Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 2, pp. 175 – 179

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.2.002

COMMUTATIVITY OF DMAP SOLVABLE TOPOLOGICAL GROUPS WHOSE CLOSURES OF IMAGES OF REPRESENTATIONS ARE CONNECTED

A. I. Shtern

February 16, 2026

Abstract. By a DMAP topological group we mean a topological group which has sufficiently many (not necessarily continuous) finite-dimensional unitary representations or, equivalently, sufficiently many (not necessarily continuous) finite-dimensional irreducible unitary representations. In the paper, improving our earlier result, we prove that a solvable DMAP topological group is commutative, and some consequences of this fact are indicated.

§ 1. Introduction

In 1934, for any given topological group G, von Neumann [1] introduced a closed normal subgroup (and even a characteristic subgroup) of G formed by all elements $g \in G$ such that $\varphi(g) = 1$ for every almost periodic function φ on G. This subset obviously coincides with the intersection of the kernels of all finite-dimensional continuous complex unitary representations of G (or, equivalently, with the intersection of the kernels of all irreducible finite-dimensional continuous complex unitary representations of G).

2010 *Mathematics Subject Classification.* Primary 22D12, Secondary 22E15, 22D99.

Key words and phrases. Divisible group, locally compact group, MAP (maximal almost periodic) topological group, DMAP topological group, solvable topological group.

The research was supported by NRC "Kurchatov Institute" - SRISA according to the project FNEF-2024-0001.

A topological group G is said to be *maximally almost periodic* (MAP) if this subgroup is the identity subgroup, which is equivalent to the condition that the irreducible finite-dimensional continuous complex unitary representations separate the points of G.

Using a 1936 result of Freudenthal (who described the connected topological MAP groups with countable open basis), Weil [2] proved that a connected locally compact group G is MAP if and only if it is a direct product of a vector group and a compact group. The proof works, in particular, because a solvable compact connected topological group is commutative.

The last fact, which is one of the most frequently used tools in the study of structure phenomena involving compact topological groups, plays an important role, for example, in "a handbook for the expert" [3]; it was multiply used by the author of the present paper in a series of results dealing with not necessarily continuous finite-dimensional unitary representations of connected locally compact groups [4]–[12]; this series heavily uses some automatic continuity results for finite-dimensional locally bounded linear representations of Lie groups ([4]–[6]).

For the "discontinuous" version of the Freudenthal–Weil theorem, see [10].

The author of the present paper somewhat extended the class of groups for which the solvability implies the commutativity.

Theorem 1 [13]**.** *A connected MAP solvable topological group is commutative.*

By a DMAP topological group we mean a topological group which has sufficiently many (not necessarily continuous) finite-dimensional representations, i.e., a topological group which is an MAP group when equipped with the discrete topology. Certainly, a MAP group is a DMAP group, and examples of commutative topological groups without nonidentity continuous characters show that the converse assertion fails to hold.

In the paper [14], the following assertion is proved.

Theorem 2. *A solvable DMAP topological group having a chief series all of whose Abelian factors are divisible is commutative.*

In the present paper, we replace the divisibility assumption for these factors by another condition, and thus prove the following theorem.

Theorem 3. *A solvable DMAP topological group having sufficiently many finite-dimensional (not necessarily continuous) irreducible unitary represen-*

tations such that the closures of their images in these representations are connected is commutative.

The condition of Theorem 2 ensures the validity of the condition of Theorem 3, which is clear from the proof of Theorem 2 in [14]. Thus, Theorem 3 is a generalization of Theorem 2.

§ 2. Proof of Theorem 3

Let G be a solvable DMAP topological group.

Since G is a DMAP group, there are sufficiently many finite-dimensional (not necessarily continuous) irreducible unitary representations π of G (where the term "sufficiently many" means, as usual, that the family of representations separates the points of G).

The image of G under a representation π is a solvable group. Therefore, the closure G_π of this image is a compact solvable Lie group. By the assumption of Theorem 3, this closure is also connected. By the generalization of the famous Lie theorem on finite-dimensional complex (not necessarily continuous) irreducible representations of a connected Lie group (see [15]), π is one-dimensional.

Since π is an arbitrary finite-dimensional (not necessarily continuous) irreducible unitary representation of G in a family separating the points of the group, it follows that the images of any two elements of the group commute.

By the assumption concerning the family of representations π, these representations separate the points of the group G. Hence, any two elements of the group G commute.

This completes the proof of the theorem. $\square$

§ 3. Concluding remarks

It should be noted that some connectivity or divisibility condition is quite necessary in any theorem of the above kind. Indeed, the symmetric group S_3 of three elements is finite (and thus compact), solvable, and noncommutative.

For an example of a pseudocompact group which is connected but not divisible, see [16].

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this note in the Proceedings of the Jangjeon Mathematical Society.

References

1. J. von Neumann, *Almost periodic functions in a group, I*, Trans. Amer. Math. Soc. **36** (1934), 445–492.
2. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, 2nd ed., Act. Sc. Ind. No. 1145, Hermann, Paris, 1953.
3. K.-H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, de Gruyter, Berlin, 1998.
4. A. I. Shtern, *Van der Waerden's continuity theorem for the commutator subgroups of connected Lie groups and Mishchenko's conjecture*, Adv. Stud. Contemp. Math. (Kyungshang) **13** (2006), no. 2, 143–158.
5. A. I. Shtern, *A version of the van der Waerden theorem and the proof of the Mishchenko conjecture for homomorphisms of locally compact groups*, Izv. Ross. Akad. Nauk Ser. Mat. **72** (2008), no. 1, 183–224; Izv. Math. **72** (2008), no. 1, 169–205.
6. A. I. Shtern, *Applications of automatic continuity results to analogs of the Freudenthal–Weil and Hochschild theorems*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 2, 203–212.
7. A. I. Shtern, *Von Neumann kernel of a connected locally compact group, revisited*, Adv. Stud. Contemp. Math. (Kyungshang) **21** (2010), no. 1, 313–318.
8. A. I. Shtern, *Homomorphic images of connected Lie groups in compact groups*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 1, 1–6.
9. A. I. Shtern, *Connected Lie groups having faithful locally bounded (not necessarily continuous) finite-dimensional representations*, Russ. J. Math. Phys. **16** (2009), no. 4, 566–567.
10. A. I. Shtern, *Freudenthal–Weil theorem for arbitrary embeddings of connected Lie groups in compact groups*, Adv. Stud. Contemp. Math. (Kyungshang) **19** (2009), no. 2, 157–164.
11. A. I. Shtern, *Von Neumann kernels of connected Lie groups, revisited and refined*, Russ. J. Math. Phys. **17** (2010), no. 2, 262–266.
12. A. I. Shtern, *The structure of homomorphisms of connected locally compact groups into compact groups*, Izv. Ross. Akad. Nauk Ser. Mat. **75** (2011), no. 6, 195–222; Izv. Math. **75** (2011), no. 6, 1281–1306.
13. A. I. Shtern, *Commutativity conditions for maximally almost periodic solvable topological groups*, Proc. Jangjeon Math. Soc. **14** (2011), no. 1, 1–6.
14. A. I. Shtern, *Commutativity conditions for DMAP solvable topological groups*, Proc. Jangjeon Math. Soc. **15** (2012), no. 2, 109–113.
15. A. I. Shtern, *Finite-dimensional Quasirepresentations of connected Lie groups and*

Mishchenko conjecture, J. Math. Sci. (NY) **159** (2009), no. 5, 653–751.

16. W. W. Comfort and J. van Mill, *Concerning connected, pseudocompact abelian groups*, Topology Appl. **33** (1989), no. 1, 21–45.

Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
NRC <Kurchatov Institute> - SRISA,
Moscow, 117312 Russia, and
Moscow Centre for fundamental and applied mathematics
Moscow, 119899 Russia
E-mail: aishtern@mtu-net.ru