

MAPPINGS BETWEEN SCHRÖDINGER EQUATIONS

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ABSTRACT. We use a theorem of Bluman–Shtelen to show mappings between the Schrödinger equations for a free particle, a linear potential, and the harmonic oscillator. Besides, we exhibit a simple procedure to deduce the Langer and Bateman *et al.*–Mavromatis transformations in order to map the Coulomb problem into the Morse and simple harmonic oscillators, respectively.

1. Introduction

The Schrödinger equation occupies a central position in quantum mechanics, serving as the fundamental dynamical law governing the evolution of microscopic physical systems. Proposed by Erwin Schrödinger in 1926, this equation provides a wave-theoretic formulation of quantum phenomena and establishes a rigorous mathematical framework for describing the behavior of particles at atomic and subatomic scales. In analogy with Newton’s equations in classical mechanics, the Schrödinger equation determines how the state of a quantum system evolves in time (see[10, 12, 3]).

At the heart of the theory lies the wave function $\psi(x, t)$, a complex-valued function whose modulus squared, $|\psi(x, t)|^2$, represents the probability density of finding a particle at position x at time t . This probabilistic interpretation, formalized by Born, marked a fundamental departure from deterministic classical mechanics and introduced intrinsic uncertainty into the physical description of nature [10, 12, 3]).

Beyond its foundational role in physics, the Schrödinger equation has profound mathematical significance. It has stimulated extensive research in spectral theory, partial differential equations, operator algebras, and semigroup theory. Moreover, generalized forms of the equation arise in diverse areas such as quantum field theory, nonlinear optics, Bose–Einstein condensation, and financial mathematics. Nonlinear variants, including the nonlinear Schrödinger equation, further extend its applicability to complex systems exhibiting wave modulation and soliton behavior [10, 12, 3]).

Thus, the Schrödinger equation stands as a cornerstone of modern science, bridging physical intuition and mathematical structure, and continues to inspire both theoretical developments and practical applications across multiple disciplines.

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The central object in quantum mechanics is the wave function $\psi(x, t)$, a complex-valued function defined on configuration space and time. For a single nonrelativistic particle of mass m moving under the influence of a potential $V(x, t)$, the time-dependent Schrödinger equation is given by (with $i = \sqrt{-1}$)

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x, t)\psi, \quad (1.1)$$

where \hbar denotes the reduced Planck constant and ∇^2 is the Laplacian operator [10, 12, 3]. The linearity of the equation encodes the principle of superposition.

From a mathematical standpoint, the Schrödinger equation is a linear partial differential equation whose associated operator

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(x, t), \quad (1.2)$$

is known as the Hamiltonian operator. The rigorous formulation of quantum dynamics in terms of self-adjoint operators on Hilbert spaces was established by von Neumann, providing a precise mathematical foundation for quantum theory. When the potential is time-independent, separation of variables leads to the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(x)\psi = E\psi, \quad (1.3)$$

which constitutes an eigenvalue problem for the Hamiltonian. The discrete spectrum of this operator corresponds to quantized energy levels, while the associated eigenfunctions describe stationary quantum states [10, 12, 3]. This spectral structure successfully explains atomic line spectra and the stability of bound systems.

Here, we consider the Schrödinger equation for a potential with a certain structure

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + [\alpha(t)x^2 + \beta(t)x + \gamma(t)] \psi = i\hbar \frac{\partial \psi}{\partial t}, \quad (1.4)$$

such that

$$\alpha = \frac{m}{2\sigma^2} (\sigma \ddot{\sigma} - 2\dot{\sigma}^2), \quad \beta = \frac{m}{\sigma^2} (\sigma \dot{\rho} - 2\dot{\sigma} \rho), \quad \gamma = \hbar \dot{\lambda} - \frac{i\hbar}{2\sigma} \dot{\sigma} - \frac{m}{2\sigma^2} \rho^2, \quad (1.5)$$

where $\sigma(t)$, $\rho(t)$, and $\lambda(t)$ are arbitrary functions with first and second derivatives.

Then the following mapping [6]

$$\tilde{x} = \sigma x + \rho, \quad \tilde{t} = \int^t \sigma^2(u) du, \quad \tilde{\psi} = e^{ig(x,t)} \psi, \quad (1.6)$$

with

$$g = \frac{m}{2\hbar\sigma} \dot{\sigma} x^2 + \frac{m}{\hbar\sigma} \dot{\rho} x + \lambda, \quad (1.7)$$

transforms Eq. (1.4) into the Schrödinger equation for a free particle

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}^2} = i\hbar \frac{\partial \tilde{\psi}}{\partial \tilde{t}}. \quad (1.8)$$

In Sec. 2 we show that an adequate selection of σ , ρ , and λ allows us to exhibit that the Schrödinger equations for the harmonic oscillator and the linear potential are equivalent to Eq. (1.8).

The Schrödinger equation for the Coulomb potential is given by [10, 12, 3])

$$\frac{1}{2} \left(\frac{d^2 R}{dr^2} - \frac{l(l+1)}{r^2} R \right) + \frac{\tau}{r} R = \eta R, \quad l = 0, \dots, n-1, \quad (1.9)$$

where

$$\tau = \frac{Zq^2}{4\pi\epsilon_0}, \quad \eta = \frac{Z^2 q^4}{32\pi^2 \epsilon_0^2 n^2}, \quad n \geq 1. \quad (1.10)$$

Here we deduce how Eq. (1.9) changes under the mapping

$$R = A \xi(u) \psi(u), \quad r = B f(u), \quad (1.11)$$

where A and B are constants. The independent variable r and the Coulomb radial wave function R are transformed in order to obtain a new Schrödinger-like equation for ψ in the variable u . Hence, the aim is to find the functions f and ξ that lead to another potential of physical interest associated with ψ .

In Sec. 3 we show that our approach gives, in natural manner, the Langer transformation [14], which allows to relate [15] the Coulomb potential to the Morse interaction [10, 12, 13] describing the vibrational motion of a diatomic molecule. Moreover, we also deduce the Bateman *et al.* [2]–Mavromatis [7] mapping, which has been investigated in the context of connecting the Coulomb problem with oscillator systems.

2. Transformations between Schrödinger equations

Here we apply the mapping (1.7) to two important cases in quantum mechanics.

(a). Let

$$\sigma = \sec(\omega t), \quad \rho = 0, \quad \lambda = -\frac{i}{2} \ln(\cos(\omega t)). \quad (2.1)$$

Then Eq. (1.5) implies

$$\alpha = \frac{m}{2} \omega^2, \quad \beta = 0, \quad \gamma = 0, \quad (2.2)$$

and hence Eq. (1.4) becomes the Schrödinger equation for the harmonic oscillator [10, 12, 3])

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{m}{2} \omega^2 x^2 \psi = i\hbar \frac{\partial \psi}{\partial t}. \quad (2.3)$$

The corresponding transformation (1.7)

$$\tilde{x} = x \sec(\omega t), \quad \tilde{t} = \frac{1}{\omega} \tan(\omega t), \quad \tilde{\psi} = \sqrt{\cos(\omega t)} \exp\left(\frac{im}{2\hbar} x^2 \tan(\omega t)\right) \psi. \quad (2.4)$$

This mapping transforms Eq. (2.3) into the free-particle Schrödinger equation (1.8).

The quantum harmonic oscillator (HO) is one of the basic cases studied in books on quantum mechanics and modern physics. The Schrödinger equation for HO can be solved

exactly by several methods. Here our aim was to add another process to the ones already known. We showed that, essentially, by means of a coordinate mapping mixing space and time variables, it is possible to relate the solutions of the Schrödinger equation for a free-particle with the solution of the Schrödinger equation for a HO.

(b). Let

$$\sigma = 1, \quad \rho = \frac{F}{2m}t^2, \quad \lambda = \frac{F^2}{6m\hbar}t^3. \quad (2.5)$$

where F is a constant. Equation (1.5) then yields

$$\alpha = 0, \quad \beta = F, \quad \gamma = 0, \quad (2.6)$$

and Eq. (1.4) reduces to the Schrödinger equation for a linear potential [10, 12, 3]

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + Fx\psi = i\hbar \frac{\partial \psi}{\partial t}. \quad (2.7)$$

The associated mapping (1.7) becomes

$$\tilde{x} = x + \frac{F}{2m}t^2, \quad \tilde{t} = t, \quad \tilde{\psi} = \exp\left[\frac{iF}{\hbar}t\left(x + \frac{F}{6m}t^2\right)\right] \psi. \quad (2.8)$$

This transformation maps Eq. (2.7) into the free-particle Schrödinger equation (1.8).

The mappings (2.4) and (2.8) were obtained in Refs. [5, 5, 1] through the Studying of the corresponding classical systems.

3. Coulomb potential

We apply the transformation (1.11) to Eq. (1.9) to obtain

$$\frac{1}{2} \left[\psi'' + \left(2\frac{\xi'}{\xi} - \frac{f''}{f'} \right) \psi' + \left(\frac{\xi''}{\xi} - \frac{f''}{f'} \frac{\xi'}{\xi} - l(l+1) \frac{(f')^2}{f^2} \right) \psi \right] + \tau B \frac{(f')^2}{f} \psi = \eta B^2 (f')^2 \psi, \quad (3.1)$$

where $\psi' = d\psi/du$.

Equation (3.1) may be a Schrödinger-like structure if the coefficient of the first-derivative term ψ' vanishes. This requirement leads to

$$\frac{f''}{f'} = 2 \frac{\xi'}{\xi}, \quad \Rightarrow \quad f' = c\xi^2, \quad c = \text{constant}. \quad (3.2)$$

Under this condition, Eq. (3.1) becomes

$$\frac{1}{2} \left[\psi'' + \left(\frac{\xi''}{\xi} - 2\frac{(\xi')^2}{\xi^2} - c^2 l(l+1) \frac{\xi^4}{f^2} \right) \psi \right] + c^2 \tau B \frac{\xi^4}{f} \psi = c^2 \eta B^2 \xi^4 \psi. \quad (3.3)$$

Equation (3.3) offers several possibilities. For example, into it we can make the coefficient of $-c^2 l(l+1)\psi$ equal to one, that is:

$$f = \xi^2. \quad (3.4)$$

Then Eq. (3.2) yields the Langer transformation [14]

$$f = e^{cu}, \quad \xi = e^{\frac{c}{2}u}. \quad (3.5)$$

With these Equations, Eq. (3.3) takes the form

$$-\frac{1}{2}\psi'' + \frac{\tau B c^2}{2} \left(\frac{2\eta B}{\tau} e^{2cu} - 2e^{cu} \right) \psi = -\frac{c^2}{2} \left(l + \frac{1}{2} \right)^2 \psi. \quad (3.6)$$

Which is attractive because permits to introduce the Morse potential [10, 12, 13].if we select the values $c = -a$ and $2\eta B/\tau = 1$,with notation

$$D = \frac{\tau B c^2}{2}, \quad (3.7)$$

we obtain from Eq. (1.10)

$$B = \frac{4\pi\varepsilon_0}{Zq^2} n^2, \quad D = \frac{a^2}{2} n^2, \quad a > 0. \quad (3.8)$$

Therefore, Eq. (3.6) becomes the vibrational Schrödinger equation for the Morse interaction

$$-\frac{1}{2}\psi'' + D \left(e^{-2au} - 2e^{-au} \right) \psi = E\psi, \quad E = -\frac{a^2}{2} \left(l + \frac{1}{2} \right)^2. \quad (3.9)$$

herea is the range parameter(associated with the width of the potential well), and D is the energy of dissociation (well-depth) [9, 11, 8].

Finally, from Eqs. (1.11), (3.5), and (3.8), we obtain [15]

$$R = A e^{-\frac{a}{2}u} \psi(u), \quad r = \frac{4\pi\varepsilon_0}{Zq^2} n^2 e^{-au}, \quad (3.10)$$

where normalization we can employ the value

$$A^2 = \frac{Zq^2}{4\pi\varepsilon_0 n \left(l + \frac{1}{2} \right)}. \quad (3.11)$$

Now we back to Eq. (3.3) and seeking an alternative to Eq. (3.4). For example,to make the coefficient of the term $c^2\tau B\psi$ equals to one, that is:

$$f = \xi^4. \quad (3.12)$$

Then Eq. (3.2) gives the Bateman *et al.* [2]–Mavromatis [7] transformation

$$f = \frac{c^2}{4} u^2, \quad \xi = \sqrt{\frac{c}{2}} u. \quad (3.13)$$

Under this transformation, Eq. (3.3) becomes

$$-\frac{1}{2} \left[\psi'' - \frac{(4l+1)(4l+3)}{4u^2} \psi \right] + \frac{c^4 \eta B^2}{4} u^2 \psi = c^2 \tau B \psi. \quad (3.14)$$

being achieved the mapping of the hydrogen-like atom into the 3-dimensional simple harmonic oscillator with certain parameters.

It is clear that Eq. (3.3)we can try different connections between f and ξ , respecting the constraint (3.2), to deduce Schrödinger-like equations associated to several potentials.

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