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SOME RELATIONS DERIVED FROM CERTAIN FAMILIES OF SPECIAL NUMBERS COMPRISE BERNOULLI, TRIANGULAR, AND OBLONG NUMBERS

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Abstract. The main objective of this article is to discover new relationships between perfect square numbers and binomial coefficients, which stem from Pascal's triangle and have been hidden in our subconscious since middle school. Therefore, in this manuscript, we investigate some certain families of special numbers with their fundamental properties. We also investigate relations among the Bernoulli numbers, the triangular numbers, and the Oblong numbers. Our aim is to discover some novel formulas derived from these relations.

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1. Introduction

While studying Pascal's triangle in middle school math class, we noticed some relationships that we pondered for a long time. This led us to examine these relationships more comprehensively and in detail. As a result of this research, we discovered that these relationships were not coincidental, but rather part of a larger idea and a new set of concepts. The aim of this article is to find new relationships between the binomial coefficients that make up Pascal's triangle and the numbers they generate, and to examine existing relationships between these numbers and other special numbers and polynomials.

Definition of the triangular number, denoted by T_n, is given by

$$T_n = 1 + 2 + \cdots + n, \tag{1}$$

which is the sum of the first n integers [7, 8]. Tthese numbers can also defined by

$$T_n = \binom{n+1}{2} \tag{2}$$

(*cf.* [2, 7–9], and see also [11]- [13]).

By using the Principle of Mathematical Induction method, Nelsen [7, 8] showed that

$$3T_n + T_{n-1} = T_{2n}.$$

The triangle numbers can be showen by the ID "A000217" in https://oeis.org/A000217.

The triangle numbers are counts objects arranged in an equilateral triangle. These numbers are a type of figurate numbers, related to the square numbers and cube numbers, and the Oblong numbers are two dimensional figurate numbers. That is, the nth triangular number is the number of dots in the triangular arrangement with n dots on each side, and is equal to the sum of the n natural numbers from 1 to n, (*cf.* [9], [11]- [13]).

A relation between the Oblong numbers O_n and the triangular numbers T_n is given by:

$$O_n = n(n+1) = 2T_n, \tag{3}$$

(*cf.* [9], [11]- [13]).

The Bernoulli numbers and polynomials are given, respectively, by

$$\begin{aligned} B_0 &= 1, \\ B_1 &= -\frac{1}{2}, \\ B_2 &= \frac{1}{6}, \\ B_3 &= 0, \\ B_4 &= -\frac{1}{30}, \\ B_5 &= 0, \\ B_6 &= \frac{1}{42}, \end{aligned}$$

so on; and

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\ B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\ B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \end{aligned}$$

and so on (cf. [3]- [5], [6], [10]).

Formula of the sums of consecutive positive integers, in terms of the Bernoulli numbers and polynomials, is given by

$$\sum_{j=1}^{k-1} j^n = \frac{B_{n+1}(k) - B_{n+1}}{n+1} \tag{4}$$

(cf. [1], [3]- [5], [6], [10]). For $n = 1, 2, 3$, using (4), we have the following formulas:

$$\sum_{j=1}^{k-1} j^0 = \frac{B_1(k) - B_1}{1} = k,$$

(5)
$$\sum_{j=1}^{k-1} j = \frac{B_2(k) - B_2}{2} = \frac{k\,(k-1)}{2},$$

and

(6)
$$\sum_{j=1}^{k-1} j^2 = \frac{B_3(k) - B_3}{3} = \frac{k\,(k-1)\,(2k-1)}{6}.$$

Here we note that we discovered that the literature on Bernoulli numbers and their definitions dates back to the 1600s (cf. [1], [3]- [5], [6], [10]). We found that very complex functions related to them exist. We obtained our results by giving several Bernoulli numbers and polynomials that we can use throughout the sections of this paper. As a result, new formulas about triangular numbers can be discovered using higher-degree Bernoulli polynomials.

In the next section, we give some novel formulas, which are including the Bernoulli numbers, the triangular numbers and the Oblong numbers.

2. Main Results

In this section, we give relations and formulas among the Bernoulli numbers, the triangular numbers and the Oblong numbers.

Theorem 2.1. *The following formula holds true:*

$$\sum_{j=1}^{k-1} (T_j + T_{j-1}) = \frac{k\,(k-1)\,(2k-1)}{6}.$$

Proof. Using the following well-known formula, we get

(7)
$$T_j + T_{j-1} = j^2$$

(cf. [9], [11], [13]). This formula give us

$$\sum_{j=1}^{k-1} (T_j + T_{j-1}) = \sum_{j=1}^{k-1} j^2$$

By combining equation (6) with the above formula we get

$$\sum_{j=1}^{k-1} (T_j + T_{j-1}) = \frac{B_3(k) - B_3}{3}.$$

Using $B_3\,(k) = k^3 - \frac{3}{2}k^2 + \frac{1}{2}k$ and $B_3 = 0$, after some calculations, we get result of theorem. □

Theorem 2.2. *The following formula holds true:*

$$\sum_{j=1}^{k-1} (O_j + O_{j-1}) = \frac{2}{3}B_3(k).$$

Proof. Using (3) and the following well-known formula, we get

$$O_j + O_{j-1} = 2j^2 \tag{8}$$

(cf. [9], [11], [13]), we get

$$\sum_{j=1}^{k-1} (O_j + O_{j-1}) = 2\sum_{j=1}^{k-1} j^2$$

By combining equation (6) with the above formula and using $B_3(k) = k^3 - \frac{3}{2}k^2 + \frac{1}{2}k$ and $B_3 = 0$, after some calculations, we get result of theorem. □

Theorem 2.3. *The following formula holds true:*

$$\sum_{j=1}^{k-1} j^3 = \frac{(B_2(k) - B_2)^2}{4}.$$

Proof. Using the following well-known formula

$$1^3 + 2^3 + \cdots + (k-1)^3 = \frac{k^2(k-1)^2}{4}$$

(cf. [1], [3], [14]) and (2), we get

$$1^3 + 2^3 + \cdots + (k-1)^3 = T_{k-1}^2$$

Combining this formula with (5), we have

$$\sum_{j=1}^{k-1} j^3 = T_{k-1}^2.$$

After some calculations, we get result of theorem. □

We observe that proof of (7) can also be given by the Pascal's triangle, is an infinite triangular array of the binomial coefficients. It is easy to see that, from the nth adjacent rows of this triangle is the perfect square. That is,

$$\begin{array}{l}
\binom{0}{0} \\
\binom{1}{0}\binom{1}{1} \\
\binom{2}{0}\binom{2}{1}\binom{2}{2} \\
\vdots \\
\binom{n}{0}\binom{n}{1}\cdots\binom{n}{n-2}\binom{n}{n-1}\binom{n}{2} \\
\binom{n+1}{0}\binom{n+1}{1}\cdots\binom{n+1}{n-1}\binom{n+1}{n}\binom{n+1}{n+1}
\end{array}$$

from the above, we have

$$\binom{n}{n-2} + \binom{n+1}{n-1} = n^2.$$

From (7) and (3), we have the following theorems, respectively:

Theorem 2.4. *The following formula holds true:*

$$\binom{n}{n-2}+\binom{n+1}{n-1}=T_{n-1}+T_n.$$

Theorem 2.5. *The following formula holds true:*

$$\binom{n}{n-2}+\binom{n+1}{n-1}=\frac{O_j+O_{j-1}}{2}.$$

3. Conclusion

In this paper, we studied on the special numbers with their some fundamental properties. We also gave some certain families of special numbers involving triangular numbers, and the Oblong numbers, and other numbers. Moreover, We investigated relations among the Bernoulli numbers, the triangular numbers, and the Oblong numbers. We proved some theorems involving many novel formulas related to the Bernoulli numbers, the triangular numbers, and the Oblong numbers. Some of these formulas are given as follows:

$$\sum_{j=1}^{k-1}(T_j+T_{j-1})=\frac{k\,(k-1)\,(2k-1)}{6},$$

$$\sum_{j=1}^{k-1}(O_j+O_{j-1})=\frac{2}{3}B_3(k),$$

$$\sum_{j=1}^{k-1}j^3=T_{k-1}^2.$$

and

$$\binom{n}{n-2}+\binom{n+1}{n-1}=\frac{O_j+O_{j-1}}{2}=T_{n-1}+T_n.$$

Our formulas can be extended to the alternating sums, which are related to the other special numbers and polynomials. Our future plane aim to find their applications and find solution them.

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