alpha(EC(D_n)) = \sum_{u_1 u_2 \in E(EC(D_n))} \{deg(u_1)deg(u_2)\}^\alpha$ is the general Randic index [20] of $EC(D_n)$.

Case 1. When $\gcd(n, 3) = 3$

Using Theorem 3.6, there are n edges with vertices on both ends having degree 2 when $\gcd(n, 3) = 3$. Thus $R_\alpha(EC(D_n)) = n \times \{2 \times 2\}^\alpha = 4^\alpha n$ when

$gcd(n, 3) = 3$.

Case 2. When $gcd(n, 3) = 1$

Using Theorem 3.6, $EC(D_n)$ is the empty graph when $gcd(n, 3) = 1$. So $E(EC(D_n))$ is the empty set. Thus $R_{\alpha}(EC(D_n)) = 0$ when $gcd(n, 3) = 1$.

(ii) $R_{-\frac{1}{2}}(EC(D_n)) = \sum_{u_1 u_2 \in E(EC(D_n))} \frac{1}{\sqrt{deg(u_1)deg(u_2)}}$ is the Randic index [21] of $EC(D_n)$.

Case 1. When $gcd(n, 3) = 3$

$R_{-\frac{1}{2}}(EC(D_n)) = 4^{-\frac{1}{2}}n = \frac{n}{2}$.

Case 2. When $gcd(n, 3) = 1$

Using Theorem 3.6, $EC(D_n)$ is the empty graph when $gcd(n, 3) = 1$. So $E(EC(D_n))$ is the empty set. Thus $R_{-\frac{1}{2}}(\Gamma_{cpg}(D_n)) = 0$. \square

Theorem 4.4. Let D_n be the dihedral group of order $2n$. Then the atomic-bond connectivity index, $ABC(EC(D_n)) = \begin{cases} \frac{n}{\sqrt{2}} & \text{when } gcd(3, n) = 3, \\ 0 & \text{when } gcd(3, n) = 1. \end{cases}$

Proof. Let D_n be the $2n$ order dihedral group. Atomic-bond connectivity index [22] of $EC(D_n)$ is

$$ABC(EC(D_n)) = \sum_{u_1 u_2 \in E(EC(D_n))} \sqrt{\frac{deg(u_1) + deg(u_2) - 2}{deg(u_1)deg(u_2)}}.$$

Case 1. When $gcd(n, 3) = 3$

Using Theorem 3.6, there are n edges with vertices on both ends having degree 2 when $gcd(n, 3) = 3$. Thus $ABC(EC(D_n)) = n \times \sqrt{\frac{2+2-2}{2 \times 2}} = \frac{n}{\sqrt{2}}$ when $gcd(n, 3) = 3$.

Case 2. When $gcd(n, 3) = 1$

Using Theorem 3.6, $EC(D_n)$ is the empty graph when $gcd(n, 3) = 1$. So $ABC(EC(D_n)) = 0$ when $gcd(n, 3) = 1$. \square

Theorem 4.5. Let D_n be the dihedral group of order $2n$. Then the geometric-arithmetic index, $GA(EC(D_n)) = \begin{cases} n & \text{when } gcd(3, n) = 3, \\ 0 & \text{when } gcd(3, n) = 1. \end{cases}$

Proof. Let D_n be the $2n$ order dihedral group.

$GA(EC(D_n)) = \sum_{u_1 u_2 \in E(EC(D_n))} \frac{2\sqrt{deg(u_1) \times deg(u_2)}}{deg(u_1) + deg(u_2)}$ is the Geometric-Arithmetic index [23] of $EC(D_n)$.

Case 1. When $gcd(n, 3) = 3$

Using Theorem 3.6, there are n edges with vertices on both ends having degree 2 when $gcd(n, 3) = 3$. Thus $GA(EC(D_n)) = n \times \frac{2\sqrt{2 \times 2}}{2+2} = n$ when $gcd(n, 3) = 3$.

Case 2. When $gcd(n, 3) = 1$

Using Theorem 3.6, $EC(D_n)$ is the empty graph when $gcd(n, 3) = 1$. So $GA(EC(D_n)) = 0$ when $gcd(n, 3) = 1$. \square

CONCLUSION

In this research structural representation of the equal-cubic graph of a finite group is given. Also structural properties such as girth, independent

number, clique number, chromatic number, chromatic polynomial, matching number, weakly perfectness, Laplacian polynomial, domination number of equal-cubic graph of a finite group are calculated. Various topological indices of the equal-cubic graph of a dihedral group are also obtained.

DECLARATION

This research work is part of the thesis submitted by the first author (Shilpa Aggarwal) to Baba Mastnath University for the award of Doctor of Philosophy.

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DEPARTMENT OF MATHEMATICS, BABA MASTNATH UNIVERSITY, ASTHAL BOHAR, ROHTAK-124021(HARYANA)
INDIA
Email address: shipsaggarwal107@gmail.com

DEPARTMENT OF MATHEMATICS, BABA MASTNATH UNIVERSITY, ASTHAL BOHAR, ROHTAK-124021(HARYANA)
INDIA
Email address: pankajrana2034@gmail.com

DEPARTMENT OF MATHEMATICS, Pt. NRS GOVT. COLLEGE, ROHTAK-124001(HARYANA) INDIA
Email address: amit_sehgal_iit@yahoo.com

DEPARTMENT OF MATHEMATICS, BABA MASTNATH UNIVERSITY, ASTHAL BOHAR, ROHTAK-124021(HARYANA)
INDIA
Email address: poojabudhiraja@bmu.ac.in