Advanced Studies in Contemporary Mathematics
36 (2026), No. 2, pp. 143 - 155
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2026.36.2.008

Graph-Theoretic Analysis of the Commuting Graph of the Semi-Dihedral Group via Degree and Distance-Based Indices

Naveen S. Sapare[1*], Sudhir R. Jog[2], V.T. Hosamath[3], V.G. Hiremath[4],
[1,3,4]Dept of Engineering Mathematics,
KLE Technological University Hubballi-580031, Karnataka, India
[2]Dept of Engineering Mathematics,
KLS Gogte Institute of Technology, Belagavi, Karnataka, India
[1,2,3,4]Visvesvaraya Technological University, Belagavi - 590018, Karnataka, India

Abstract:

The mathematical description of molecular structures is useful in the explanation of their physicochemical and biological behaviour. Algebraic graph theory provides a useful model of describing such structures by relating algebraic concepts with graphical representations. This paper explores commuting graphs of semi-dihedral groups with an aim of examining their structural and topological characteristics. This is done by giving a closer scrutiny of few topological indices with degree-distance based such as the Zagreb indices, modified Zagreb indices, the refined Zagreb index, the forgotten index, the degree-distance index, the Gutman index, the first Zagreb degree-distance index, the generated degree-distance index. These descriptors are obtained systematically to obtain the underlying connectivity and distance-based properties of the graphs. The received results give a good insight into the structural behaviour of semi-dihedral groups and prove their possible helpfulness in chemical graph theory and other mathematical modelling usage.

Keywords: Commuting graph, Semi-dihedral group, Topological indices, Degree and distance based indices.

1. Introduction

Graph theory is emerging as a valuable instrument of the study of chemical and algebraic systems. The correlation of molecular composition and physical behaviour has always been known to be significant in the study of chemistry and molecular science. Molecular structures can be best examined using topological indices. They are numbers that reveal significant structural data of graphs without having to understand everything about the system. The physical, chemical and biological characteristics of a chemical composition are directly influenced by the algebraic value of the composition. This is how mathematical analysis of such buildings is necessary, both in theory and practice [1].

Commuting graph studies in finite non-abelian groups has now become a leading subject of research at the intersection of abstract algebra and graph theory. Commuting graphs graphically and algebraically indicate the relationship between elements in a group, by stating those elements that commute with others. Topological indices of commuting graphs have been recently studied in depth in groups of finite non-abelian groups, establishing necessary techniques and methods of future exploration of this direction. Such studies have demonstrated that commuting graphs possess much algebraic information that may be quantified with various topological graph indices [2].

This has been extended by researchers who have studied relative g-noncommutating graphs and other commuting structures of dihedral groups. The topological indices of these special graphs have been calculated and analysed, to gain more knowledge concerning the structure of the underlying group [4]. Besides, numerous studies have been reported on non-commuting graphs of dihedral groups and in this case, various topological indices have been systematically computed and compared [5]. These investigations have produced major information on how various graph structures obtained on the same algebraic object could possess distinct topological properties.

This has been extended to include more important group families including the generalized quaternion groups. Complete studies of the topological indices of non-commuting graphs in generalised quaternion groups have revealed some unique patterns and connections relevant to this group of groups [6]. Simultaneously, studies of power graphs on dihedral groups have also helped us to understand connectivity indices and their algebraic significance [7]. Moreover, the topological indices have been used to explore the Structural Features of cubic power graphs of dihedral groups, and this analysis has produced better insights into the relationship between algebraic parameters and the graph-theoretic measures [8].

Topological indices have been extended to analysing order divisor graphs of cyclic groups and its properties [9]. The wider studies of various graph topologies themselves which are of algebraic objects have revealed that topological indices are practical in the description and distinction of various algebraic families. This has been particularly assisted by the degree based topological indices, when applied to strong double graphs and other complex graph structures [10]. Those mathematical concepts have also been demonstrated to be applicable in practice in the development and application of reverse degree-based topological indices to molecular structures [11].

Topics in higher graph theory Topological graph theory has developed further, and product graphs have been studied in terms of neighbourhood Zagreb indices [12]. These developments indicate how the topological index theory has become more complicated and applicable in many areas. Meanwhile, the study of the commuting graph has been the subject of theoretical study that has taught us more of the fundamental properties of the graph, as well as how it relates to the linear algebra and the combinatorics [13]. The literature has provided vast material on the topic of molecular descriptors of cheminformatics which has illuminated the bigger picture of these indices being functional tools [14].

Authors have studied centrality of subgraphs and their relevance to explain clustering phenomena within complex networks on top of traditional indexes [15]. Refinements have been made to the first Zagreb index, which is a popular topological measure, and made more potent [16]. Graph indices based on character-based graphs, such as indices based on molecular branching patterns, have also provided additional information on the structure of graphs [17]. The recent comprehensive research has not only broadened the theoretical framework of classical Zagreb indices but also of a broader range of related indices [18].

The historic work on the structure determination using graph-theoretic techniques, which dates to classical cheminformatics [19], has evolved into an established field of research boasting a collection of state-of-the-art indices and analysis devices. Most recent works have also built on these concepts in semigroup structures, such as the symmetric inverse semigroup, in which commuting graphs and their properties still bring forth new information [20].

Considering the vast amount of literature on the topic of commuting graphs, topological indices and their use in algebraic structures, the paper will be a comprehensive analysis of the Graphs defined by commutativity in semi-dihedral groups. Semi-dihedral groups form an important subclass of the more general classification of 2-groups and its commuting graph properties have not been fully recorded. The research aims to fill this gap by computing and analysing carefully different topological graph indices, such as the First and Second Zagreb index, its Second modified and variations, the Refined modified Zagreb index, the Forgotten index, degree-distance based indices, and generalizations of degree-distance based indices to commuting graphs of semi-dihedral groups. By doing this we are contributing to the body of knowledge that has been growing over the years, the one that relates algebraic structure to graph-theoretic properties.

2. Preliminaries

This section gives a short summary of the main ideas and important findings in graph theory. As we get deeper into the subject in the rest of the post, it will be a helpful reference point. The graphs examined in this study are basic graphs, namely unweighted simple graphs devoid of loops or numerous edges. Let u and v be two points in a graph γ. d_u is the number of edges which are adjacent to u. This is called its degree. And nh(u) represents the neighbourhood of a vertex u. The $d_{ve}(u)$ is the total of all the different edges that connect every vertex in nh(u). A full graph K_n is one in which every pair of distinct vertices is connected by an edge that is unique from all others. dis (u, v) is the distance between u and v along the shortest path.

Topological Indices	Symbol	Formula
First Zagreb index [21]	$M_1(\gamma)$	$\sum_{u,v\in G}(d_u + d_v)$
Second Zagreb index [21]	$M_2(\gamma)$	$\sum_{u,v\in G}(d_u . d_v)$
Second modified Zagreb index [22]	$mM_2(\gamma)$	$\sum_{u,v\in G}\left(\frac{1}{d_u . d_v}\right)$
Refined modified Zagreb index [22]	$RM_3(\gamma)$	$\sum_{u,v\in G}(d_u - 1)(d_v - 1)$
Forgotten index [23]	$F(\gamma)$	$\sum_{u,v\in G}(d_v)^3$
Degree-distance index [24]	$DD(\gamma)$	$\sum_{u,v\in G}(d_u + d_v). d(u,v)$
Gutman Degree-Distance index [25]	$DD_G(\gamma)$	$\sum_{u,v\in G}(d_u . d_v) . d(u,v)$
First Zagreb Degree-distance index [26]	$ZDD_1(\gamma)$	$\sum_{u,v\in G}[(d_u)^3 + (d_v)^3]. dis(u,v)$
Generalised Degree-distance index [26]	$DD_\alpha(\gamma)$	$\sum_{u,v\in G}(d_u . d_v)^{\alpha}. dis(u,v)$

Table-1: List of Topological Indices

2.1 Structural Properties of $\gamma(SD_{8p})$

Definition 2.1.1: Consider group G and let β be a nonempty subset of G. The commuting graph of G with respect to β, denoted by γ (G, β), is the graph whose vertex set is β. Two different vertices x_1, x_2 ∈ β are said to be adjacent if and only if $a_1b_2 = a_2a_1$ in G. For $p \geq 2$, a semi-dihedral group with cardinality $8p$ is defined by the presentation $SD_{8p} = \langle u, v \mid u^{4p} = e, v^2 = e, vuv = u^{2p-1}\rangle$.

If p is odd, then the center of SD_{8p} is $Z(SD_{8p}) = \{e, u^p, u^{2p}, u^{3p}\}$. In this case, SD_{8p} can be decomposed as $\varphi_1 = \{e, u, u^2, \ldots, u^{4p-1}\}$ and $\varphi_2 = \{v, vu, vu^2, \ldots, vu^{4p-1}\} = \bigcup_{i=0}^{p-1} {\varphi_2}^i$, where ${\varphi_2}^i = \{vu^i, vu^{p+i}, vu^{2p+i}, vu^{3p+i}\}$, and $\varphi_3 = \varphi_1 \setminus Z(SD_{8p})$.

If p is even, then $Z(SD_{8p}) = \{e, u^2p\}$. In this case, define $\psi_1 = \{e, u, u^2, \ldots, u^{4p-1}\}$ and $\psi_2 = \{v, uv, u^2v, \ldots, u^{4p-1}v\} = \bigcup_{i=0}^{2p-1} {\psi_2}^i$, where ${\psi_2}^i = \{vu^i, vu^{2p+i}\}$, and $\psi_3 = \psi_1 \setminus Z(SD_{8p})$.

Lemma 2.1.2[27]: The Graph structure of the commuting elements of the semi-dihedral group SD_{8p} is given by.
$\gamma(SD_{8p}) = K_4 \vee (pK_4 \cup K_{4p-4})$, if p is odd,
$\gamma(SD_{8p}) = K_2 \vee (2pK_2 \cup K_{4p-2})$, if p is even.

In the commuting graph $\gamma(SD_{8m})$, The vertex set is broken up into three groups: the centre $Z(SD_{8p})$ and two non-central groups (φ_2, $\varphi_3$when p is odd, or ψ_2, ψ_3 when p is even). And they are listed on section 2.1 and Table-2. Every vertex in the centre is next to every other vertex in the graph. This means that the distance between any centre vertex and any non-central vertex is one. Vertices in the same full subgraph, such K_4, K_2, or K_{4m-4}, are also one unit apart from each other. But vertices that are in distinct non-central partitions are not next to each other. The shortest path between them always goes through a centre vertex, which means they are two units apart. So, the commuting graph's diameter is two, and all pairs of vertices are either one or two units apart.

Topological indices are important factors for studying the structural behaviour of networks. and find various applications in physical chemistry. Numerous indices have been studied for different algebraic graphs, as indicated in the present research [28, 29,30]. In this paper, the concept is extended to address selected topological properties of commuting graphs corresponding to finite semi-dihedral groups.

Preposition 2.1.3.: All conceivable distances of $\gamma(SD_{8p})$ is given by dis (SD_{8p}, k) = $\begin{cases} 4p(2p+5), & if\ k=1, \\ 48p(p-1), & if\ k=2 \end{cases}$ When p is odd

dis (SD_{8p}, k) = $\begin{cases} 8p(p+1), & if\ k=1, \\ 8p(4p-2), & if\ k=2 \end{cases}$ When p is even

Preposition 2.1.4: The degree of a ∈ $\gamma(SD_{8p})$ is given by

$d_a = \begin{cases} 8p-1, & if\ a \in Z(SD_{8p}) \\ 7, & if\ a \in \varphi_2 \\ 4p-1 & if\ a \in \varphi_3 \end{cases}$ When p is odd

$d_a = \begin{cases} 8p-1, & if\ a \in Z(SD_{8p}) \\ 3, & if\ a \in \psi_2 \\ 4p-1, & if\ a \in \psi_3 \end{cases}$ When p is even

Table-2 provides a concise representation of the propositions and lemmas related to the vertex set and vertex degrees of the commuting graph $\gamma(SD_{8p})$.

Partition	Symbol		When p is Odd		When p is Even	
	p is Odd	p is even	Vertex count	Vertex degrees	Vertex count	Vertex degrees
Centre Vertices	$Z[SD_{8p}]$	$Z[SD_{8p}]$	4	8p-1	2	8p-1
Vertex part 2	Φ_2	Ψ_2	4p	7	4p	3
Vertex part 3	Φ_3	Ψ_3	4p-4	4m-1	4p-2	4p-1
Total Vertices		8p				

Table-2: Vertex Partition and degree of vertices

3. Evaluation of Topological Indices

In this section, we analyse topological aspects of commuting graphs of finite semi-dihedral groups. Based on the structural properties given in Section 2.1, the following topological indices are obtained.

Theorem 3.1: First Zagreb index $M_1(\gamma)$ of the graph of commuting elements of the semi-dihedral group is

$M_1(\gamma) = 64p^3 + 160p^2 + 168p$, When p is odd,
$M_1(\gamma) = 64p^3 + 64p^2 + 24p$, When p is Even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.4 and together with the $M_1(\gamma)$, we derive
Case 1: When p is odd

$M_1(\gamma)=6[8p-1+8p-1]+6p[7+7]+(4p-4)_{c_2}[4p-1+4p-1]+16p[8p-1+7]+16(p-1)[8p-1+4p-1]$
$= 6[16p-2+6p(14)+\frac{(4p-4)(4p-5)(8p-2)}{2}+16p(8p+6)+(16p-16)(12p-2)$
$= 96p-12+84p+\frac{(16p^2-20p-16p+20)(8p-2)}{2}+128p^2+96p+192p^2-32p-192p+32$
$=\frac{1}{2}[640p^2+104p+40+128p^3-32p^2-160p^2+40p-128p^2+32p+160p-40]$
$=\frac{1}{2}[128p^3+320p^2+336p]$
$M_1(\gamma)= 64p^3+160p^2+168p$

Case 2: When p is even

$M_1(\gamma)=[8p-1+8p-1]+2p(3+3)+(4p-2)_{c_2}(4p-1+4p-1)+8p(8p-1+3)+4(2p-1)(8p-1+4p-1)$

$= 16p-2+12p+\dfrac{(4p-2)(4p-3)(8p-2)}{2}+8p(8p+2)+(8p-4)(12p-2)$

$= 28p-2+\dfrac{(16p^2-20p+6)(8p-2)}{2}+64p^2+16p+96p^2-64p+8$

$= 160p^2-20p+6+64p^3-16p^2-80p^2+20p+24p-6$

$M1(\gamma)=64p^3+64p^2+24p$

Theorem 3.2: Second Zagreb index $M_2(\gamma)$ of the graph of commuting elements of the semi-dihedral group is

$M_2(\gamma)= 128p^4+160p^3+888p^2+196p$, When p is odd,
$M_2(\gamma)= 128p^4+32p^3+168p^2$, When p is Even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.4 and together with $M_2(\gamma)$, we derive
Case 1: When p is odd

$M_2(\gamma)=6(8p-1)^2+6p(7^2)+(4p-4)_{c_2}(4p-1)^2+16p[(8p-1)7]+16(p-1)[(8p-1)(4p-1)]$

$= 6(64p^2-16p+1)+294p+\frac{(4p-4)(4p-5)}{2}(16p^2-8p+1)+112p(8p-1)+(16p-16)(32p^2-12p+1)$

$= 384p^2+6-96p+294p+(8p^2-18p+10)(16p^2-8p+1)+896p^2-112p+512p^3-192p^2+16p-512p^2+192p-16$

$= 512p^3+576p^2+294p-10+128p^4+8p^2-64p^3-288p^2-18p+144p^2+160p^2+10-80p$

$M_2(\gamma)= 128p^4+160p^3+888p^2+196p$

Case 2: When p is even.
$M_2(\gamma)=(8p-1)^2+2p(3\times 3)+(4p-2)_{c_2}(4p-1)^2+8p[3(8p-1)]+4(2p-1)[(8p-1)(4p-1)$

$= 64p^2-16p+1+18p+\frac{(4p-2)(4p-3)(16p^2-8p+1)}{2}+192p^2-24p+(8p-4)(32p^2-12p+1)$

$= 256p^2-22p+1+(2p-1)(64p^3+4p-32p^2-48p^2-3+24p)+256p^3-96p^2+8p-128p^2+48p-4$

$= 256p^3+32p^2+34p-3+128p^4-160p^3+56p^2-6p-64p^3+80p^2-28p$

$M_2(\gamma)= 128p^4+32p^3+168p^2$

Theorem 3.3: Second modified Zagreb index $mM_2(\gamma)$ of the graph of commuting elements of the semi-dihedral group is

$mM_2(\gamma) = \frac{6144p^5+34816p^4-45344p^3+17720p^2-988p}{50176p^4-3763\quad{}^3+1019\quad{}^2-1176p+49}$, When p is odd,
$mM_2(\gamma)= = \frac{2048p^5+6144p^4-6112p^3+1704p^2-112p}{9216\quad{}^4-6912p^3+1872\quad{}^2-216p+9}$, When p is Even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.4 and together with $mM_2(\gamma)$, we derive
Case 1: When p is odd

mM$_2$(γ)= $\frac{6}{(8p-1)^2}+\frac{6p}{7\times7}+(4p-4)_{c_2}\frac{1}{(4p-1)^2}+\frac{16p}{7(8p-1)}+\frac{16(p-1)}{(8p-1)(4p-1)}$

$=\frac{6}{(8p-1)^2}+\frac{6p}{49}+\frac{(4p-4)(4p-5)}{(4p-1)^2}+\frac{16p}{7(8p-1)}+\frac{16(p-1)}{(8p-1)(4p-1)}$

$=\frac{6\times49(4p-1)^2+6p(8p-1)^2(4p-1)^2+(2p-2)(4p-5)49(8p-1)^2+16p\times7\times(8p-1)(4p-1)^2+16(p-1)49(8p-1)(4p-1)}{49(8p-1)^2(4p-1)^2}$

$=\frac{(4p-1)^2[294+6\ (64p^2-16p+1)+49(8p^2-18p+10)(64p^2-16p+1)+(8p-1)(4p-1)[112p(4p-1)+784p-784}{49(8p-1)^2(4p-1)^2}$

$=\frac{(16p^2-8p+1)(294+384p^3+6p-96p^2)+49(512p^4+8p^2-128p^3-1152p^3-18p+288p^2+640p^2+10-160p)+(32p^2-12p+\ )(448p^2+672p-784)}{[7(8p-1)(4p-1)]^2}$

$=\frac{6144p^5+34816p^4-45344p^3+17720p^2-988p}{(224p^2-84p+7)^2}$

mM$_2$(γ) $\frac{6144p^5+34816p^4-45344p^3+17720p^2-988p}{50176p^4-37632p^3+10192p^2-1176p+49}$

Case 2: When p is even.

mM$_2$(γ)= $\frac{1}{(8p-1)^2}+\frac{2p}{3\times3}+(4p-2)_{c_2}\frac{1}{(4p-1)^2}+\frac{8p}{3(8p-1)}+\frac{4(2p-1)}{(8p-1)(4p-1)}$

$=\frac{1}{(8p-1)^2}+\frac{2p}{9}+\frac{(2p-1)(4p-3)}{(4p-1)^2}+\frac{8p}{3(8p-1)}+\frac{4(2p-1)}{(8p-1)(4p-1)}$

$=\frac{9(4p-1)^2+2p(8p-1)^2(4p-1)^2+(2p-1)(4p-3)9(8p-1)^2+8p\times3\times(8p-1)(4p-1)^2+(8p-4)9(8p-1)(4p-1)}{9(8p-1)^2(4p-1)^2}$

$=\frac{(4p-1)^2[9+2p(64\ \ ^2-16p+1)+9(8p^2-10p+3)(64p^2-16p+1)+(8p-1)(4p-1)[24p(4p-1)+72p-36}{9(8p-1)^2(4p-1)^2}$

$\frac{(16p^2-8p+1)(9+128p^3+2p-32p^2)+9(512p^4+8p^2-128p^3-640p^3-10p+160p^2+192p^2+3-48p)+(32p^2-12p+1)(96p^2+48p-36)}{9(64p^2-16p+1)(16p^2-8p+1)}$

mM$_2$(γ)= $\frac{2048p^5+6144\ \ ^4-6112p^3+1704p^2-112p}{9216p^4-6912p^3+1872p^2-216p}$

Theorem 3.4: Second modified Zagreb index RM$_3$(γ) of graph of commuting elements of the semi-dihedral group is

RM$_3$(γ)= $128p^4+96p^3+736p^2+48p$, When p is odd,
RM$_3$(γ)= $128p^4-32p^3+112p^2-16p$, When p is Even

Proof: By employing the structural description of γ(SD_{8p}) given in Lemma 2.1.2 Proposition 2.1.4 and together with RM$_3$(γ) , we derive

Case 1: When p is odd

RM$_3$(γ)= $6[(8p-1-1)(8p-1-1)]+6p[(7-1)(7-1)]+(4p-4)_{c_2}[(4p-1-1)(4p-1-1)+16p[(8p-1-1)(7-1)]+16(p-1)[(8p-1-1)(4p-1-1)]$

= $6[(8p-2)(8p-2)]+6p(36)+\frac{(4p-4)(4p-5)}{2}\times(4p-2)(4p-2)+16p[(8p-2)(6)]+16(p-1)[(8p-2)(4p-2)]$

= $6(64p^2-32p+4)+216p+(8p^2-18p+10)(16p^2-16p+4)+16p(48p-12)+(16p-16)(32p^2-24p+4)$

= $384p^2-192p+24+216p+128p^4-128p^3+32p^2-288p^3+288p^2-72p+160p^2-160p+40+768p^2-192p+512p^3-384p^2+64p-512p^2+384p-64$

RM$_3$(γ)= $128p^4+96p^3+736p^2+48p$

Case 2: When p is even

RM$_3$(γ)= $[(8p-1-1)(8p-1-1)]+2p[(3-1)(3-1)]+(4p-2)_{c_2}[(4p-1-1)(4p-1-1)+8p[(8p-1-1)(3-1)]+4(2p-1)[(8p-1-1)(4p-1-1)]$

$= [(8p-2)(8p-2)] + 8p + \frac{(4p-2)(4p-3)}{2} \times (4p-2)(4p-2) + 16p(8p-2) +$
$(8p-4)[(8p-2)(4p-2)]$
$= 64p^2 - 32p + 4 + 8p + (8p^2 - 10p + 3)(16p^2 - 16p + 4) + 128p^2 - 32p + (8p-4)(32p^2 -$
$24p + 4)$
$= 192p^2 - 56p + 4 + 128p^4 - 128p^3 + 32p^2 - 160p^3 + 160p^2 - 40p + 48p^2 - 48p + 12 +$
$+256p^3 - 192p^2 + 32p - 128p^2 + 96p - 16$
$RM_3(\gamma) = 128p^4 - 32p^3 + 112p^2 - 16p$

Theorem 3.5: Forgotten index F(γ) of the graph of commuting elements of the semi-dihedral group is

$F(\gamma) = 768p^4 - 640p^3 + 264p^2 + 1319p + 4$, When p is odd,
$F(\gamma) = 768p^4 - 512p^3 + 168p^2 + 79p + 2$, When p is Even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.4 and together with F(γ), we derive
Case 1: When p is odd

$F(\gamma) = p(8p-1)^3 + 4p(7)^3 + (4p-4)(4p-1)^3$
$= p(512p^3 - 192p^2 + 24p - 1) + 1372p + (4p-4)(64p^3 - 48p^2 + 12p - 1)$
$= 512p^4 - 192p^3 + 24p^2 - p + 1372p + 256p^4 - 192p^3 + 48p^2 - 4p - 256p^3$
$+192p^2 - 48p + 5$
$F(\gamma) = 768p^4 - 640p^3 + 264p^2 + 1319p + 4$

Case 2: When p is even

$F(\gamma) = p(8p-1)^3 + 4p(3)^3 + (4p-2)(4p-1)^3$
$= p(512p^3 - 192p^2 + 24p - 1) + 108p + (4p-2)(64p^3 - 48p^2 + 12p - 1)$
$= 512p^4 - 192p^3 + 24p^2 - p + 108p + 256p^4 - 192p^3 + 48p^2 - 4p - 128p^3$
$+96p^2 - 24p + 2$
$F(\gamma) = 768p^4 - 512p^3 + 168p^2 + 79p + 2$

Theorem 3.6: Degree-Distance Index DD(G) of the graph of commuting elements of the semi-dihedral group is

$DD(G) = 192p^3 + 224p^2 - 24p$, When p is odd
$DD(G) = 192p^3 + 64p^2 - 8p$, When p is even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.3 & 2.1.4 and together with DD(G), we derive

Case 1: When p is odd

$DD(G) = 4_{c_2}[8p - 1 + 8p - 1] + (4p-4)_{c_2}[4p - 1 + 4p - 1] + 6p[7 + 7] + 4(4p-4)[8p - 1 +$
$4p - 1] + 16p[8p - 1 + 7] + 16p(p-1)(4p+6)(2)$
$= 6[16p - 2] + \frac{(4p-4)(4p-5)}{2}[8p - 2] + 6p[14] + [16p - 16][12p - 2] + 16p[8p + 6] + 224p^2 -$
$224p + 32p(p-1)(4p+6)$
$= 96p - 12 + (4p-4)(4p-5)(4p-1) + 84p + (16p-16)(12p-2) + 16p(8p+6) +$
$(32p^2 - 32p)(4p+6)$
$= 96p - 12 + [16p^2 - 36p + 20](4p-1) + 84p + 192p^2 - 32p - 192p + 32 + 128p^2 + 96p +$
$128p^3 + 192p^2 - 128p^2 - 192p$
$= 96p - 12 + 64p^3 - 16p^2 - 144p^2 + 36p + 80p - 20 + 84p + 192p^2 - 32p - 192p + 32 +$
$128p^2 + 96p + 128p^3 + 64p^2 - 192p$
$DD(G) = 192p^3 + 224p^2 - 24p$

Case 2: When p is even

DD(G)=$2p[3+3]+2_{c_2}[8p-1+8p-1]+(4p-2)_{c_2}[4p-1+4p-1]+8p[8p-1+3]+(8p-4)[8p-1+4p-1]+4p(4p-2)(4p+2)(2)$
$=2p[6]+[16p-2]+\frac{(4p-2)(4p-3)}{2}[8p-2]+8p[8p+2]+(8p-4)(12p-2)+(16p^2-8p)(8p+4)$
$=12p+16p-2+(4p-2)(4p-3)(4p-1)+8p(8p+2)+(8p-4)(12p-2)+128p^3+64p^2-64p^2-32p$
$=12p+16p-2+64p^3-80p^2+24p-16p^2+20p-6+64p^2+16p+96p^2-16p-48p+8+128p^3-32p$
DD(G) $=192p^3+64p^2-8p$

Theorem 3.7: Gutman-Degree distance $\mathrm{DD_G(G)}$ of the graph of commuting elements of the semi-dihedral group is

$\mathrm{DD_G(G)}=128p^4+1056p^3-168p^2+164p-18$, When p is odd
$\mathrm{DD_G(G)}=128p^4+416p^3-120p^2+48p$, When p is even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.3 & 2.1.4 and together with $\mathrm{DD_G(G)}$, we derive

Case 1: When p is odd

$$\mathrm{DD_G(G)}=4_{c_2}[(8p-1)(8p-1)]+(4p-4)_{c_2}[(4p-1)(4p-1)]+6p[(7)(7)]$$
$$+4(4p-4)[(8p-1)(4p-1)+16p[7(8p-1)]+4p(4p-4)[(7)(4p+1)](2)$$
$=6[16p-2]+\frac{(4p-4)(4p-5)}{2}(4p-1)^2+6p[49]+4(4p-4)(8p-1)(4p-1)+16p(56p-7)+(16p^2-16p)[28p+7](2)$
$=96p-12+(2p-2)(4p-5)(4p-1)^2+294p+(16p-16)(32p^2-12p+1)+896p^2-112p+(32p^2-32p)(28p+7)$
$=96p-12+128p^4+8p^2-64p^3-288p^3-18p+144p^2+160p^2+10-80p+294p+512p^3-192p^2+16p-512p^2+192p-16+896p^2-112p+896p^3+224p^2-896p^2-224p$
$\mathrm{DD_G(G)}=128p^4+1056p^3-168p^2+164p-18$

Case 2: When p is even

$$\mathrm{DD_G(G)}=2p[(3)(3)]+2_{c_2}[(8p-1)(8p-1)+(4p-2)_{c_2}[(4p-1)(4p-1)]+8p[3(8p-1)]$$
$$+(8p-4)[(8p-1)(4p-1)]+4p(4p-2)[(3)(4p-1)](2)$$
$=2p[9]+(64p^2-8p-8p+1)+\frac{(4p-2)(4p-3)}{2}(16p^2-8p+1)+8p[24p-3]+(8p-4)(32p^2-12p+1)+(16p^2-8p)[24p-6]$
$=18p+64p^2-16p+1+(2p-1)(4p-3)(16p^2-8p+1)+8p[24p-3]+(8p-4)(32p^2-12p+1)+384p^3-96p^2-192p^2+48p$
$=18p+64p^2-16p+1+(8p^2-6p-4p+3)(16p^2-8p+1)+8p[24p-3]+(8p-4)(32p^2-12p+1)+384p^3-288p^2+48p$
$=18p+64p^2-16p+1++128p^4-64p^3+8p^2-160p^3+80p^2-10p+48p^2-24p+3+192p^2-24p+256p^3-96p^2+8p-128p^2+48p-4+384p^3-288p^2+48p$
$=128p^4+416p^3-120p^2+48p$

Theorem 3.8: First Zagreb Degree-Distance Index $ZDD_1(G)$ of the graph of commuting elements of the semi-dihedral group is
$ZDD_1(G)=768p^4+1216p^3+1136p^2+1016p-6$, When p is odd

$ZDD_1(G)= 768p^4 + 192p^3 + 208p^2 - 32p$, When p is even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.3 & 2.1.4 and together with $ZDD_1(G)$, we derive

Case 1: When p is odd

$ZDD_1(G) = 4_{c_2}[(8p-1)^2 + (8p-1)^2] + 4p - 4_{c_2}[(4p-1)^2 + (4p-1)^2] + 6p[49+49]$
$+ 4(4p-4)[(8p-1)^2 + (4p-1)^2] + 16p[49 + (8p-1)^2] + 4p(4p-1)[(7^2)$
$+ (4p-1)^2](2)$

$= 6[64p^2 + 1 - 16p + 64p^2 + 1 - 16p] + \frac{(4p-4)(4p-5)}{2}[(16p^2 - 8p + 1) + (16p^2 - 8p + 1)] +$
$6p[98] + 16p[49 + 64p^2 + 1 - 16p] + (16p^2 - 4p)[49 + 16p^2 + 1 - 8p](2)$

$= 6[128p^2 - 32p + 1] + (8p^2 - 18p + 10)(32p^2 - 16p + 2) + 6p[98] + (16p - 16)(80p^2 -$
$24p + 2) + 16p[64p^2 - 16p + 50] + (16p^2 - 4p)(32p^2 - 16p + 100)$

$= 768p^2 - 192p + 6 + 256p^4 - 128p^3 + 16p^2 - 576p^3 + 288p^2 - 36p + 320p^2 - 160p +$
$20 + 588p + 1280p^3 - 384p^2 + 32p - 1280p^2 + 384p - 32 + 1024p^3 - 256p^2 + 800p +$
$512p^4 - 256p^3 + 1600p^2 - 128p^3 + 64p^2 - 400p$

$ZDD_1(G) = 768p^4 + 1216p^3 + 1136p^2 + 1016p - 6$

Case 2: When p is even

$ZDD_1(G) == 2p[9+9] + 2_{c_2}[(8p-1)^2 + (8p-1)^2] + 4p - 2_{c_2}[(4p-1)^2 + (4p-1)^2] +$
$8p[9 + (8p-1)^2] + (8p-4)[(8p-1)^2 + (4p-1)^2] + 4p(4p-2)[(3^2) + (4p-1)^2](2)$

$= 2p[18] + [64p^2 + 1 - 16p + 64p^2 + 1 - 16p] + \frac{(4p-2)(4p-3)}{2}[(16p^2 - 8p + 1) +$
$(16p^2 - 8p + 1)] + 8p[9 + 64p^2 + 1 - 16p] + (8p - 4)[64p^2 + 1 - 16p + 16p^2 + 1 - 8p] +$
$(16p^2 - 8p)[9 + 16p^2 + 1 - 8p](2)$

$= 36p + [128p^2 - 32p + 2] + (8p^2 - 10p + 3)(32p^2 - 16p + 2) + 8p[64p^2 - 16p + 10] +$
$(8p-4)[80p^2 - 24p + 2] + (32p^2 - 16p)(16p^2 - 8p + 10)$

$= 128p^2 + 4p + 2 + 256p^4 - 128p^3 + 16p^2 - 320p^3 + 160p^2 - 20p + 96p^2 - 48p + 6 +$
$512p^3 - 128p^2 + 80p + 640p^3 - 192p^2 + 16p - 320p^2 + 96p - 8 + 512p^4 - 256p^3 +$
$320p^2 - 256p^3 + 128p^2 - 160p$

$= 768p^4 + 192p^3 + 208p^2 - 32p$

Theorem 3.9: Generalized-Degree Distance Index $DD_\alpha(G)$ of the graph of commuting elements of the semi-dihedral group is

$DD_\alpha(G)=6(8p-1)^{\alpha}+6p.7^{2\alpha}+16p(56p-7)^{\alpha}+2(p-1)[(4p-1)(4p-1)^{2\alpha}+8(32p^2-12p+1)^{\alpha}+16p(28p-7)^{\alpha}]$,
When p is odd

$DD_\alpha(G)=(8p-1)^2\alpha+(32p^2-10p+3)(4p-1)^{2\alpha}+2p[9^{\alpha}+4(12p-3)^{\alpha}]+4(2p-1)[(32p^2-12p+1)^{\alpha}+4p(12p-3)^{\alpha}]$,
When p is even

Proof: By employing the structural description of $\gamma(SD_{8p})$ given in Lemma 2.1.2 Proposition 2.1.3 & 2.1.4 and together with $DD_\alpha(G)$, we derive

Case 1: When p is odd

$DD_\alpha(G) = 4_{c_2}[(8p-1)(8p-1)]^\alpha + 4p - 4_{c_2}[(4p-1)(4p-1)]^\alpha + 6p[7\times 7]^\alpha$
$+ 4(4p-4)[(8p-1)(4p-1)]^\alpha + 16p[7(8p-1)]^\alpha + 4p(4p$
$- 4)[7(4p-1)]^\alpha(2)$

$= 6[(8p-1)]^{2\alpha} + \frac{(4p-4)(4p-5)}{2}[(4p-1)]^{2\alpha} + 6(7)^{2\alpha}p + 4(4p-4)(8p-1)^\alpha(4p-1)^\alpha +$
$16p[7^\alpha(8p-1)^\alpha] + (16p^2 - 16p)(7^\alpha)(4p-1)^\alpha(2)$

$= 6(8p-1)^{2\alpha} + (2p-2)(4p-5)(4p-1)^{2\alpha} + 6p\times 7^{2\alpha} + 4(4p-4)(8p-1)^\alpha(4p-1)^\alpha +$
$16p[7^\alpha(8p-1)^\alpha] + (32p^2 - 32p)(7^\alpha)(4p-1)^\alpha$

$= 6(8p-1)^{2\alpha} + 6p(49)^\alpha + (8p^2 - 18p + 10)(4p-1)^{2\alpha} + (16p-16)(8p-1)^\alpha(4p-1)^\alpha +$
$16p[7^\alpha(8p-1)^\alpha] + (32p^2 - 32p)(7^\alpha)(4p-1)^\alpha$

$= 6(8p-1)^{2\alpha} + 6p(49)^{\alpha} + (8p^2 - 18p + 10)(4p-1)^{2\alpha} + (16p-16)[32p^2 - 12p + 1]^{\alpha} + 16p[56p-7]^{\alpha} + (32p^2 - 32p)(28p-7)^{\alpha}$

$DD_{\alpha}(G)=6(8p-1)^{\alpha}+6p.7^{2\alpha}+16p(56p-7)^{\alpha}+2(p-1)[(4p-1)(4p-1)^{2\alpha}+8(32p^2-12p+1)^{\alpha}+16p(28p-7)^{\alpha}]$

Case 2: When p is even

$$DD_{\alpha}(G) = 2m[(3)(3)]^{\alpha} + 2_{c_2}[(8p-1)(8p-1)]^{\alpha} + 4p - 2_{c_2}[(4p-1)(4p-1)]^{\alpha} + 8p[3(8p-1)]^{\alpha} + (8p-4)[(8p-1)(4p-1)]^{\alpha} + 4p(4p-2)[3(4p-1)]^{\alpha}(2)$$

$= 2p(9)^{\alpha} + (8p-1)^{2\alpha} + \frac{(4p-2)(4p-3)}{2}(4p-1)^{2\alpha} + 8p[3^{\alpha}(8p-1)^{\alpha}] + (8p-4)(8p-1)^{\alpha}(4p-1)^{\alpha} + (16p^2 - 8p)(3^{\alpha})(4p-1)^{\alpha}(2)$

$=2p(9)^{\alpha} + (8p-1)^{2\alpha} + (2p-1)(4p-3)(4p-1)^{2\alpha} + 8p[(24p-3)^{\alpha}] + (8p-4)[32p^2 - 12p + 1]^{\alpha} + (32p^2 - 16p)(3^{\alpha})(4p-1)^{\alpha}$

$=2p(9)^{\alpha} + (8p-1)^{2\alpha} + (32p^2 - 10p + 3)(4p-1)^{2\alpha} + 8p[(24p-3)^{\alpha}] + (8p-4)[32p^2 - 12p + 1]^{\alpha} + (32p^2 - 16p)(12p-3)^{\alpha}$

$DD_{\alpha}(G)=(8p-1)^2\alpha+(32p^2-10p+3)(4p-1)^{2\alpha}+2p[9^{\alpha}+4(12p-3)^{\alpha}]+4(2p-1)[(32p^2-12p+1)^{\alpha}+4p(12p-3)^{\alpha}]$

4. Interpretation of Results

Findings in this paper give a quantitative explanation of the commutative property of the semi-dihedral group SD_{8p} in terms of graph-theoretic invariants. The topological indices obtained are all expressed as the polynomials of the parameter p which is directly proportional to the order of the group and dictates the commutating graph magnitude. This is a natural dependence which is linear because the degrees of the vertex, partitions of the edges and the distributions of distances with the commuting graph are linear functions of p.

One of the structural characteristics that affects all the outcomes is the occurrence of non-trivial centre in SD_{8p}. The central elements of a group are vertices that are commutative with all the other elements in the group and thus contain a complete subgraph of all other remaining elements. The high vertex degree of central vertices makes them the most heavily weighted vertex in the indices that are based on degrees like the First and Second Zagreb index and the Forgotten index. Thus, the most high-degree terms in the resulting polynomials express the prevalence of centre terms and give information on how much the group is not abelian.

The cubic and quartic growth of some indices show that the commutation relations get increasingly complicated with an increase in group order. Specifically, the Second Zagreb index and Forgotten index that include products or higher powers of the degrees of the vertex, enhance the impact of the highly connected vertices. This proves that SD_{8p} commuting graph is very centralized with majority of the commutation paths mediated by the centre of the group.

The commuting graph distance indices, distance based and degree-based indices, also help to understand the global structure of the commuting graph. The graph diameter is equal to two, so any two non-neighbouring nodes are linked with a central one. Consequently, distance contributions are limited to one or two, which makes the expressions of the analysis simpler but at the same time expressive of useful algebraic information. The polynomial expressions of the Degree-Distance index, Gutman index, and Zagreb Degree -Distance index obtained are the reflection of how central elements influence indirect commutation relationships between non-central elements.

The results also indicate a clear-cut of odd with even values of p. The size of the centre and partitioning of non-central elements incur a change in degrees of the vertices and distributions of edges which subsequently alter the coefficients of the expressions as polynomials. Though the general level of the polynomials is the same in both instances, the variation of coefficients carries the minute variations in the internal symmetry and commutation behaviour of the group.

Comprehensively, the algebraic structure of the semi-dihedral group is well represented by the obtained polynomial expressions of the different topological indices. They show that commuting graphs can play the role of an efficient intermediary between group theory and graph theory, in which complicated commutative phenomena can be expressed as positive quantities that are easily computable. The results that are obtained not only contribute to the knowledge of the structural properties of SD_{8p}. It is also enlightening to the applicability of topological indices in the study of the algebraic graphs formed by non-abelian group structures.

Conclusion:

This paper outlines a graphical analysis of the commuting graph of SD_{8p} semi-dihedral group, and the purpose of the paper is to learn more about the structural and commutative nature of the graph in terms of graph theorized quantities. Through a discovery of explicit vertex partitions and set of vertex degrees and distance relations, a definitive structural characterization of the commuting graph has been discovered in both odd and even values of the parameter p.

Based on these structural foundations, some key degree-distance based topological indices were obtained in closed form. These are the Zagreb indices and their variations, the Forgotten index and the Degree distance indices. Each indexed up is a described as a polynomial function of p, showing the direct relation of the commuting graph to the order of the group on which it depends. The most salient terms of these polynomials indicate the prevailing powers of central elements that commute with all the elements of the group and form highly connected nodes in the graph.

The findings also indicate that the commuting graph is diameter-two which is a key factor in determining the indices based on distance. The non-central non-commutative vertices are always linked to one another by central ones, where the central elements are highlighted the most noticeable medium between the commutative structure. The oddness and evenness of the case of pare that was present in the coefficients of the expressions of the polynomials, suggested slight changes in the internal symmetry and commutation properties of the group.

Generally, this article shows that commuting graphs and the corresponding topological indices are an effective method of converting algebraic commutation properties into quantitative graph invariants. The findings are an extension of already existing work on algebraic graphs of non-abelian groups and has a part to play in devising more insights into semi-dihedral group structures. The approach and outcomes provided here can be extended to other families of finite groups and the derived indices can be potentially used in chemical graph theory and other similar fields in which a significant role is played by algebraic graph models.

References:

1. I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chemical Physics Letters 17 (1972) 535–538, https://doi.org/10.1016/0009-2614(72)85099-1. (ScienceDirect)

2. F. Ali, Z. Raza and A. Shabbir, *On the topological indices of commuting graphs for finite non-abelian groups*, Symmetry 14 (2022) 1266, https://doi.org/10.3390/sym14061266. (MDPI)

3. W. Ahmed, A. Ali and M. Javaid, *A comprehensive study on topological indices and entropy*, Discrete Applied Mathematics (2025), in press.

4. Nur Ain Supu, Intan Muchtadi-Alamsyah and Erma Suwastika, *Topological indices of relative g-noncommuting graph of dihedral groups*, Journal of the Indonesian Mathematical Society 29 (2023) 271–288, https://doi.org/10.22342/jims.29.3.1594.271-288. (jims-a.org)

5. N. I. Alimon, N. H. Sarmin and A. Erfanian, *Topological indices of non-commuting graph of dihedral groups*, Malaysian Journal of Fundamental and Applied Sciences 14 (2018) 473–479, https://doi.org/10.11113/mjfas.v14n0.1270. (mjfas.utm.my)

6. N. I. Alimon, N. H. Sarmin and A. Erfanian, *Topological indices of the non-commuting graph of generalized quaternion groups*, Bulletin of the Malaysian Mathematical Sciences Society 43 (2020) 2871–2891.

7. P. Munandar, *Connectivity indices of power graphs over dihedral groups*, Journal Sains Dasar 14 (2025) 45–55.

8. P. Rana, A. Ahmad and M. F. Nadeem, *Topological indices and structural properties of cubic power graphs of dihedral groups*, Contemporary Mathematics 8 (2024) 1–15.

9. A. S. Bawana and Y. Susanti, *Some topological indices of order divisor graphs of cyclic groups*, Journal of the Indonesian Mathematical Society 31 (2025) 89–104.

10. M. Rafiullah, *Degree-based topological indices for strong double graphs*, Journal of Graph Theory 98 (2021) 112–128, https://doi.org/10.1002/jgt.22635.

11. A. Hakeem, M. Imran and W. Khalid, *Reverse degree-based topological indices of molecular structures*, Frontiers in Physics 12 (2024) 1422098, https://doi.org/10.3389/fphy.2024.1422098.

12. S. Mondal, N. De and A. Pal, *On neighborhood Zagreb index of product graphs*, Chemical Physics Letters 755 (2020) 137754, https://doi.org/10.1016/j.cplett.2020.137754.

13. X. Ma, Y. Wang and L. Zhang, *Aspects of the commuting graph*, Linear Algebra and its Applications 676 (2023) 45–63, https://doi.org/10.1016/j.laa.2023.01.007.

14. R. Todeschini and V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley-VCH, Weinheim, 2009.

15. E. Estrada and J. A. Rodríguez-Velázquez, *Subgraph centrality and clustering in complex networks*, Physical Review E 71 (2005) 056103, https://doi.org/10.1103/PhysRevE.71.056103.

16. A. Ali and N. Trinajstić, *A novel/old modification of the first Zagreb index*, Discrete Mathematics, Algorithms and Applications 10 (2018) 1850044, https://doi.org/10.1142/S1793830918500444.

17. M. Randić, *Characterization of molecular branching*, Journal of the American Chemical Society 97 (1975) 6609–6615, https://doi.org/10.1021/ja00850a008.

18. I. Gutman, *Beyond the Zagreb indices*, Arabian Journal of Chemistry 13 (2020) 192–205, https://doi.org/10.1016/j.arabjc.2019.05.006.

19. H. Wiener, *Structural determination of paraffin boiling points*, Journal of the American Chemical Society 69 (1947) 17–20, https://doi.org/10.1021/ja01145a002.

20. J. Araújo, W. Bentz and J. Konieczny, *The commuting graph of the symmetric inverse semigroup*, Electronic Journal of Combinatorics 19 (2012) P34.

21. Gutman, I., & Trinajstić, N. (1972). *Graph theory and molecular orbitals. III. Total π-electron energy of alternant hydrocarbons.* Chemical Physics Letters, 17(4), 535–538. https://doi.org/10.1016/0009-2614(72)85099-1

22. Miličević, A., Nikolić, S., & Trinajstić, N. (2004). *On reformulated Zagreb indices.* Molecular Diversity, 8(4), 393–399. https://doi.org/10.1023/B:MODI.0000047504.14261.2a

23. Furtula, B., & Gutman, I. (2015). *A forgotten topological index.* Journal of Mathematical Chemistry, 53(4), 1184–1190. https://doi.org/10.1007/s10910-015-0480-z

24. Dobrynin, A. A., & Kochetova, A. A. (1994). *Degree distance of a graph: A degree analogue of the Wiener index.* Journal of Chemical Information and Computer Sciences, 34(5), 1082–1086. https://doi.org/10.1021/ci00021a008

25. Gutman, I. (1994). *Selected properties of the Schultz molecular topological index.* Journal of Chemical Information and Computer Sciences, 34(5), 1087–1089. https://doi.org/10.1021/ci00021a009

26. Deutsch, E., & Klavžar, S. (2015). *M-polynomial and degree-based topological indices.* Iranian Journal of Mathematical Chemistry, 6(2), 93–102. https://doi.org/10.22052/ijmc.2015.10106 (ijpam.eu)

27. T. Cheng, M. Dehmer, F. Emmert-Streib, Y. Li and W. Liu, Properties of commuting graphs over semidihedral groups, Symmetry 13 (2021) #103, https://doi.org/10.3390/sym13010103

28. F. Ali, B. A. Rather, N. Fatima, M. Sarfraz, A. Ullah, K. A. M. Alharbi and R. Dad, On the topological indices of commuting graphs for finite non-abelian groups, Symmetry 14 (2022) #1266, https://doi.org/10.3390/sym14061266.

29. F. Ali, B. A. Rather, M. Sarfraz, A. Ullah, N. Fatima and W. K. Mashwani, Certain topological indices of non-commuting graphs for finite non-abelian groups, Molecules 27 (2022) #6053, https://doi.org/10.3390/molecules27186053.

30. F. Ali, K. Raza and T. Rubab, *Topological properties of commuting graphs over semi-dihedral groups*, Iranian Journal of Mathematical Chemistry 16 (2025) 109–126, https://doi.org/10.22052/IJMC.2024.254494.1840