

## THE TRANSCENDENTAL VALUES OF SPECIAL FUNCTIONS

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**ABSTRACT.** In general, there is no framework for the irrationality or transcendency of the analytic functions unless some special functions. Some of consequences are widely known on the special functions satisfying differential equations. Firstly, we consider E-functions and then the G-functions introduced by Siegel in 1921. Secondly, we consider the meromorphic functions satisfying differential equations as in the so called Schneider-Lang Theorem. Lastly, we deal with analytic functions satisfying functional equations.

### 1. WEIERSTRASS DREAM

E. Strauss (1886) tried to prove that a transcendental function which is analytic in an open domain  $D$  of  $\mathbb{C}$  containing 0 cannot take rational values at all points of  $D$ . According to P. Stäckel, K. Weierstrass sent him a letter where he supplied him with a counterexample.

For a transcendental function  $f$ , denote  $E_f$  the set of algebraic numbers  $\alpha$  such that  $f(\alpha)$  is algebraic. Thanks to the work by A. Hurwitz (1891), P. Stäckel (1895), G. Faber (1904), C.G. Lekkerkerker (1949), A.O. Gel'fond (1965), K. Mahler (1965), it is known that if  $S$  is a countable subset of  $\mathbb{C}$  and  $T$  is dense subset of  $\mathbb{C}$ , there exist transcendental functions  $f$  mapping  $S$  into  $T$ , as well as its derivatives.

**Theorem 1.1** (A.J. van der Poorten [18]). *There are transcendental entire functions  $f$  such that  $D^k f(\alpha) \in \mathbb{Q}(\alpha)$  for all  $k \geq 0$  and all algebraic  $\alpha$ .*

**Theorem 1.2** (J. Huang, D. Marques, M. Merb [19]). *For each countable subset  $A$  of  $\mathbb{C}$  and each family of dense subsets  $E_{\alpha,s}$  of  $\mathbb{C}$  indexed by  $(\alpha, s) \in A \times \mathbb{N}$ , there exists a transcendental entire function  $f: \mathbb{C} \rightarrow \mathbb{C}$  such that  $f^{(s)}(\alpha) \in E_{\alpha,s}$ .*

**Theorem 1.3** (D. Marques and G. Moreira (2014)). *There exists many transcendental entire functions  $f$  with the property that both  $f$  and its inverse function assume algebraic value at algebraic points.*

### 2. RIEMANN ZETA FUNCTION

**Definition 2.1.** (Euler-Mascheroni Constant) *The Euler-Mascheronic constant defined by the following limit:*

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log N \right) = \gamma = 0.5777\ 215\ 664\ 9\dots$$

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**The Basel Problem (1644):**  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

In 1644, Pietro Mengoli (1626 - 1686) asked the exact value of the sum

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = 1.644934\cdots$$

Euler found the exact sum to be  $\pi^2/6$  and announced this discovery in 1735

**2.1. Euler's Proof.** The sum of the inverse of the roots of a polynomial  $f$  with  $f(0) = 1$  is  $-f'(0)$ : For

$$1 + a_1z + a_2z^2 + \cdots + a_nz^n = (1 - \alpha_1z)\cdots(1 - \alpha_nz)$$

we have  $\alpha_1 + \cdots + \alpha_n = -a_1$ .

Write

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Set  $z = x^2$ .

The zeroes of the function

$$\frac{\sin \sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots$$

are  $\pi^2, 4\pi^2, 9\pi^2, \dots$  hence the sum of the inverse of these numbers is

$$\sum_{n \geq 1} \frac{1}{n^2 \pi^2} = \frac{1}{6}$$

$$\frac{\sin x}{x} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2 \pi^2}\right), \quad \frac{\sin x}{x} = 1 - \frac{x^2}{6} + \cdots \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

His arguments were based on manipulations that were not justified at the time, although he was later proven correct, and it was not until 1741 that he was able to produce a truly rigorous proof.

## 2.2. Special Values of Zeta Function.

**Definition 2.2.** (Riemann Zeta Function) *The complex function which is defined for  $\Re(s) > 1$  by the Dirichlet series*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

**Definition 2.3.** (Bernoulli numbers) *The Bernoulli numbers  $B_n$  are a sequence of signed rational numbers that can be defined by the exponential generating function*

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n \geq 1} (-1)^{n+1} B_n \frac{t^{2n}}{(2n)!},$$

$$\bullet B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, \dots$$

**Theorem 2.1** (Euler's formula).

$$\zeta(2n) = 2^{2n-1} \frac{B_n}{(2n)!} \pi^{2n} \quad (n \geq 1).$$

$$\bullet \zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}.$$

**Theorem 2.2.**  $\zeta(s)$  has a meromorphic continuation to  $\mathbb{C}$  with a unique pole in  $s = 1$  of residue 1.

$$\lim_{s \rightarrow 1} \left( \zeta(s) - \frac{1}{s-1} \right) = \gamma.$$

**Definition 2.4.** (Euler Gamma Function)

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \frac{(1 + 1/n)^s}{1 + s/n} = \int_0^{\infty} x^{s-1} e^{-x} dx.$$

**Theorem 2.3.** (Beta Function)

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

**Theorem 2.4.** (Functional Equation of the Riemann Zeta Function)

Connection between  $\zeta(s)$  and  $\zeta(1-s)$  :

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s).$$

- Trivial zeroes of the Riemann zeta function -2, -4, -6, ...
- **Riemann Hypothesis** : The non-trivial zeroes of the Riemann zeta function have real part 1/2.

**Theorem 2.5** ([9], F. Lindemann).  $\pi$  is a transcendental number, hence  $\zeta(2k)$  also for  $k \geq 1$ .

**Theorem 2.6** ([9], Hermite-Lindemann). Transcendence of  $\log \alpha$  and  $e^\beta$  for  $\alpha$  and  $\beta$  non-zero algebraic numbers with  $\log \alpha \neq 0$ .

**2.3. Diophantine Question.** Determine all algebraic relations among the numbers

$$\zeta(2), \zeta(3), \zeta(5), \zeta(7), \dots$$

**Conjecture 2.1.** There is no algebraic relation among these numbers ; the numbers

$$\zeta(2), \zeta(3), \zeta(5), \zeta(7), \dots$$

are algebraically independent.

- In particular the numbers  $\zeta(2n+1)$  and  $\zeta(2n+1)/\pi^{2n+1}$  for  $n \geq 1$  are expected to be transcendental.

**Theorem 2.7** ([20], R. Apéry). The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1.202056903159594285399738161511\dots$$

is irrational.

**Theorem 2.8** ([16], T.Rivoal). Let  $\varepsilon > 0$ . For any sufficiently large odd integer  $a$ , the dimension of the  $\mathbb{Q}$ -vector space spanned by the numbers  $1, \zeta(3), \zeta(5), \dots, \zeta(a)$  is at least

$$\frac{1-\varepsilon}{1+\log 2} \log a.$$

- In [21], W. Zudilin : At least one of the four numbers

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11)$$

is irrational.

**Theorem 2.9** (S. Fischler, W. Zudilin).

*There exists odd integers  $j_1$  and  $j_2$  with  $5 \leq j_1 \leq 139$  and  $5 \leq j_2 \leq 1961$  such that the four numbers  $1, \zeta(3), \zeta(j_1), \zeta(j_2)$  are linearly independent over  $\mathbb{Q}$ .*

### 3. MAHLER'S FUNCTIONS

The name *Fredholm series* is often wrongly attributed to the power series

$$\chi_2(z) = \sum_{n \geq 0} z^{2^n}$$

According to [23], Ferdholm studied rather the theta series

$$\sum_{n \geq 0} z^{n^2}$$

**Theorem 3.1** ([22], A. J. Kempner). *The number  $\chi_2(1/2)$  is transcendental number*

In [23], Much more general results were achieved by K. Mahler in 1930, and then in 1969 he get the following theorem.

**Theorem 3.2** (K. Mahler (1969)). *The transcendence of the values at algebraic points of the function*

$$\chi_d(z) = \sum_{n \geq 0} z^{d^n} \quad \text{for } d \geq 2$$

The point is that this function satisfies a functional equation

$$\chi_d(z) = z + \chi_d(z^d) \quad (|z| < 1).$$

In order to prove algebraic independence of of values, it suffices to prove the algebraic independence of the functions.

**Theorem 3.3** ([4], P. Bundschuh and K.Väänänen ). *For  $l > 1$ , denote by  $\Phi_l$  the  $l$ -th cyclotomic polynomial and set  $\Phi_1(x) = 1 - x$ . For  $d \geq 2$ , define*

$$F_{d,l}(x) = \prod_{j \geq 0} \Phi_l(z^{d^j})$$

*Given positive integers  $d \geq 2$  and  $l \geq 1$ . The the following statements are equivalent*

- (i)  $d$  is composite or does not divide  $l$*
- (ii)  $F_{d,l}$  is hypertranscendental.*
- (iii)  $F_{d,l}$  is not a rational function.*

Using the functional equation

$$F_{d,l}(z) = \Phi_l(z) F_{d,l}(z^d)$$

together with Mahler's method, they deduce deep results of algebraic independence on the values of this infinite product and its derivatives.

4. *E*-FUNCTIONS

In [1-3], consider *E*-functions and then the *G*-functions introduced by Siegel in 1929.

**Definition 4.1.** (*E*-function) Consider a power series

$$f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n \in \mathbb{Q}[[z]]$$

such that

- $a_n$  increases at most exponentially in  $n$
- $f$  satisfies a linear differential equation with coefficients in  $\mathbb{Q}(z)$
- The common denominator of  $a_0, a_1, \dots, a_n$  increase at most exponentially in  $n$ .
  - The general definition replaces the rational numbers  $a_n$  by algebraic numbers.
  - Examples of *E*-functions are algebraic constants, polynomials with algebraic coefficients,  $e^z$ ,  $\cos z$  and  $\sin z$

Bessel's function of index 0 is also a *E*-function

$$J_0(z) = \sum_{n \geq 0} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

which is a solution of the Bessel differential equation

$$y'' + \frac{1}{z}y' + y = 0.$$

Generally, for  $\lambda \in \mathbb{C}$ , if we define

$$K_\lambda(z) = \sum_{n \geq 0} \frac{(-1)^n}{(\lambda + 1)_n n!} \left(\frac{z}{2}\right)^{2n},$$

then the function

$$J_\lambda(z) = \sum_{n \geq 1} \frac{(-1)^n (z/2)^{2n+\lambda}}{n! \Gamma(n+1+\lambda)} = \frac{1}{\Gamma(\lambda+1)} \left(\frac{z}{2}\right)^\lambda K_\lambda(z)$$

is a solution of the differential equation

$$z^2 y'' + z y' + (z^2 - \lambda^2) y = 0,$$

and  $J_{-\lambda}(z)$  is a solution of the same differential equation. Modified Bessel functions of the first kind are

$$I_\lambda(z) = \sum_{n \geq 1} \frac{(z/2)^{2n+\lambda}}{n! \Gamma(n+1+\lambda)} = i^{-\lambda} J_\lambda(iz).$$

Further examples of *E*-functions are Siegel hypergeometric *E*-functions. Let  $a_1, \dots, a_l, b_1, \dots, b_m$  be rational numbers with  $m > l$  and  $b_1, \dots, b_m$  not in  $\{0, -1, -2, \dots\}$  and  $b_m = 1$ .

Define

$$c_n = \frac{(a_1)_n \cdots (a_l)_n}{(b_1)_n \cdots (b_m)_n}$$

Set  $t = m - l$ . Then

$$f(z) = \sum_{n \geq 1} c_n z^{tn}$$

is an *E*-function.

**Theorem 4.1.** *Let  $K$  be a number field,  $E_1, E_2, \dots, E_n$  be  $E$ -functions which are algebraically independent over  $K(z)$  and satisfy a system of linear differential equations*

$$y_i' = \sum_{j=1}^n f_{ij}(x)y_j \quad (i = 1, \dots, n)$$

*with  $f_{ij} \in K[z]$  and  $\alpha$  be a non-zero algebraic number in  $K$  not pole of the  $E_i$ . Then  $E_1(\alpha), E_2(\alpha), \dots, E_n(\alpha)$  are algebraically independent.*

5. G-FUNCTIONS

**Definition 5.1.** *Consider a power series*

$$g(z) = \sum_{n \geq 0} a_n z^n \in \mathbb{Q}[[z]]$$

*such that*

- *$g$  has a positive radius of convergence*
- *$g$  satisfies a linear differential equation with coefficients in  $\mathbb{Q}(z)$ .*
- *The common denominators of  $a_0, a_1, \dots, a_n$  increase at most exponentially in  $n$*

The general definition replaces the rational numbers  $a_n$  by algebraic numbers. From the definitions, it is clear that  $\sum_{n \geq 0} a_n z^n$  is a  $G$ -function if and only if  $\sum_{n \geq 0} (a_n/n!)z^n$  is an  $E$ -function. Examples of  $G$ -functions are algebraic functions, Gauss hypergeometric functions  $F_1^2 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} z^n$  with rational parameters  $a, b, c$ , solutions of Picard-Fuchs equations over  $\mathbb{Q}(z)$ .

In [6-7], Fischler and Rivoal introduce the set  $\mathbb{G}$  of all values taken by any analytic continuation of any  $G$ -function at any algebraic point. They prove that  $\mathbb{G}$  is a countable subring of  $\mathbb{C}$  which contains the field  $\overline{\mathbb{Q}}$  of algebraic numbers and the logarithms of algebraic numbers. Conjecturally,  $\mathbb{G}$  is not a field. According to a conjecture of Bombieri and Dwork,  $\mathbb{G}$  should coincide with the set of periods of algebraic varieties defined over  $\overline{\mathbb{Q}}$ .

It is expected that  $\gamma$  does not belong to the field of fractions of  $\mathbb{G}$ .

It is natural to ask.

*Let  $g$  be a  $G$ -function which is not algebraic. Is it true that  $g(\alpha)$  is algebraic for at most finitely many algebraic  $\alpha$ ?*

In [25], for Gauss hypergeometric functions, the answer is given by a result of Wolfart : *Let  $f(z) = F_1^2 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| z \right)$  with  $a, b, c \in \mathbb{Q}$ . Let  $\Delta$  be the monodromy group and*

$$\mathbb{E} = \{ \alpha \in \overline{\mathbb{Q}} \mid f(\alpha) \in \overline{\mathbb{Q}} \}$$

- (1) *If  $f$  is algebraic ( $\Delta$  finite), then  $\mathbb{E} = \overline{\mathbb{Q}}$*
- (2) *If  $f$  is arithmetic, then  $\mathbb{E}$  is dense in  $\overline{\mathbb{Q}}$*
- (3) *Otherwise,  $\mathbb{E}$  is finite.*

Let  $a = 1/12, b = 5/12, c = 1/2$ . Then the quaternion algebra is  $M(2, \mathbb{Q})$  and the monodromy group is  $SL(2, \mathbb{Z})$ . It can be show that

$$F_1^2 \left( \begin{matrix} 1/12, 5/12 \\ 1/2 \end{matrix} \middle| 1 - \frac{1}{J(\tau)} \right)^4 = \frac{E_4(\tau)}{E_4(i)}$$

where

$$E_4(\tau) = 1 + 240 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n}, \quad \Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}$$

and  $q = e^{2\pi i \tau}$ ,  $J(\tau) = E_4(\tau)^3 / 1728 \Delta(\tau)$ . In particular  $J(i) = 1$ . From CM-theory it follows that if  $\tau_0 \in \mathbb{Q}(i)$ ,  $\text{Im}(\tau_0) > 0$ , then both  $J(\tau_0)$  and  $E_4(\tau_0)/E_4(i)$  are algebraic.

### 6. SCHNEIDER-LANG THEOREM

**Theorem 6.1** (Schneider-Lang (1966)). *Let  $d \geq 2$  be an integer and  $f_1, \dots, f_d$  be meromorphic functions of finite order growth. Assume  $f_1$  and  $f_2$  are algebraically independent. Let  $K$  be a number field. Assume that for  $1 \leq i \leq d$ , the derivative  $f_i'$  of  $f_i$  belongs to the ring  $K[f_1, \dots, f_d]$ . Then the set of  $w \in \mathbb{C}$  which are not pole of  $f_1, \dots, f_d$  and such that  $f_i(w) \in K$  for  $i = 1, 2, \dots, d$  is finite.*

An extension of the Schneider-Lang Theorem in several complex variables includes a number of further results, in particular Baker's Theorem on linear independence of logarithms of algebraic numbers, as well as a number of results related with abelian functions and algebraic groups. Among these results is the Theorem of Schneider (1941) on the transcendence of  $B(a, b)$  for rational numbers  $a$  and  $b$  such that  $a + b$  is not a negative integer.

### 7. LINEAR AND ALGEBRAIC INDEPENDENCE

The transcendence of  $\Gamma(a/b)$  for  $a/b \in \mathbb{Q}$  is known only for a restricted set of values of  $a/b$ : in the interval  $(0, 1)$ , we know that the numbers

$$\Gamma(1/6), \Gamma(1/4), \Gamma(1/3), \Gamma(1/2), \Gamma(2/3), \Gamma(3/4), \Gamma(5/6)$$

are transcendental, but we do not know any further irrational value of  $\Gamma$ . It is conjectured by Deligne, Rohrlich and Lang that any algebraic relations among Gamma values at rational points should belong to the ideal of relations generated by the standard relations, namely

*Translation:*

$$\Gamma(a + 1) = \Gamma(a),$$

*Reflection:*

$$\Gamma(a)\Gamma(1 - a) = \frac{\pi}{\sin(\pi a)}$$

and

*Multiplication:* For any positive number  $n$ ,

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na)$$

An example is

$$\frac{\Gamma(1/24)\Gamma(11/24)}{\Gamma(5/24)\Gamma(7/24)} = \sqrt{3}\sqrt{2+\sqrt{3}}$$

a root of  $X^4 - 12X^2 + 9$ .

It is not known whether the three numbers  $\Gamma(1/5), \Gamma(2/5)$  and  $e^{\pi\sqrt{5}}$  are algebraically independent: this would follow from the above mentioned conjecture of Deligne-Rohrlhc-Lang, as shown by F. Adiceam, who also deduces from Nesternko's result that each of the three numbers

$$\begin{aligned} & \Gamma(1/20)\Gamma(3/20)\Gamma(7/20)\Gamma(9/20) \\ & \Gamma(1/5)\Gamma(7/20)\Gamma(9/2) \\ & \Gamma(1/5)^{-1}\Gamma(1/20)\Gamma(3/20) \end{aligned}$$

is transcendental over the field  $\mathbb{Q}(\pi, e^{\pi\sqrt{5}})$ .

## 8. CONCLUSION

In summary, the journey from Euler's pioneering work on the Basel problem to the modern understanding of  $E$ -functions,  $-G$ -functions, and the values of the Gamma function reveals the profound depth of transcendental number theory. While significant milestones have been achieved—such as Apéry's irrationality proof and the transcendence results by Mahler and Siegel—many fundamental questions, including the algebraic independence of various zeta values, remain completely open. Exploring these special values not only bridges diverse domains of pure mathematics but also continually challenges our deep understanding of the boundary between algebraic structures and analytic transcendence.

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