

## DEGENERATE GENERALIZED HERMITE POLYNOMIALS: OPERATIONAL AND MONOMIALITY PROPERTIES

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**ABSTRACT.** In the current study, we derive certain properties of the degenerate 2-parameter 3-variable Hermite polynomials, degenerate 1-parameter 2-variable Hermite polynomials and multi-index degenerate Hermite polynomials through the generating formula along with their series definition and derivative relations and then we explore the operational identities for these polynomials. Also, we derive the monomiality identities for the degenerate 2-parameter 3-variable Hermite polynomials and degenerate multi-index Hermite polynomials.

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**KEYWORDS AND PHRASES.** Degenerate Generalized Hermite polynomials, Operational identities, Monomiality principle.

### 1. INTRODUCTION

Hermite polynomials propose adaptable simple solutions to boundary value problems, which have a wide range of uses in applied sciences. Dattoli and colleagues discovered applications for Hermite polynomials in optic wave transfer and quantum mechanics theory [9, 7]. Dattoli and Torre [10] established two-variable, one-parameter Hermite polynomials, which have since been extended by others. Subuhi with Rehana Khan defined the Hermite polynomials with three variables and two parameters [19, 18]. The multi-variable and multi-index Hermite polynomials have been employed to the model quantum-phase-space mechanics as it relates to wave propagation, as well as the investigation of charged-beam transport concerns in classical mechanics [8]. Several branches of mathematics are essential across various fields and exhibit a keen interest in special functions. Hermite polynomials constitute a prominent subset of these distinguished polynomial sequences. Mathematicians have investigated degenerate variants of particular Hermite polynomials and numbers to enhance their analytical characteristics and broaden their range of applications (e.g., see here [32, 25, 16, 22, 14, 24, 26, 27, 15]).

For the rest of this section, we will construct certain symbols and revisit several well-known results that are important throughout this article.

For any non-zero  $\lambda \in \mathbb{R}$ , the degenerate exponential function is provided as [4, 5, 28, 13]:

$$(1) \quad e_{\lambda}^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{\theta=0}^{\infty} (x)_{\theta, \lambda} e_{\lambda}(t) = (1 + \lambda t)^{\frac{1}{\lambda}} = e_{\lambda}^1(t) = \sum_{\theta=0}^{\infty} (1)_{\theta, \lambda} \frac{t^{\theta}}{\theta!},$$

where  $(x)_{\theta, \lambda}$  denotes the  $\lambda$ -falling sequence supplied via

$$(2) \quad (x)_{\theta, \lambda} = 1, \quad (x)_{\theta, \lambda} = x(x - \lambda) \dots (x - (\theta - 1)\lambda), \quad \theta \geq 1.$$

The degenerate two-variable Hermite polynomials, represented as  $H_{\theta}(x, y|\lambda)$ , were examined via their respective generating function and series expansion [16]:

$$(3) \quad (1 + \lambda)^{\frac{xt}{\lambda}} (1 + \lambda)^{\frac{yt^2}{\lambda}} = \sum_{\theta=0}^{\infty} H_{\theta}(x, y|\lambda) \frac{t^{\theta}}{\theta!}, \quad x, y \in \mathbb{C},$$

$$(4) \quad H_{\theta}(x, y|\lambda) = \theta! \sum_{\omega=0}^{[\theta/2]} \frac{1}{(\theta - 2\omega)!} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{\theta - \omega} x^{\theta - 2\omega} y^{\omega}, \quad 0 \leq \omega \leq \frac{\theta}{2}.$$

The degenerate three-variable Hermite polynomials  $H_{\theta}(x, y, z|\lambda)$  were explored through the generating relation and series expansion [16]:

$$(5) \quad (1 + \lambda)^{\frac{xt}{\lambda}} (1 + \lambda)^{\frac{yt^2}{\lambda}} (1 + \lambda)^{\frac{zt^3}{\lambda}} = \sum_{\theta=0}^{\infty} H_{\theta}(x, y, z|\lambda) \frac{t^{\theta}}{\theta!}, \quad x, y \in \mathbb{C}.$$

The degenerate three-variable Hermite polynomials  $H_{\theta}(x, y, z|\lambda)$  were explored through the series definition [16]:

$$(6) \quad H_{\theta}(x, y, z|\lambda) = \theta! \sum_{r=0}^{[\theta/3]} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^r \frac{z^r H_{\theta - 3r}(x, y|\lambda)}{r!(\theta - 3r)!}, \quad 0 \leq r \leq \frac{\theta}{3}.$$

or, equivalently  $(0 \leq \omega \leq \frac{\theta - 3r}{3}, 0 \leq r \leq \frac{\theta}{3})$

$$(7) \quad H_{\theta}(x, y, z|\lambda) = \theta! \sum_{r=0}^{[\theta/3]} \sum_{\omega=0}^{[(\theta - 3r)/2]} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{\theta - \omega - 2r} \frac{x^{\theta - 3r - 2\omega} y^{\omega} z^r}{r! \omega! (\theta - 3r)! (\theta - 3r - 2\omega)!}.$$

The next partial derivatives with  $H_{\theta}(x, y, z|\lambda)$  remain valid [16]:

$$(8) \quad \frac{\partial}{\partial x} H_{\theta}(x, y, z|\lambda) = \theta \frac{\log(1 + \lambda)}{\lambda} H_{\theta - 1}(x, y, z|\lambda), \quad \theta \geq 1,$$

$$\frac{\partial^m}{\partial x^m} H_{\theta}(x, y, z|\lambda) = \frac{\theta!}{(\theta - m)!} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^m H_{\theta - m}(x, y, z|\lambda), \quad 0 \leq m \leq \theta.$$

The concept of monomiality is an effective tool for studying polynomials and their characteristics. The concept of monomiality originated in the early nineteenth century, with J. F. Steffensen [31] proposed the definition of poweroid. Dattoli [12] developed and expanded the concept of quasi-monomiality.

A polynomial set  $\{p_\theta(x)\}_{\theta \in \mathbb{N}}$  is described as quasi-monomial when we can identify two special operators,  $\hat{M}$  and  $\hat{P}$ . These operators, known as multiplicative and derivative operators, play a crucial role in the structure of the polynomial set through:

$$(9) \quad \hat{M} \{p_\theta(x)\} = p_{\theta+1}(x)$$

and

$$(10) \quad \hat{P} \{p_\theta(x)\} = \theta p_{\theta-1}(x).$$

The commutation relation that results is fulfilled by the operators  $\hat{M}$  and  $\hat{P}$ :

$$(11) \quad [\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = 1.$$

If  $\hat{M}$  and  $\hat{P}$  have differential realizations, the differential equation fulfilled by  $p_\theta(x)$  may be derived via a particular relation:

$$(12) \quad \hat{M}\hat{P} \{p_\theta(x)\} = \theta p_\theta(x).$$

Since the sequel  $p_0(x) = 1$ , therefore  $p_n(x)$  can be expressed as:

$$(13) \quad p_\theta(x) = \hat{M}^\theta \{p_0(x)\} = \hat{M}^\theta \{1\}.$$

Motivated by the possibility and significance for applications of numerous types of special polynomials and numbers in various problems within combinatorics, number theory, differential equations, numerical analysis, and other areas of mathematics and physics, which have recently been examined by several researchers. Also, motivated by the classes of degenerate special functions that provide an analogue of the classical polynomials, offering a foundational framework for mathematical investigation across various disciplines, including umbral calculus and combinatorics. Furthermore, we are motivated by the principal benefit of employing degenerate special polynomials, which lies in their extensive applicability as a robust mathematical tool for modeling complex systems across various disciplines, including physics, engineering, and computer science, as well as in providing a comprehensive framework for theoretical investigation in pure mathematics. I have laid out the following portions of this paper accordingly: Section 2 presents the degenerate 2-parameter 3-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate 1-parameter 2-variable Hermite polynomials  $H_n(x, y; s_1|\lambda)$  through the generating formula, along with their series definition and derivative relations. Section 3 explores the operational rules for the degenerate three variables Hermite polynomials  $H_\theta(x, y, z|\lambda)$ , degenerate two parameters two variables Hermite polynomials  $H_\theta(x, y; s_1|\lambda)$ , degenerate two parameters three variables Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ . In Section 4, I explore the monomiality properties for degenerate 2-parameter 3-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ .

## 2. DEGENERATE GENERALIZED HERMITE POLYNOMIALS

In this part, we create degenerate 2-parameter 3-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate 1-parameter 2-variable Hermite polynomials  $H_n(x, y; s_1|\lambda)$  through generating formula along with their series definition and derivative relations. Also, we present certain results for degenerate multi-index Hermite polynomials with their associated formalism.

### 2.1. Degenerate Two Parameters Three Variables Hermite Polynomials.

We create degenerate two parameters three variables Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  via the corresponding generating value:

$$(14) \quad (1 + \lambda)^{\frac{xt}{\lambda}} (1 + \lambda)^{\frac{ys_1 t^2}{\lambda}} (1 + \lambda)^{\frac{zs_2 t^3}{\lambda}} = \sum_{\theta=0}^{\infty} H_\theta(x, y, z; s_1; s_2|\lambda) t^\theta, \quad x, y \in \mathbb{C}.$$

Expanding the left part of the formula (14), then by establishing the appropriate powers of  $t$  for every factor in the subsequent expression, we gain the next series description of  $H_\theta(x, y, z; s_1; s_2|\lambda)$ :

$$(15) \quad H_\theta(x, y, z; s_1; s_2|\lambda) = \sum_{r=0}^{[\theta/3]} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^r (zs_2)^r \frac{H_{\theta-3r}(x, y; s_2|\lambda)}{r!}, \quad 0 \leq r \leq \frac{\theta}{3},$$

or, equivalently for  $0 \leq \omega \leq \frac{\theta-3r}{3}, 0 \leq r \leq \frac{\theta}{3}$

$$(16) \quad H_\theta(x, y, z; s_1; s_2|\lambda) = \sum_{r=0}^{[\theta/3]} \sum_{\omega=0}^{[(\theta-3r)/2]} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{\theta-\omega-2r} \frac{(x)^{\theta-3r-2\omega} (ys_1)^\omega (zs_2)^r}{r! \omega! (\theta - 3r - 2\omega)!}.$$

**Theorem 2.1.** *For degenerate two parameters three variables Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$ , the next facts about partial derivatives are valid:*

$$(17) \quad \frac{\partial}{\partial x} H_\theta(x, y, z; s_1; s_2|\lambda) = \frac{\log(1 + \lambda)}{\lambda} H_{\theta-1}(x, y, z; s_1; s_2|\lambda), \quad \theta \geq 1,$$

$$(18) \quad \frac{\partial^m}{\partial x^m} H_\theta(x, y, z; s_1; s_2|\lambda) = \left( \frac{\log(1 + \lambda)}{\lambda} \right)^m H_{\theta-m}(x, y, z; s_1; s_2|\lambda), \quad \theta \geq m,$$

$$(19) \quad \frac{\partial}{\partial y} H_\theta(x, y, z; s_1; s_2|\lambda) = s_1 \frac{\log(1 + \lambda)}{\lambda} H_{\theta-2}(x, y, z; s_1; s_2|\lambda), \quad \theta \geq 2,$$

$$(20) \quad \frac{\partial}{\partial z} H_\theta(x, y, z; s_1; s_2|\lambda) = s_2 \frac{\log(1 + \lambda)}{\lambda} H_{\theta-3}(x, y, z; s_1; s_2|\lambda) \quad \theta \geq 3,$$

$$(21) \quad \frac{\partial}{\partial s_1} H_\theta(x, y, z; s_1; s_2|\lambda) = y \frac{\log(1 + \lambda)}{\lambda} H_{\theta-2}(x, y, z; s_1; s_2|\lambda), \quad \theta \geq 2,$$

$$(22) \quad \frac{\partial}{\partial s_2} H_\theta(x, y, z; s_1; s_2|\lambda) = z \frac{\log(1 + \lambda)}{\lambda} H_{\theta-3}(x, y, z; s_1; s_2|\lambda), \quad \theta \geq 3.$$

*Proof.* Considering the partial derivative for each aspect of (14) with respect to  $x$ , we receive

$$(23) \quad \sum_{\theta=0}^{\infty} \frac{\partial}{\partial x} H_\theta(x, y, z; s_1; s_2|\lambda) t^\theta = t \frac{\log(1 + \lambda)}{\lambda} (1 + \lambda)^{\frac{x t}{\lambda}} (1 + \lambda)^{\frac{y s_1 t^2}{\lambda}} (1 + \lambda)^{\frac{z s_2 t^3}{\lambda}}.$$

Implementing the equation (14) upon the right portion of the equation (23), we yield

$$\sum_{\theta=0}^{\infty} \frac{\partial}{\partial x} H_\theta(x, y, z; s_1; s_2|\lambda) t^\theta = \frac{\log(1 + \lambda)}{\lambda} \sum_{\theta=0}^{\infty} H_\theta(x, y, z; s_1; s_2|\lambda) t^{\theta+1}.$$

Thereby, when the matching quantities of  $t$  on each part are compared, we determine assertion (17).

Furthermore, by calculating the  $m^{th}$  partial derivative value of each part of (14) with regard to  $x$  and continuing the preceding procedures of proving equation (17), we obtain statement (18).

Afterward, considering the partial derivatives for each part of the equation (14) concerning  $y, z, s_1$  and  $s_2$ . Then again, carry out the steps in the equation's proof (17). This will yield the assertions (19), (20), (21) and (22), in that order.

Theorem 2.1 has been fully proved. □

**Remark 2.1.** (i) Taking  $x = z = 0$  and  $x = y = 0$  in equation (14), we obtain the initial conditions for  $H_\theta(x, y, z; s_1; s_2|\lambda)$  as follows ( $\theta = 0, 1, 2, \dots$ ):

$$(24) \quad H_\theta(0, y, 0; s_1; s_2|\lambda) = \left( \frac{\log(1 + \lambda)}{\lambda} \right)^\theta (s_1 y)^{[\theta/2]} \text{ and} \\ H_\theta(0, 0, z; s_1; s_2|\lambda) = \left( \frac{\log(1 + \lambda)}{\lambda} \right)^\theta (s_2 z)^{[\theta/3]},$$

(ii) Moreover, we obtain the initial conditions for  $H_\theta(x, y, z; s_1; s_2|\lambda)$  by taking  $y = z = 0$  and  $x = 0$  in equation (14) in formula (14) one by one [22, 16]:

$$(25) \quad H_\theta(x, 0, 0; s_1; s_2|\lambda) = \left( \frac{\log(1 + \lambda)}{\lambda} \right)^\theta x^\theta \text{ and } H_\theta(x, y, 0; s_1; s_2|\lambda) = H_\theta(x, y; s_1|\lambda).$$

(iii) By setting  $y=0$  within expression (14), we immediately derive the next series:

$$(26) \quad H_{\theta}(x, z; s_2|\lambda) = \sum_{r=0}^{[\theta/3]} \left( \frac{\log(1+\lambda)}{\lambda} \right)^{\theta-2r} \frac{(x)^{\theta-3r} (zs_2)^r}{r!(\theta-3r)!}, \quad \theta = 0, 1, 2, \dots$$

(iv) Finally, swapping  $x = 2x$ ,  $y = -1$  and  $z = 0$  into equation (14) yields [22]:

$$(27) \quad H_{\theta}(x, -1, 0; s, s_2|\lambda) = H_{\theta}(x|\lambda).$$

Once more, for  $m = 2, 3$  in formula (18) and utilizing formulas (19), (20), we obtain the next partial differential equations for  $H_{\theta}(x, y, z; s_1; s_2|\lambda)$ :

**Corollary 2.2.** *The partial differential equations for  $H_{\theta}(x, y, z; s_1; s_2|\lambda)$  are given below:*

$$(28) \quad \frac{\partial^2}{\partial x^2} H_{\theta}(x, y, z; s_1; s_2|\lambda) = \frac{\log(1+\lambda)}{\lambda} \frac{1}{s_1} \frac{\partial}{\partial y} H_{\theta}(x, y, z; s_1; s_2|\lambda)$$

and

$$(29) \quad \frac{\partial^3}{\partial x^3} H_{\theta}(x, y, z; s_1; s_2|\lambda) = \left( \frac{\log(1+\lambda)}{\lambda} \right)^2 \frac{1}{s_2} \frac{\partial}{\partial z} H_{\theta}(x, y, z; s_1; s_2|\lambda).$$

**Remark 2.2.** *We obtain the generating function for degenerate two parameters two variables Hermite polynomials  $H_{\theta}(x, y; s_1|\lambda)$  by taking  $z = 0$  in equation (14) as follows:*

$$(30) \quad (1+\lambda)^{\frac{xt}{\lambda}} (1+\lambda)^{\frac{ys_1 t^2}{\lambda}} = \sum_{\theta=0}^{\infty} H_{\theta}(x, y; s_1|\lambda) t^{\theta}, \quad x, y \in \mathbb{C}.$$

Expanding a previous equation via (30) then juxtaposing identical powers of  $t$  on the each part, provides the series description for  $H_{\theta}(x, y; s_1|\lambda)$ :

$$(31) \quad H_{\theta}(x, y; s_1|\lambda) = \sum_{\omega=0}^{[\theta/2]} \left( \frac{\log(1+\lambda)}{\lambda} \right)^{\theta-\omega} \frac{(x)^{\theta-2\omega} (ys_1)^{\omega}}{\omega!(\theta-2\omega)!}.$$

By differentiating equation (30), with respect to  $x, y, s_1$ , we obtain the respective subsequent derivatives for  $H_{\theta}(x, y; s_1|\lambda)$ :

$$(32) \quad \frac{\partial}{\partial x} H_{\theta}(x, y; s_1|\lambda) = \frac{\log(1+\lambda)}{\lambda} H_{\theta-1}(x, y; s_1|\lambda), \quad \theta \geq 1,$$

$$(33) \quad \frac{\partial}{\partial y} H_{\theta}(x, y; s_1|\lambda) = s_1 \frac{\log(1+\lambda)}{\lambda} H_{\theta-2}(x, y; s_1|\lambda), \quad \theta \geq 2$$

and

$$(34) \quad \frac{\partial}{\partial s_1} H_{\theta}(x, y; s_1|\lambda) = y \frac{\log(1+\lambda)}{\lambda} H_{\theta-2}(x, y; s_1|\lambda), \quad \theta \geq 2.$$

**2.2. Degenerate Multi-index Hermite Polynomials.**

We generate incomplete degenerate Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$  via a subsequent generating formula:

$$(35) \quad \sum_{\theta,\omega=0}^{\infty} h_{\theta,\omega}(x, y, \tau|\lambda) \frac{u^\theta v^\omega}{\theta! \omega!} = (1 + \lambda)^{\frac{xu}{\lambda}} (1 + \lambda)^{\frac{yv}{\lambda}} (1 + \lambda)^{\frac{\tau uv}{\lambda}}.$$

By extending the right part of the formula (35) and then matching the corresponding values of  $t$  on each part of the resultant formula, we obtain the subsequent series formula.

$$(36) \quad h_{\theta,\omega}(x, y, \tau|\lambda) = \theta! \omega! \sum_{s=0}^{\min[\theta,\omega]} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{\theta+\omega-s} \frac{\tau^s x^{\theta-s} y^{\omega-s}}{(\theta - s)! (\omega - s)! s!}.$$

The boundaries condition for  $h_{\theta,\omega}(x, y, \tau|\lambda)$  is derived by inserting  $\tau = 0$  into the formula (35):

$$(37) \quad h_{\theta,\omega}(x, y, 0|\lambda) = \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{\theta+\omega} x^\theta y^\omega.$$

The theorem that is employed for demonstrating the partial derivatives for  $h_{\theta,\omega}(x, y, \tau|\lambda)$ :

**Theorem 2.3.** *For  $h_{\theta,\omega}(x, y, \tau|\lambda)$ , the subsequent partial derivatives are valid:*

$$(38) \quad \frac{\partial}{\partial \tau} h_{\theta,\omega}(x, y, \tau|\lambda) = \frac{\log(1 + \lambda)}{\lambda} \theta \omega h_{\theta-1,\omega-1}(x, y, \tau), \quad \theta, \omega \geq 1,$$

$$(39) \quad \frac{\partial}{\partial x} h_{\theta,\omega}(x, y, \tau|\lambda) = \frac{\log(1 + \lambda)}{\lambda} \theta h_{\theta-1,\omega}(x, y, \tau|\lambda), \quad \theta \geq 1,$$

$$(40) \quad \frac{\partial}{\partial y} h_{\theta,\omega}(x, y, \tau|\lambda) = \frac{\log(1 + \lambda)}{\lambda} \omega h_{\theta,\omega-1}(x, y, \tau|\lambda), \quad \omega \geq 1.$$

*Proof.* Considering the partial derivative of every part on formula (35) with the parameter  $\tau$ , we receive

$$(41) \quad \sum_{\theta,\omega=0}^{\infty} \frac{\partial}{\partial \tau} h_{\theta,\omega}(x, y, \tau|\lambda) \frac{u^\theta v^\omega}{\theta! \omega!} = uv \frac{\log(1 + \lambda)}{\lambda} (1 + \lambda)^{\frac{xu}{\lambda}} (1 + \lambda)^{\frac{yv}{\lambda}} (1 + \lambda)^{\frac{\tau uv}{\lambda}}.$$

So putting the formula (35) onto the right half within the formula (41), that we gain

$$\sum_{\theta,\omega=0}^{\infty} \frac{\partial}{\partial \tau} h_{\theta,\omega}(x, y, \tau|\lambda) \frac{u^\theta v^\omega}{\theta! \omega!} = \frac{\log(1 + \lambda)}{\lambda} \sum_{\theta,\omega=1}^{\infty} h_{\theta-1,\omega-1}(x, y, \tau|\lambda) \frac{u^{\theta-1} v^{\omega-1}}{(\theta - 1)! (\omega - 1)!}.$$

As a result, comparing the connected  $u, v$  values across every part produces statement (38).

Afterward, we calculate the partial derivatives for every value of the formula (35) in terms of  $x$  and  $y$ . This offers us

$$(42) \quad \sum_{\theta, \omega=0}^{\infty} \frac{\partial}{\partial x} h_{\theta, \omega}(x, y, \tau | \lambda) \frac{u^{\theta} v^{\omega}}{\theta! \omega!} = u \frac{\log(1 + \lambda)}{\lambda} (1 + \lambda)^{\frac{xu}{\lambda}} (1 + \lambda)^{\frac{yv}{\lambda}} (1 + \lambda)^{\frac{\tau uv}{\lambda}},$$

$$(43) \quad \sum_{\theta, \omega=0}^{\infty} \frac{\partial}{\partial y} h_{\theta, \omega}(x, y, \tau | \lambda) \frac{u^{\theta} v^{\omega}}{\theta! \omega!} = v \frac{\log(1 + \lambda)}{\lambda} (1 + \lambda)^{\frac{xu}{\lambda}} (1 + \lambda)^{\frac{yv}{\lambda}} (1 + \lambda)^{\frac{\tau uv}{\lambda}}.$$

Then, to produce assertions (39) as well as (40), respectively, we replicate the procedure described in the equation's proof (38).

Theorem 2.3 has been fully proved.  $\square$

As  $\lambda \rightarrow 0$ , formulas (14), (30) and (35) give the generating relations of  $H_{\theta}(x, y, z; s_1; s_2)$ ,  $H_{\theta}(x, y; s_1)$  [19, 18] and  $h_{\theta, \omega}(x, y, \tau)$  [8].

The application of operational forms has clearly facilitated the study of special polynomials, since we gain the operational definition for degenerate generalized Hermite polynomials  $H_{\theta}(x, y, z | \lambda)$ ,  $H_{\theta}(x, y; s_1 | \lambda)$ ,  $H_{\theta}(x, y, z; s_1, s_2 | \lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta, \omega}(x, y, \tau | \lambda)$ .

### 3. OPERATIONAL IDENTITIES

Through this portion, we discover that the application of operational forms has eased the study of special polynomials. We can explore the operational rules for the degenerate three variables Hermite polynomials  $H_{\theta}(x, y, z | \lambda)$ , degenerate two-parameter two-variable Hermite polynomials  $H_{\theta}(x, y; s_1 | \lambda)$ , degenerate two-parameter three-variable Hermite polynomials  $H_{\theta}(x, y, z; s_1, s_2 | \lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta, \omega}(x, y, \tau | \lambda)$ .

We recall the following operational representation for the degenerate two-variable Hermite polynomials  $H_{\theta}(x, y | \lambda)$  satisfied the below property [16]:

$$(44) \quad H_{\theta}(x, y | \lambda) = \exp\left(\frac{\lambda}{\log(1 + \lambda)} y \frac{\partial^2}{\partial x^2}\right) \left(\frac{x \log(1 + \lambda)}{\lambda}\right)^{\theta}.$$

**Theorem 3.1.** *The upcoming operational form is demonstrated through the  $H_{\theta}(x, y, z | \lambda)$ :*

$$(45) \quad H_{\theta}(x, y, z | \lambda) = \exp\left(\left(\frac{\lambda}{\log(1 + \lambda)}\right)^2 z \frac{\partial^3}{\partial x^3}\right) \exp\left(\frac{\lambda}{\log(1 + \lambda)} y \frac{\partial^2}{\partial x^2}\right) \left(\frac{x \log(1 + \lambda)}{\lambda}\right)^{\theta},$$

where  $\frac{\partial^2}{\partial x^2}$  and  $\frac{\partial^3}{\partial x^3}$  are the 2<sup>nd</sup> and 3<sup>rd</sup> derivative operators.

*Proof.* From formula (3), it is easy to see

$$(46) \quad \frac{\partial^{3r}}{\partial x^{3r}} H_\theta(x, y|\lambda) = \frac{\theta!}{(\theta - 3r)!} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{3r} H_{\theta-3r}(x, y|\lambda), \quad 0 \leq m \leq \frac{\theta}{3}.$$

Applying the equation (46) in the right region of the formula (6), we attain

$$(47) \quad H_\theta(x, y, z|\lambda) = \sum_{r=0}^{\infty} \frac{\left( \frac{\lambda}{\log(1+\lambda)} z \frac{\partial^3}{\partial x^3} \right)^r}{r!} H_\theta(x, y|\lambda)\{1\},$$

or, correspondingly

$$H_\theta(x, y, z|\lambda) = \exp\left( \frac{\lambda}{\log(1 + \lambda)} z \frac{\partial^3}{\partial x^3} \right) H_\theta(x, y|\lambda)\{1\}.$$

Applying formula (44) to the right half of the aforementioned equation results in the assertion (45). □

Presently, we arrive at the subsequent outcome:

**Theorem 3.2.** *The degenerate two-variable one-parameter Hermite polynomials  $H_\theta(x, y, z; s_1|\lambda)$  and degenerate three-variable two-parameter Hermite polynomials  $H_\theta(x, y, z; s_1; s_2|\lambda)$  achieve via the next respective operational identities:*

$$(48) \quad H_\theta(x, y; s_1|\lambda) = \exp\left( \frac{\lambda}{\log(1 + \lambda)} y s_1 \frac{\partial^2}{\partial x^2} \right) \frac{1}{\theta!} \left( \frac{x \log(1 + \lambda)}{\lambda} \right)^\theta$$

and

$$(49) \quad H_\theta(x, y, z; s_1; s_2|\lambda) = \exp\left( \left( \frac{\lambda}{\log(1 + \lambda)} \right)^2 z s_2 \frac{\partial^3}{\partial x^3} \right) \exp\left( \frac{\lambda}{\log(1 + \lambda)} y s_1 \frac{\partial^2}{\partial x^2} \right) \frac{1}{\theta!} \left( \frac{x \log(1 + \lambda)}{\lambda} \right)^\theta.$$

*Proof.* It is clear that

$$\begin{aligned} \frac{\partial^{2\omega}}{\partial x^{2\omega}} \left( x \frac{\log(1 + \lambda)}{\lambda} \right)^\theta &= \frac{\theta!}{(\theta - 2\omega)!} \left( \frac{x \log(1 + \lambda)}{\lambda} \right)^{\theta-2\omega} \\ &= \frac{\theta!}{(\theta - 2\omega)!} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{\theta-\omega} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{-\omega} x^{\theta-2\omega}. \end{aligned}$$

Utilizing the preceding equation to the right aspect of the formula (31), we acquire

$$H_\theta(x, y; s_1|\lambda) = \sum_{\omega=0}^{\infty} \left( \frac{\lambda}{\log(1 + \lambda)} \right)^\omega \frac{\left( y s_1 \frac{\partial^2}{\partial x^2} \right)^\omega}{\omega!} \frac{1}{\theta!} \left( x \frac{\log(1 + \lambda)}{\lambda} \right)^\theta.$$

Consequently, we arrive at the statement (48).

In view of equation (32), we have

$$\frac{\partial^{3\omega}}{\partial x^{3\omega}} H_\theta(x, y; s_1|\lambda) = \left( \frac{\log(1 + \lambda)}{\lambda} \right)^{3\omega} H_{\theta-3\omega}(x, y; s_1|\lambda).$$

Utilizing the preceding formula on the right part of the formula (15), we acquire

$$H_{\theta}(x, y, z; s_1; s_2|\lambda) = \sum_{\omega=0}^{\infty} \frac{\left(\left(\frac{\lambda}{\log(1+\lambda)}\right)^2 z s_2 \frac{\partial^3}{\partial x^3}\right)^{\omega} H_{\theta}(x, y; s_1|\lambda)}{\theta! \omega!},$$

or, equivalently

$$H_{\theta}(x, y, z; s_1; s_2|\lambda) = \exp\left(\left(\frac{\lambda}{\log(1+\lambda)}\right)^2 z s_2 \frac{\partial^3}{\partial x^3}\right) \frac{1}{\theta!} H_{\theta}(x, y; s_1|\lambda).$$

Using equation (48), we get the statement (49).

Theorem 3.2 has been fully proved.  $\square$

The operational definitions (45), (48) and (49) greatly simplifies the study of the properties of  $H_{\theta}(x, y, z|\lambda)$ ,  $H_{\theta}(x, y; s_1|\lambda)$  and  $H_{\theta}(x, y, z; s_1, s_2|\lambda)$  and their generalizations as well as from these definitions. We can now establish the analogous operational identities for  $h_{\theta, \omega}(x, y, \tau|\lambda)$ .

**Theorem 3.3.** *Regarding  $h_{\theta, \omega}(x, y, \tau|\lambda)$ , the operational definition afterward is true:*

$$(50) \quad h_{\theta, \omega}(x, y, \tau|\lambda) = \exp\left(\tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial x} \frac{\partial}{\partial y}\right) \left(x \frac{\log(1+\lambda)}{\lambda}\right)^{\theta} \left(y \frac{\log(1+\lambda)}{\lambda}\right)^{\omega}.$$

where  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  are the derivative operators.

*Proof.* It is clear that

$$\begin{aligned} \frac{\partial^s}{\partial x^s} \frac{\partial^s}{\partial y^s} \left(x \frac{\log(1+\lambda)}{\lambda}\right)^{\theta} \left(y \frac{\log(1+\lambda)}{\lambda}\right)^{\omega} &= \frac{\theta!}{(\theta-s)!} \frac{\omega!}{(\omega-s)!} \\ &\left(\frac{\log(1+\lambda)}{\lambda}\right)^{\theta-s} \left(\frac{\log(1+\lambda)}{\lambda}\right)^{\omega} \left(\frac{\log(1+\lambda)}{\lambda}\right)^{-s} x^{\theta-s} y^{\omega-s}. \end{aligned}$$

With the use of the previous equation upon the right part of the formula (36), we acquire

$$h_{\theta, \omega}(x, y, \tau) = \sum_{s=0}^{\infty} \frac{\left(\frac{\lambda}{\log(1+\lambda)}\right)^s \tau^s \frac{\partial^s}{\partial x^s} \frac{\partial^s}{\partial y^s}}{s!} \left(x \frac{\log(1+\lambda)}{\lambda}\right)^{\theta} \left(y \frac{\log(1+\lambda)}{\lambda}\right)^{\omega},$$

or, equivalently

$$h_{\theta, \omega}(x, y, \tau) = \sum_{s=0}^{\infty} \frac{\left(\tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial x} \frac{\partial}{\partial y}\right)^s}{s!} \left(x \frac{\log(1+\lambda)}{\lambda}\right)^{\theta} \left(y \frac{\log(1+\lambda)}{\lambda}\right)^{\omega}.$$

We gain the statement (50) through the use of the series expansion of exponential function into the right part of the previous formula.

We have finished proving Theorem 3.3.  $\square$

To illustrate our section, we give actual examples by expressing the operational identities of the degenerate generalized Hermite polynomials  $H_\theta(x, y, z|\lambda)$ ,  $H_\theta(x, y; s_1|\lambda)$ ,  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ .

**Example 3.1.** *By applying formulas (44), we derive the results that follow:*

$$H_3(x, y, z|\lambda) = \exp\left(\frac{\lambda}{\log(1+\lambda)} z \frac{\partial^3}{\partial x^3} x\right) \exp\left(\frac{\lambda}{\log(1+\lambda)} y \frac{\partial^2}{\partial x^2}\right) \left(\frac{x \log(1+\lambda)}{\lambda}\right)^3.$$

**Example 3.2.** *By applying formulas (48), we derive the results that follow:*

$$H_5(x, y; s_1|\lambda) = \exp\left(\frac{\lambda}{\log(1+\lambda)} y s_1 \frac{\partial^2}{\partial x^2}\right) \frac{1}{5!} \left(\frac{x \log(1+\lambda)}{\lambda}\right)^5.$$

**Example 3.3.** *By applying formulas (49), we derive the result that follows:*

$$H_7(x, y, z; s_1; s_2|\lambda) = \exp\left(\frac{\lambda}{\log(1+\lambda)} y s_1 \frac{\partial^2}{\partial x^2}\right) \exp\left(\frac{\lambda}{\log(1+\lambda)} z s_2 \frac{\partial^3}{\partial x^3}\right) \frac{1}{7!} \left(\frac{x \log(1+\lambda)}{\lambda}\right)^7.$$

**Example 3.4.** *By applying formulas (50), we derive the result that follows:*

$$h_{3,7}(x, y, \tau|\lambda) = \exp\left(\tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial x} \frac{\partial}{\partial y}\right) \left(x \frac{\log(1+\lambda)}{\lambda}\right)^3 \left(y \frac{\log(1+\lambda)}{\lambda}\right)^7.$$

The concept of quasi-monomials to special functions has enabled research into their features, particularly several operators and other key identities. In this part, we cultivate the  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and  $h_{\theta,\omega}(x, y, \tau|\lambda)$  are quasi-monomials.

#### 4. QUASI-MONOMIALITY PROPERTIES

In this section, we explore monomiality properties for degenerate two-parameter three-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ .

The monomiality of degenerate two-parameter three-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  is established by achieving the next result:

**Theorem 4.1.** *The degenerate two-parameter three-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  are quasi-monomial via regard for the next multiplicative and differential operators:*

$$(51) \quad \hat{M} = \frac{1}{(\theta+1)} \left( \frac{\log(1+\lambda)}{\lambda} x + 2y s_1 \frac{\partial}{\partial x} + 3 \frac{\lambda}{\log(1+\lambda)} z s_2 \frac{\partial^2}{\partial x^2} \right).$$

and

$$(52) \quad \hat{P} = \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial x}.$$

*Proof.* By calculating the partial derivative of each facet of equation (14) concerning  $t$ , we can easily obtain

$$(53) \quad \sum_{\theta=1}^{\infty} H_{\theta}(x, y, z; s_1, s_2 | \lambda) \theta t^{\theta-1} = \frac{\log(1+\lambda)}{\lambda} (x + 2ys_1t + 3zs_2t^2) \\ (1+\lambda)^{\frac{xt}{\lambda}} (1+\lambda)^{\frac{ys_1t^2}{\lambda}} (1+\lambda)^{\frac{zs_2t^3}{\lambda}}, \quad x, y \in \mathbb{C}.$$

Using equation (14), we acquire

$$(54) \quad \sum_{\theta=1}^{\infty} H_{\theta}(x, y, z; s_1, s_2 | \lambda) \theta t^{\theta-1} = \frac{\log(1+\lambda)}{\lambda} x \sum_{\theta=0}^{\infty} H_{\theta}(x, y, z; s_1, s_2 | \lambda) t^{\theta} \\ + 2 \frac{\log(1+\lambda)}{\lambda} y s_1 \sum_{\theta=1}^{\infty} H_{\theta-1}(x, y, z; s_1, s_2 | \lambda) t^{\theta} \\ + 3 \frac{\log(1+\lambda)}{\lambda} z s_2 \sum_{\theta=2}^{\infty} H_{\theta-2}(x, y, z; s_1, s_2 | \lambda) t^{\theta}, \quad x, y \in \mathbb{C}.$$

From equation (17), we achieve

$$(55) \quad \frac{\partial^2}{\partial x^2} H_{\theta}(x, y, z; s_1, s_2 | \lambda) = \left( \frac{\log(1+\lambda)}{\lambda} \right)^2 H_{\theta-2}(x, y, z; s_1, s_2 | \lambda), \quad \theta \geq 2.$$

Applying formulae (17) and (55), we obtain

$$(\theta+1)H_{\theta+1}(x, y, z; s_1, s_2 | \lambda) = \left( x + 2ys_1 \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial x} \right. \\ \left. + 3zs_2 \left( \frac{\lambda}{\log(1+\lambda)} \right)^2 \frac{\partial^2}{\partial x^2} \right) \frac{\log(1+\lambda)}{\lambda} H_{\theta}(x, y, z; s_1, s_2 | \lambda).$$

or, equivalently

$$H_{\theta+1}(x, y, z; s_1, s_2 | \lambda) = \frac{1}{\theta+1} \left( \frac{\log(1+\lambda)}{\lambda} x + 2ys_1 \frac{\partial}{\partial x} \right. \\ \left. + 3zs_2 \frac{\lambda}{\log(1+\lambda)} \frac{\partial^2}{\partial x^2} \right) H_{\theta}(x, y, z; s_1, s_2 | \lambda).$$

The monomiality rule (9) generates an expression for the multiplicative operator  $\hat{M}$  (51).

Assertion (52) originates straightaway from equations (17) and (10).  $\square$

**Remark 4.1.** Altering the multiplicative and derivative operators at expressions (51) and (52) into monomiality formula (12), we gain the next differential equation of degenerate 3-variable Hermite polynomials  $H_{\theta}(x, y, z | \lambda)$ :

$$(56) \quad \left( 3 \left( \frac{\lambda}{\log(1+\lambda)} \right)^2 z s_2 \frac{\partial^3}{\partial x^3} + 2 \frac{\lambda}{\log(1+\lambda)} y s_1 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - \theta \right) H_\theta(x, y, z; s_1, s_2 | \lambda) = 0.$$

Also, from equations (13) and (51), it follows that

$$(57) \quad \left( \frac{1}{(\theta+1)} \left( \frac{\log(1+\lambda)}{\lambda} x + 2y s_1 \frac{\partial}{\partial x} + 3 \frac{\lambda}{\log(1+\lambda)} z s_2 \frac{\partial^2}{\partial x^2} \right) \right)^\theta \{1\} = H_\theta(x, y, z; s_1, s_2 | \lambda),$$

which verifies series expansion (15) for degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau | \lambda)$ .

**Theorem 4.2.** *The quasi-monomials for degenerate multi-index Hermite polynomials  $h_{\theta,\omega}(x, y, \tau | \lambda)$  are affected via the subsequent multiplicative and derivative operators:*

$$(58) \quad \hat{M}_{xh} = \frac{\log(1+\lambda)}{\lambda} \left( x + \tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial y} \right),$$

$$(59) \quad \hat{M}_{yh} = \frac{\log(1+\lambda)}{\lambda} \left( y + \tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial x} \right)$$

and

$$(60) \quad \hat{P}_h = \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial \tau}.$$

*Proof.* Considering the partial derivative of every portion of equation (35) with respect to  $u$ , we acquire

$$\begin{aligned} \sum_{\theta,\omega=0}^{\infty} h_{\theta,\omega}(x, y, \tau | \lambda) \frac{\partial}{\partial u} \cdot \frac{u^\theta v^\omega}{\theta! \omega!} &= \frac{\partial}{\partial u} (1+\lambda)^{\frac{xu}{\lambda}} (1+\lambda)^{\frac{yv}{\lambda}} (1+\lambda)^{\frac{\tau uv}{\lambda}} \\ &\quad + (1+\lambda)^{\frac{xu}{\lambda}} (1+\lambda)^{\frac{yv}{\lambda}} \frac{\partial}{\partial u} (1+\lambda)^{\frac{\tau uv}{\lambda}}, \end{aligned}$$

which gives

$$\begin{aligned} \sum_{\theta=1, \omega=0}^{\infty} h_{\theta,\omega}(x, y, \tau | \lambda) \frac{u^{\theta-1} v^\omega}{(\theta-1)! \omega!} &= x \frac{\log(1+\lambda)}{\lambda} (1+\lambda)^{\frac{xu}{\lambda}} (1+\lambda)^{\frac{yv}{\lambda}} (1+\lambda)^{\frac{\tau uv}{\lambda}} \\ &\quad + \tau v \frac{\log(1+\lambda)}{\lambda} (1+\lambda)^{\frac{xu}{\lambda}} (1+\lambda)^{\frac{yv}{\lambda}} (1+\lambda)^{\frac{\tau uv}{\lambda}}. \end{aligned}$$

Using equations (35), (42) and (43) in the right part of above equation, we get

$$(61) \quad \sum_{\theta=1, \omega=0}^{\infty} h_{\theta,\omega}(x, y, \tau | \lambda) \frac{u^{\theta-1} v^\omega}{(\theta-1)! \omega!} = \frac{\log(1+\lambda)}{\lambda} \sum_{n=0}^{\infty} (x + \tau v) h_{\theta,\omega}(x, y, \tau | \lambda) \frac{u^\theta v^\omega}{\theta! \omega!}.$$

From formula (43) then we compare the coefficients that have same powers of  $t$  on each part of formula (61), we gain

$$(62) \quad h_{\theta+1,\omega}(x, y, \tau|\lambda) = \frac{\log(1+\lambda)}{\lambda} \left( x + \tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial y} \right) h_{\theta,\omega}(x, y, \tau|\lambda),$$

which from formula (9), we get assertion (58).

Moreover, to obtain the claim (59), we employ the partial derivative of each side of formula (35) for  $v$  and apply the procedure for claim (58).

From equations (10) and (38), gives assertion (60).

Theorem 4.2 has been fully proved.  $\square$

**Remark 4.2.** *Substituting the multiplicative and derivative operators at expressions (58) or (59), (60) into monomiality formula (12), we deduce the next differential equation of the degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ :*

$$(63) \quad \left( x \frac{\partial}{\partial \tau} + \tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial y} \frac{\partial}{\partial \tau} - \theta \omega \right) h_{\theta,\omega}(x, y, \tau|\lambda) = 0,$$

or, equivalently

$$(64) \quad \left( y \frac{\partial}{\partial \tau} + \tau \frac{\lambda}{\log(1+\lambda)} \frac{\partial}{\partial x} \frac{\partial}{\partial \tau} - \theta \omega \right) h_{\theta,\omega}(x, y, \tau|\lambda) = 0,$$

Also, from equations (13) and (51), it follows that

$$(65) \quad \left( x \frac{\log(1+\lambda)}{\lambda} + \tau \frac{\partial}{\partial y} \right)^\theta \{1\} = h_{\theta,\omega}(x, y, \tau|\lambda),$$

or, equivalently

$$(66) \quad \left( y \frac{\log(1+\lambda)}{\lambda} + \tau \frac{\partial}{\partial x} \right)^\theta \{1\} = h_{\theta,\omega}(x, y, \tau|\lambda),$$

which verifies series expansion (15) for and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ .

## 5. CONCLUSIONS

Hermite polynomials are useful for solving boundary value problems in various fields, including numerical analysis and combinatorics. Their applications are in the chaotic light, quantum harmonic oscillators, optic wave transfer and the theory of quantum mechanics problems. Several branches of mathematics are important in many fields and have a strong interest in special functions. Hermite polynomials are a significant subset of these distinguished polynomial sequences. Mathematicians have studied degenerate forms of specific Hermite polynomials and numbers to improve their analytical properties. In this explanation, we demonstrate a variety of new properties associated with degenerate generalized and multi-index Hermite polynomials, as well as the related formalism. In previous sections, we

discussed how to employ the degenerate generalized and multi-index Hermite for enhancing the scenario. This paper explored the degenerate 2-parameter 3-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate 1-parameter 2-variable Hermite polynomials  $H_\theta(x, y; s_1|\lambda)$  through generating formula along with their series definitions and derivative relations. Also, we presented the operational rules for the degenerate three variables Hermite polynomials  $H_\theta(x, y, z|\lambda)$ , degenerate two parameters two variables Hermite polynomials  $H_\theta(x, y; s_1|\lambda)$ , degenerate two parameters three variables Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ . Moreover, we explored the monomiality properties for degenerate two-parameter three-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ . To illustrate our conclusions, we gave actual examples by expressing the degenerate operational identities of the generalized Hermite polynomials  $H_\theta(x, y, z|\lambda)$ ,  $H_\theta(x, y; s_1|\lambda)$ ,  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate incomplete Hermite polynomials  $h_{\theta,\omega}(x, y, \tau|\lambda)$ .

This study could be valuable both theoretically and practically, particularly in fields involving distributions and combinatorics. Here are some possible applications for this study: Combined Analysis and Degenerate Theory: degenerate 2-parameter 3-variable Hermite polynomials  $H_\theta(x, y, z; s_1, s_2|\lambda)$  and degenerate 1-parameter 2-variable Hermite polynomials  $H_\theta(x, y; s_1|\lambda)$  represent the number of partition of degenerate generalized Hermite polynomials.

- (1) Advantages of solving complex challenges: These polynomials aid in the resolution of intricate mathematical issues that cannot be addressed using standard approaches. For example, they can be found as solutions to differential equations in physics and engineering.
- (2) Modeling Physical Phenomena: They are especially important in mathematical physics, where they have been employed to describe the quantum harmonic oscillator and Schrödinger equation solutions, as well as in information theory and statistical mechanics.
- (3) Versatile Applications across Disciplines: Their usefulness extends to practical applications in signal processing, image processing, and financial mathematics, where they may simulate variables such as stock prices and interest rates.
- (4) Theoretical Insights and Frameworks: Investigating degenerate forms of classical polynomials (such as Legendre, Laguerre and Hermite) yields deeper insights into their fundamental structure, behavior, and underlying mathematical concepts. They frequently result in the discovery of novel symmetry and combinatorial features.
- (5) Facilitating Advanced Techniques: They are frequently built utilizing sophisticated operational formalisms and umbral calculus, which provide systematic ways for manipulating functions and efficiently deriving novel conclusions and identities.
- (6) The study of degenerate special polynomials is a dynamic and expanding field of study because it combines theoretical mathematical studies with the need for varied and powerful answers to real-world,

multidisciplinary problems. The capacity to generate hybrid and multivariable versions of these polynomials broadens their potential for modeling even more complicated, multidimensional systems and phenomena in a variety of scientific and engineering fields.

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#### REFERENCES

- [1] R. Alyusof, *Quasi-monomiality principle and certain properties of degenerate hybrid special polynomials*, *Symmetry* 15 (2023), 407.
- [2] G. E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, Cambridge 71, (1999).
- [3] P. Appell, *Sur une classe de polynômes*, *Ann Sci École Norm Sup.* 9 (1880), 119–144.
- [4] L. Carlitz, *A degenerate Staudt-Clausen theorem*, *J. Archiv der Mathematik* 7 (1956), 25–33.
- [5] L. Carlitz, *Degenerate stirling, bernoulli and eulerian numbers*, *Utilitas Math.* 15 (1979), 31–83.
- [6] J. Choi, *Notes on formal manipulations of double series*, *Commun. Korean Math. Soc.* 18 (2003), 781–789.
- [7] G. Dattoli, G. Chiccoli, S. Lorenzutta, G. Maino, A. Torre, *Generalized Bessel functions and generalized Hermite polynomials*, *J. Math. Anal. Appl.* 178 (1993), 509–516.
- [8] G. Dattoli, A. Torre, S. Lorenzutta, G. Maino, G. Chiccoli, *Multivariable Hermite polynomials and phase-space dynamics*, In: *Proc. 3rd Internat. Workshop on Squeezed States and Uncertainty Relations*, U.B.C., Baltimore, Maryland-USA, (1993).
- [9] G. Dattoli, G. Chiccoli, S. Lorenzutta, G. Maino, A. Torre, *Theory of generalized Hermite polynomials*, *Comput. Math. Appl.* 28 (1994), 71–83.
- [10] G. Dattoli, A. Torre, M. Carpanese, *Operational rules and arbitrary order Hermite generating functions*, *J. Math. Anal. Appl.* 227 (1998), 98–111.
- [11] G. Dattoli, *Generalized polynomials, operational identities and their applications*, *J. Comput. Appl. Math.* 118 (2000), 111–123.
- [12] G. Dattoli, *Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle*, *Advanced Special Functions and Applications* (Melfi, 1999), 147–164, *Proc. Melfi Sch. Adv. Top. Math. Phys.*, 1, Aracne, Rome, (2000).
- [13] D. V. Dolgy, T. Kim, *Some explicit formulas of degenerate Stirling numbers associated with the degenerate special numbers and polynomials*, *Proc. Jangjeon Math. Soc.* 2 (2018), 309–317.
- [14] K.-W. Hwang, Y.-S. Seol, C.-S. Ryoo, *Explicit identities for 3-variable degenerate Hermite Kampé de Fériet polynomials and differential equation derived from generating function*, *Symmetry* 13 (2020), 7.
- [15] K.-W. Hwang, C. S. Ryoo, *Some identities involving two-variable partially degenerate Hermite polynomials induced from differential equations and structure of their roots*, *Mathematics* 8 (2020), 632.
- [16] K.-W. Hwang, C. S. Ryoo, *Differential equations associated with two variable degenerate Hermite polynomials*, *Mathematics* 8 (2022), 228.
- [17] S. Khan, G. Yasmin, N. Ahmad, *On a new family related to truncated exponential and Sheffer polynomials*, *J. Math. Anal. Appl.* 418 (2014), 921–937.
- [18] R. Khan, S. Khan, *Lie-theoretic generating relations involving multi-variable Hermite-Tricomi functions*, *Integral Transforms Spec. Funct.* 20 (2009), 365–375.

- [19] R. Khan, S. Khan, *Operational calculus associated with certain families of generating functions*, Commun. Korean Math. Soc. 30 (2015), 429–438.
- [20] T. Kim, D. San Kim, H. Lee, J. Kwon, *Degenerate binomial coefficients and degenerate hypergeometric functions*, Adv. Difference Equ. 1 (2020), 17pp.
- [21] T. Kim, D. San Kim, *Degenerate Laplace transform and degenerate gamma function*, Russ. J. Math. Phys. 24 (2017), 241–248.
- [22] T. Kim, D. San Kim, L-C. Jang, H. Lee, H. Kim, *Representations of degenerate Hermite polynomials*, Adv. in Appl. Math. 139 (2022), 102359, 18 pp.
- [23] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960, reprinted by Chelsea Publ. Co., Bronx, New York, (1971).
- [24] N. Raza, U. Ahmad, S. Khan, *Operational-umbral approach to bivariate degenerate Hermite polynomials and their partial orthogonality*, arXiv preprint arXiv:2508.21338 (2025).
- [25] M. Riyasat, T. Nahid, S. Khan, *An algebraic approach to degenerate Appell polynomials and their hybrid forms via determinants*, Acta Math. Sci. 43 (2023), 719–735.
- [26] C. S. Ryoo, *Some Identities Involving Hermite Kampé de Fériet Polynomials Arising from Differential Equations and Location of Their Zeros*, Mathematics 7 (2018), 23.
- [27] C. S. Ryoo, J.-Y. Kang, *Some identities involving degenerate  $q$ -Hermite polynomials arising from differential equations and distribution of their zeros*, Symmetry 14 (2022), 706.
- [28] D. San Kim, G. W. Jang, *A note on degenerate gamma function and degenerate Stirling number of the second kind*, Adv. Stud. Contemp. Math. 28 (2018), 207–214.
- [29] D. San Kim, T. Kim, J. Kwon, H. Lee, *A recursive formula for sums of values of degenerate falling factorials*, Contemp. Math. 6 (2025), 1138–1149
- [30] H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, Ellis Horwood Ltd., Chichester; Halsted Press, New York, (1984).
- [31] J. F. Steffensen, *The poweroid, an extension of the mathematical notion of power*, Acta Math. 73 (1941), 333–366.
- [32] M. Zayed, S. Wani, *Exploring the versatile properties and applications of multidimensional degenerate Hermite polynomials*, Aims Math. 8 (2023), 30813–30826.

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