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SOME NEW IDENTITIES INVOLVING TRUNCATED EXPONENTIAL BASED APOSTOL-GENOCCHI POLYNOMIALS

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Abstract. This study defines the generating function of a mixed-type polynomial called the Truncated Exponential-based Apostol-Genocchi polynomials of order γ. We additionally identify various important characteristics and identities, such as the implicit summation formula, the correlation formula, and the formulas for their partial derivatives. In addition, we research various connections with Stirling numbers.

1. Introduction and Preliminaries

The present study aims to present the vital characteristics of $G^{(k)}(x)$, the generalized Genocchi polynomials of order k. When $k = 1$, we obtain the classical Genocchi polynomials:

$$G^{(1)}(x) = G_n(x).$$

If $x = 0$, we obtain the generalized Genocchi numbers of order k:

$$G^{(k)}(0) = G^{(k)}.$$

When $k = 1$, these reduce to the usual Genocchi numbers:

$$G_n(0) = G_n.$$

For details on the generalized Bernoulli numbers $B^{(k)}(x)$ and the generalized Euler polynomials $E^{(k)}(x)$, which are closely related to $G^{(k)}(x)$, the reader may refer to the classical works of Nörlund, who has extensively studied these families of polynomials. Several references to Genocchi's work can also be found in [10]. Although Genocchi polynomials share many properties with Bernoulli and Euler polynomials, they possess certain unique and distinctive characteristics. Indeed, Genocchi polynomials are interesting in their own right.

This article presents new results concerning Genocchi polynomials, and some of the observations on Genocchi numbers may also be new.

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Genocchi Numbers

The definition of Genocchi numbers (see [10]) is as follows:

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\qquad |t|<\pi. \tag{1.1}$$

From this generating function, we observe that $G_0=0$.

Since $\tanh t$ is an odd function, the function $\frac{t}{\tanh t}$ is even in t. Thus, we conclude that

$$G_1=1,\qquad G_{2n+1}=0\quad (n=1,2,3,\dots). \tag{1.2}$$

The first few Genocchi numbers are (see [10]):

n	0	1	2	4	6	8	10	12	14	16	18	20
G_n	0	1	−1	1	−3	17	−155	2073	−38227	929567	−28820619	1109652905

Designating the Bernoulli numbers by B_n (i.e., $B_n = B^{(1)}(0)$), we know (see [10]) that:

Genocchi's Theorem.

$$G_{2m}=2m\,E_{2m-1}(0), \tag{1.3}$$

where $E_n(x)$ denotes the Euler polynomials.

According to [10], the Euler numbers E_n are defined by

$$E_n=2^nE_n\left(\tfrac{1}{2}\right),$$

and not by the value $E_n(0)=E^{(1)}(0)$.

A recent appearance of Genocchi numbers occurs in [10], where the author investigates integers arising from the Bessel function $J_1(z)$. In this context, Genocchi numbers appear as part of the value of $\sigma_{2n}(\nu)$, the Rayleigh function.

Generalized Genocchi Polynomials

Definition and Basic Properties. The generalized Genocchi polynomials of order k, denoted by $G_n^{(k)}(x)$, are defined by

$$\left(\frac{2t}{e^t+1}\right)^k e^{xt}=\sum_{n=0}^{\infty}G_n^{(k)}(x)\frac{t^n}{n!},\qquad (k=0,1,2,\dots). \tag{1.4}$$

This definition implies that

$$G_n^{(k)}(x)=0\qquad (n<k), \tag{1.5}$$

and moreover,

$$G_n^{(0)}(x)=x^n.$$

Polynomials and special number sequences play important roles in various scientific disciplines, including pure and applied mathematics, physics, and engineering. They are frequently encountered in areas such as fluid dynamics, number theory, quantum mechanics, differential equations, and mathematical physics (see [4, 16, 17]). Duran et al. [8] studied Bell-based Bernoulli polynomials and their applications. Husain et al. [12] investigated the properties of Bell-based Apostol–Bernoulli polynomials, while Khan et al. [13] explored Bell-based Euler polynomials and their applications.

Inspired by these developments, the present study examines the Apostol–Genocchi polynomials of degree η in the framework of Bell polynomials. We establish several properties, including correlation formulas, derivative identities, implicit summation formulas, relations with Stirling numbers, and several special cases. Furthermore, we derive symmetric identities and discuss their connections to known identities in the literature.

To introduce a new family of Apostol–Genocchi polynomials of order γ, we make use of the truncated exponential polynomials $e_m(x)$ (see [1, 6, 19, 20]) defined as follows:

$$e_m(x) = \sum_{r=0}^{m} \frac{x^r}{r!} \quad (m \in \mathbb{N}_0) \tag{1.6}$$

which are the first $m+1$ terms of the Maclaurin series for e^x.

The gamma function[1, 20])

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\mathbb{R}(z) > 0) \tag{1.7}$$

in particularly,

$$\Gamma(m+1) = m! = \int_0^\infty e^{-t} t^{m}\, dt \quad (m \in \mathbb{N}_0) \tag{1.8}$$

we obtain(see,[6])

$$e_m(x) = \frac{1}{m!} \int_0^\infty e^{-t} (x+t)^{m}\, dt \quad (m \in \mathbb{N}_0) \tag{1.9}$$

The Truncated Exponential polynomials $e_m(u)$ are defined by the generating function (see [6],

$$\frac{e^{xt}}{(1-t)!} = \sum_{m=0}^{\infty} e_m(x) t^m, \quad (|t| < 1). \tag{1.10}$$

The generating relation (1.10)can be easily derived by taking the Cauchy product of the two Maclaurin series e^{xt} and $\frac{1}{1-t}$. Differentiating both sides of identity(1.10) with respect to the variables t and x, respectively, yields the following differential- recurisve relation (see [6]

$$\frac{d}{dx} e_m(x) = e_{m-1}(x), \quad (m \in \mathbb{N}) \tag{1.11}$$

and

$$e_{m+1}(x) = \left(1 + \frac{x}{(m+1)!}\left(1 - \frac{d}{dx}\right)\right) e_m(x) \quad (m \in \mathbb{N}_0). \tag{1.12}$$

These two relations are incorporated to give the second-order differential equation (see [6].

$$\left[x\frac{d^2}{dx^2} - (m+x)\frac{d}{dx} + 1\right] e_m(x) = 0, \quad (m \in \mathbb{N}_0). \tag{1.13}$$

The applications of Genocchi polynomials can be observed in number theory and classical analysis. They play important role in integral representation of differentiable periodic functions. They are also used to approximate the differentiable periodic functions in terms of polynomials.[22, 3, 11].

The Genocchi polynomials $G_n(x)$ and the Genocchi numbers $G_n(0)$ are defined by the following generating function (see [3, 21]):

$$e^{xt}\left(\frac{2t}{e^t+1}\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}G_n(x), \quad (|t|<\pi) \tag{1.14}$$

If we take $x=0$, then the Genocchi number $G_n(0)=G_n$ is defined by

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}G_n(x), \quad (|t|<\pi) \tag{1.15}$$

The generating function of the Genocchi polynomials of order γ (see [21, 23]) is as follows:

$$\sum_{n=0}^{\infty}G_n^{(\gamma)}(x\frac{t^n}{n!}=e^{xt}\left(\frac{2t}{e^t+1}\right)^{\gamma}, \quad (|t|<\pi;\ 1^{\gamma}:=1) \tag{1.16}$$

If we take $x=0$ in (1.14), i.e., $G_n^{(\gamma)}(0)=G_n^{(\gamma)}$, they are called the Genocchi numbers, defined as follows:

$$\sum_{n=0}^{\infty}G_n^{(\gamma)}\frac{t^n}{n!}=\left(\frac{2t}{e^t+1}\right)^{\gamma}, \tag{1.17}$$

The generating function of the generalized Apostol-Genocchi polynomial $G_n(x,\lambda)$ of order $\gamma\in\mathbb{C}$ (see [5, 18, 23]) is as follows:

$$\{e^{xt}\}\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma}=\sum_{n=0}^{\infty}\frac{t^n}{n!}G_n^{(\gamma)}(x;\lambda), \quad (t+\log\lambda<\pi;\ 1^{\gamma}:=1), \tag{1.18}$$

with

$$G_n^{(\gamma)}(x;1)=G_n^{(\gamma)}(x), \tag{1.19}$$

and

$$G_n^{(\gamma)}(0;\lambda)=G_n^{(\gamma)}(\lambda), \tag{1.20}$$

They are known as the Apostol-Genocchi numbers $G_n^{(\gamma)}(\lambda)$ of order γ. The second-kind Stirling polynomials, denoted as $S_2(n,k;x)$, and Stirling numbers, denoted as $S_2(n,k)$, have generating functions defined as (see[1])

$$\sum_{n\geq 0}S_2(n,k;x)\frac{t^n}{n!}=\frac{(e^t-1)^k}{k!}, \tag{1.21}$$

when $x = 0$ in (1.21), i.e., $S_2(n,k)$ are known as the Stirling numbers (exponential generating function), the generating function is as follows (see[1])

$$\sum_{n\geq 0} S_2(n,k)\frac{t^n}{n!} = \frac{e^t-1}{k!}. \tag{1.22}$$

2. New Family of truncated exponential based Apostol-Genocchi polynomials

Here, we define the generating function of the new family of truncated exponential based Genocchi-Apostol polynomials of order γ. We also derive correlation formulae for truncated exponential based Genocchi-Apostol polynomials of order γ.

Definition 2.1 The generating function of truncated exponential based Apostol-Genocchi polynomials of order γ can be defined in the following way.
For arbitrary $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$, we defined truncated Exponential Based Apostol Genocchi polynomials of order γ by means of following generating function:

$$\frac{e^{xt}}{(1-yt)}.\left(\frac{2t}{\lambda e^t-1}\right)^{\gamma} = \sum_{n=0}^{\infty}({}_{\mathrm{e}}G_n^{(\gamma)})(x,y;\lambda),(|t+\log\lambda|<\pi;1^{\gamma}:=1), \tag{2.1}$$

Subsituting x = 0 and y = 1 in (2.1), let us define a Truncated Exponential based Apostol Genocchi polynomials of order γdefine as:

$$\sum_{n=0}^{\infty}({}_{\mathrm{e}}G_n^{(\gamma)})(0,1;\lambda)\frac{t^n}{n!} = \frac{1}{(1-yt)}.\left(\frac{2t}{\lambda e^t-1}\right)^{\gamma} \tag{2.2}$$

Remark 2.1 if we choose $\gamma = 0$ in 2.1 we have to reduce Truncated Exponential based Apostol Genocchi polynomials of order γ reduce to the truncated exponential polynomial.

$$\frac{e^{xt}}{(1-yt)} = \sum_{n=0}^{\infty} {}_{\mathrm{e}}G_n^{(0)}(x,y;\lambda)\frac{t^n}{n!}. \tag{2.3}$$

Remark 2.2 if we choose y = 0 , and λ= 1 in 2.1 we obtain familiar Genocchi polynomials of order $G_n^(\gamma)(x)$ of order γ (see [14,15] :

$$\sum_{n=0}^{\infty} G_{n,\mathrm{e}}^{(0)}(x,0;1) = e^{xt}\left(\frac{2t}{e^t-1}\right)^{\gamma} = \sum_{n=0}^{\infty} G_n^{(\gamma)}(x)\frac{t^n}{n!} \tag{2.4}$$

Remark 2.3 if we choose $y = 0, \gamma = 1$ and $\lambda = 1$ in 2.1 Truncated Exponential based Apostol Genocchi polynomials of order ${}_{\mathrm{e}}G_n^{(\gamma)}(x,y)$ reduces to usual Genocchi polynomial $B_n(x)$

$$\sum_{n=0}^{\infty} G_{n,\mathrm{e}}^{(1)}(x,0;1) = e^{xt}\left(\frac{2t}{e^t-1}\right) = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \tag{2.5}$$

Remark 2.4 If we choose $\lambda = 1$ then 2.1 reduces to truncated based polynomials of order γ .

$$\sum_{n=0}^{\infty} G_{n,\mathrm{e}}^{(\gamma)}(x,y) = \left(\frac{e^{xt}}{1-yt}\right)\left(\frac{2t}{e^t-1}\right)^{\gamma} = \sum_{n=0}^{\infty} G_{n,\mathrm{e}}^{(\gamma)}(x,y)\frac{t^n}{n!}. \tag{2.6}$$

Theorem 1. *For $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$, the following relation holds true:*

$$_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda) = \sum_{k=0}^{n}\binom{n}{k} G_k^{(\gamma)}(x;\lambda)e_{n-k}(y). \tag{2.7}$$

Proof. By using relation (2.1), we have:

$$\sum_{n\geq 0} {}_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma}\cdot\frac{e^{xt}}{1-yt}.$$

$$= \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} e^{xt}\cdot\frac{e^{xt}}{1-yt}.$$

$$= \left(\sum_{k\geq 0} G_k^{(\gamma)}(x;\lambda)\frac{t^k}{k!}\right)\cdot\left(\sum_{n\geq 0} E_n(y)\frac{t^n}{n!}\right). \tag{2.8}$$

Applying series rearrangement, we obtain

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} G_k^{(\gamma)}(x;\lambda)E_{n-k}(y)\right)\frac{t^n}{n!}. \tag{2.9}$$

After simplification using series rearrangement, we obtain the desired result (2.7). □

Theorem 2. *Consider $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$, the following relation holds true:*

$$_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda) = \sum_{k=0}^{\infty}\binom{n}{k} G_k^{(\gamma)}(\lambda)E_{n-k}(x,y) \tag{2.10}$$

Proof. By using result (2.1), we obtain

$$_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma}\cdot\frac{e^{2xt}}{(1-yt)}.$$

$$= \left\{\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma}\right\}\cdot\left\{\frac{e^{2xt}}{(1-yt)}\right\}.$$

$$= \left\{\sum_{k\geq 0} G_k^{(\gamma)}(\lambda)\frac{t^k}{k!}\right\}\cdot\left\{\sum_{n\geq 0} E_n(x,y)\frac{t^n}{n!}\right\}. \tag{2.11}$$

Applying the series rearrangement, we obtain

$$= \sum_{n\geq 0}\left\{\sum_{k=0}^{\infty}\binom{n}{k} G_k^{(\gamma)}(\lambda)E_{n-k}(y)\right\}\frac{t^n}{n!}. \tag{2.12}$$

After simplification using series rearrangement, we obtain the result (2.10).

□

Theorem 3. *For $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$, following relation holds true:*

$$ {}_eG_n^{(\gamma)}(x,y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} {}_eG_k^{(\gamma)}(y;\lambda) \cdot 2x^{n-k}. \tag{2.13}$$

Proof. By using result (2.1), we obtain

$$\begin{aligned} {}_eG_n^{(\gamma)}(x,y;\lambda) \cdot \frac{t^n}{n!} &= \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \cdot \frac{e^{2xt}}{(1-yt)}. \\ &= \left\{\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma}\right\}\left\{\frac{1}{(1-yt)}\right\} \cdot e^{2xt}. \\ &= \left\{\sum_{\substack{k\geq 0\\ \mathrm{e}}} G_k^{(\gamma)}(y;\lambda)\frac{t^k}{k!}\right\} \cdot \left\{\sum_{n\geq 0} \frac{2xt^n}{n!}\right\}. \\ &= \left\{\sum_{n\geq 0}\sum_{\substack{k\geq 0\\ \mathrm{e}}} G_k^{(\gamma)}(y;\lambda)\frac{t^{n+k}}{k!} \cdot \frac{x^n}{n!}\right\}. \end{aligned} \tag{2.14}$$

Applying the series rearrangement, we obtain

$$= \sum_{n\geq 0}\left\{\sum_{k=0}^{n}\binom{n}{k} {}_{\mathrm{e}}G_k^{(\gamma)}(y;\lambda) \cdot 2x^{n-k} \cdot y^k\right\}\frac{t^n}{n!}. \tag{2.15}$$

After simplification by using series rearrangement, we obtain the result (2.13). □

Theorem 4. *Consider $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$, the following relation holds true:*

$$ {}_eG_n^{(\gamma)}(x+y,z;\lambda) = \sum_{k=0}^{n}\binom{n}{k} G_k^{(\gamma)}(x;\lambda)E_{n-k}(y,z). \tag{2.16}$$

Proof. By using result (2.1), we obtain

$$\begin{aligned} {}_{\mathrm{e}}G_n^{(\gamma)}(x+y,z;\lambda) &= \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \cdot \frac{e^{(2x+y)t}}{(1-zt)}. \\ &= \left\{\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \cdot\right\}\left\{\frac{e^{(2x+y)t}}{(1-zt)}\right\}. \\ &= \left\{\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \cdot e^{2xt}\right\}\left\{\frac{e^{2yt}}{(1-zt)}\right\}. \\ &= \left\{\sum_{n\geq 0} G_k^{(\gamma)}(x;\lambda)\frac{t^k}{k!}\right\} \cdot \left\{\sum_{n\geq 0} E_n(y;z)\frac{t^n}{n!}\right\}. \end{aligned} \tag{2.17}$$

After simplification by using series rearrangement, we obtained the result (2.16). □

3. Implicit Summation Formulae

This section devoted to derive the useful identities such as the implicit summation formulae for the truncated exponential based Apostal polynomials of order γ.

Theorem 5. *For arbitrary $n \in \mathbb{N}$ and $\gamma_1, \gamma_2 \in \mathbb{C}$, we have the following identity:*

$$ {}_eG_n^{(\gamma_1+\gamma_2)}(x_1+x_2, y; \lambda) = \sum_{k=0}^{n} \binom{n}{k} {}_eG_k^{(\gamma_1)}(x_1, y; \lambda)\, {}_eG_{n-k}^{(\gamma_2)}(x_2, y; \lambda) \tag{3.1}$$

Proof. By using the following identity in relation (2.1):

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma_1+\gamma_2} \frac{e^{2(x_1+x_2)t}}{1-yt} = \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma_1} \frac{e^{2x_1t}}{1-yt} \left(\frac{t}{\lambda e^t-1}\right)^{\gamma_2} \frac{e^{2x_2t}}{1-yt}$$

$$= \left(\sum_{k\geq 0} {}_eG_k^{(\gamma_1)}(x_1, y; \lambda)\frac{t^k}{k!}\right) \cdot \left(\sum_{n\geq 0} {}_eG_n^{(\gamma_2)}(x_2, y; \lambda)\frac{t^k}{k!}\right)$$

$$= \sum_{n\geq 0}\sum_{k\geq 0} {}_eG_k^{(\gamma_1)}(x_1, y; \lambda)\frac{t^k}{k!} \cdot {}_eG_n^{(\gamma_2)}(x_2, y; \lambda)\frac{t^n}{n!}$$

$$= \sum_{n\geq 0}\sum_{k\geq 0} {}_eG_k^{(\gamma_1)}(x_1, y; \lambda)\frac{t^k}{k!}\, {}_eG_n^{(\gamma_2)}(x_2, y; \lambda)\frac{t^n}{n!}$$

$$= \sum_{n\geq 0}\left\{\sum_{k\geq 0}\binom{n}{k} {}_eG_k^{(\gamma_1)}(x_1, y; \lambda)\, {}_eG_{n-k}^{(\gamma_2)}(x_2, y; \lambda)\right\}\frac{t^n}{n!} \tag{3.2}$$

After simplification by using series rearrangement, we obtain the result (3.1).

$$= \sum_{k=0}^{n}\binom{n}{k} {}_eG_k^{(\gamma_1)}(x_1, y; \lambda)\, {}_eG_{n-k}^{(\gamma_2)}(x_2, y; \lambda) \tag{3.3}$$

□

Theorem 6. *For arbitrary $n \in \mathbb{N}$ and $\gamma_1, \gamma_2 \in \mathbb{C}$, we have the following identity:*

$$ {}_e\bigcap G_{k+l}^{(\gamma)}(x, y; \lambda) = \sum_{n=0}^{k}\sum_{m=0}^{l}\binom{k}{n}\binom{l}{m}(x-z)^{n+m}\, {}_eG_{k+l-n-m}^{(\gamma)}(x, y; \lambda), \tag{3.4}$$

Proof. We use the following well-known series manipulation formula:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{3.5}$$

By replacing t with $t+v$ in (2.1), we obtain:

$$\left\{\left(\frac{2(t+v)}{\lambda e^{t+v}+1}\right)^{\gamma}\right\}\left\{e^{y(e^{t+v}-1)}\right\} = e^{-2x(t+v)}\sum_{k,l=0}^{\infty} {}_eG_{k+l}^{(\gamma)}(x, y; \lambda)\frac{(t+v)^n}{n!}.$$

$$\left\{\left(\frac{2(t+v)}{\lambda e^{t+v}+1}\right)^{\gamma}\right\}\left\{e^{y(e^{t+v}-1)}\right\} = e^{-2x(t+v)}\sum_{k,l=0}^{\infty} {}_{\mathrm{e}}G_{k+l}^{(\gamma)}(x,y;\lambda)\frac{t^k}{k!}\frac{v^l}{l!}. \tag{3.6}$$

Now, replacing x with z in the equation above:

$$\left\{\left(\frac{2(t+v)}{\lambda e^{t+v}+1}\right)^{\gamma}\right\}\left\{e^{y(e^{t+v}-1)}\right\} = e^{-2z(t+v)}\sum_{k,l=0}^{\infty} {}_{\mathrm{e}}G_{k+l}^{(\gamma)}(z,y;\lambda)\frac{t^k}{k!}\frac{v^l}{l!}. \tag{3.7}$$

From equations (6) and (3.7), we get:

$$\sum_{k,l=0}^{\infty} {}_{\mathrm{e}}G_{k+l}^{(\gamma)}(x,y;\lambda)\frac{t^k}{k!}\frac{v^l}{l!} = e^{2(x-z)(t+v)}\sum_{k,l=0}^{\infty} {}_{\mathrm{e}}G_{k+l}^{(\gamma)}(z,y;\lambda)\frac{t^k}{k!}\frac{v^l}{l!}. \tag{3.8}$$

Next, we have:

$$= \sum_{n,m=0}^{\infty}(x-z)^{n+m}\frac{t^n}{n!}\frac{v^m}{m!}\sum_{k,l=0}^{\infty} {}_{\mathrm{e}}G_{k+l}^{(\gamma)}(x,y;\lambda)\frac{t^k}{k!}\frac{v^l}{l!}. \tag{3.9}$$

Finally, simplifying the expression:

$$= \sum_{k=0}^{\infty}\sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(x-z)^{n+m}\frac{t^n}{n!}\frac{v^m}{m!}\,{}_{\mathrm{e}}G_{k+l-n-m}^{(\gamma)}(x,y;\lambda), \tag{3.10}$$

which proves the result. □

Theorem 7. *For arbitrary $n \in \mathbb{N}$ and $\gamma_1, \gamma_2 \in \mathbb{C}$, we have the following identity:*

$${}_{e}G_{n+1}^{(\gamma)}(x+1,y;\lambda) - {}_{e}G_{n+1}^{(\gamma)}(x,y;\lambda) = \sum_{k=0}^{n} 2^n\binom{n+1}{k}\,{}_{e}G_{k}^{(\gamma)}(x,y;\lambda). \tag{3.11}$$

Proof. Using the relation (2.1), we get

$${}_{\mathrm{e}}G_{n+1}^{(\gamma)}(x+1,y;\lambda)\frac{t^n}{n!} - {}_{\mathrm{e}}G_{n}^{(\gamma)}(x,y;\lambda)\frac{t^n}{n!}. \tag{3.12}$$

Applying the generating function,

$$= \left(\frac{2t}{\lambda e^t+1}\right)^{(\gamma)}\frac{e^{(2x+1)t}}{1-yt} - \left(\frac{2t}{\lambda e^t+1}\right)^{(\gamma)}\frac{e^{2xt}}{1-yt}. \tag{3.13}$$

Expanding the series,

$$= \left(\frac{2t}{\lambda e^t+1}\right)^{(\gamma)}\frac{e^{(2x+1)t}}{1-yt} - \left(\frac{2t}{\lambda e^t+1}\right)^{(\gamma)}\frac{e^{2xt}}{1-yt}\sum_{n=1}^{\infty}\frac{(2t)^n}{(n-1)!}. \tag{3.14}$$

Rewriting the sum representation,

$$= \sum_{k\geq 0} {}_{\mathrm{e}}G_k(x,y;\lambda)\frac{t^k}{k!}\sum_{n\geq 0}\frac{t^n+k}{(n+1)!}\cdot 2^{n+k}. \tag{3.15}$$

Finally, simplifying leads to:

$$= \sum_{k=0}^{n} 2^n \binom{n+1}{k} {}_{\mathrm{e}}G_k^{(\gamma)}(x, y; \lambda). \tag{3.16}$$

□

Theorem 8. *For $\gamma = 1$ and $n \in \mathbb{N}$, the following relation hold true:*

$$e_n(x, y) = \frac{1}{n+1} \{\lambda_e G_{n+1}(x+1, y; \lambda) -_e G_{n+1}(x, y; \lambda)\} \tag{3.17}$$

Proof. using the result (2.1) for $\gamma = 1$ and defination of bivariate bell polynomials ,we get

$$\sum_{n\geq 0} E_n(x, y)\frac{t^n}{n!} = \frac{e^{xt}}{1 - yt}.$$

we have ,

$$\sum_{\substack{n\geq 0\\ \mathrm{e}}} G_n(x, y; \lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t + 1}\left\{\frac{e^{2xt}}{1 - yt}\right\}.$$

$$= \left(\frac{\lambda e^t - 1}{t}\right)\left\{\sum_{\substack{n\geq 0\\ \mathrm{e}}} G_n(x, y; \lambda)\frac{t^n}{n!}\right\}.$$

$$= \frac{\lambda e^t - 1}{t}\left\{\frac{2t}{\lambda e^t + 1}\frac{e^{2xt}}{1 - yt}\right\}$$

$$= \frac{1}{t}\left\{\lambda\left(\frac{2t}{\lambda e^t + 1}\right)\frac{e^{(2x+1)t}}{1 - yt} - \left(\frac{2t}{\lambda e^t + 1}\right)\frac{e^{2xt}}{1 - yt}\right\}.$$

$$= \frac{1}{t}\left\{\lambda\sum_{\substack{n\geq 0\\ \mathrm{e}}} G_n(x+1, y; \lambda)\frac{t^n}{n!} - G_{n,\mathrm{e}}(x, y; \lambda)\frac{t^n}{n!}\right\}. \tag{3.18}$$

$$= \frac{1}{n+1}\sum_{n\geq 0}\{\lambda_{\mathrm{e}} G_n(x+1, y; \lambda) -_{\mathrm{e}} G_n(x, y; \lambda)\}\frac{t^n}{n!} \tag{3.19}$$

by equating the both side , we obtained the result(3.8) □

Theorem 9. *If $n \geq 0$ and $\gamma \in \mathbb{C}$, the following relation holds:*

$${}_{\mathrm{e}}G_n^{(\gamma)}(x, y; \lambda) = \sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j} 2^k (x)_k S_2(j, k)\, {}_{\mathrm{e}}G_{n-j}^{(\gamma)}(y; \lambda), \tag{3.20}$$

where $S_2(j, k)$ denotes the Stirling numbers of the second kind.

Proof. We begin with the generating function:

$$\sum_{n=0}^{\infty} {}_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \frac{e^{2xt}}{1-yt}.$$

Rewrite $e^{2xt} = (1+(e^{2t}-1))^x$:

$$\sum_{n\geq 0} {}_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \frac{1}{1-yt}\sum_{k=0}^{\infty}(x)_k\frac{(e^{2t}-1)^k}{k!}.$$

Now apply the Stirling expansion

$$(e^{2t}-1)^k = \sum_{j=0}^{\infty} S_2(j,k)(2t)^j,$$

to obtain

$$\sum_{n\geq 0} {}_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda)\frac{t^n}{n!} = \left(\sum_{m=0}^{\infty} {}_{\mathrm{e}}G_m^{(\gamma)}(y;\lambda)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}(x)_k\sum_{j=0}^{\infty}S_2(j,k)\frac{(2t)^j}{j!}\right).$$

Rearranging the product of exponential generating functions gives

$$\sum_{n=0}^{\infty} {}_{\mathrm{e}}G_n^{(\gamma)}(x,y;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}2^k(x)_kS_2(j,k)\,{}_{\mathrm{e}}G_{n-j}^{(\gamma)}(y;\lambda)\right)\frac{t^n}{n!}.$$

Comparing coefficients of $t^n/n!$ yields the desired identity. □

4. Derivative Formula

Theorem 10. *for all n∈ $\mathbb{N}$ the partial derivative Truncated Exponential based Apostol Bernoulli polynomials of orderγ w.r.t as follows:*

$$\frac{\partial}{\partial x}\left\{ {}_{e}G_n^{(\gamma)}(x;y;\lambda)\right\} = 2x\,{}_{e}G_n^{(\gamma)}(x;y;\lambda) \tag{4.1}$$

Proof. Using the following relation:

$$\frac{\partial}{\partial x}\left(\frac{e^{2xt}}{1-yt}\right) = \frac{2t\,e^{2xt}}{1-yt}, \tag{4.2}$$

From equations (4.2) and (2.1), we have:

$$\frac{\partial}{\partial x}\left({}_{\mathrm{e}}G_n^{(\gamma)}(x;y;\lambda)\right) = 2n\,{}_{\mathrm{e}}G_{n-1}^{(\gamma)}(x;y;\lambda). \tag{4.3}$$

□

Theorem 11. *The partial derivative w.r.t y of Truncated Exponential based Bernoulli polynomials of order γ as follows:*

$$\frac{\partial}{\partial x}\left\{ {}_{e}G_n^{(\gamma)}(x;y;\lambda)\right\} = \sum_{k\geq 0}^{\infty} k\binom{n}{k}\,{}_{e}G_{n-k}^{(\gamma)}(x;y;\lambda)e_{k-1}(y) \tag{4.4}$$

Proof. First, we compute the derivative of $\frac{1}{(1-yt)}$:

$$\frac{\partial}{\partial x}\left\{\frac{1}{(1-yt)}\right\} = \frac{-1}{(1-yt)^2} \times (-t) = \frac{t}{(1-yt)^2} \tag{4.5}$$

$$= \frac{t}{(1-yt)} \cdot \frac{t}{(1-yt)} \tag{4.6}$$

Now, using the relation (2.1), we take the derivative of the series:

$$\frac{\partial}{\partial x}\left\{\sum_{n\geq 0} {}_{\mathrm{e}}G_n^{(\gamma)}(x;y;\lambda)\frac{t^n}{n!}\right\} = \frac{\partial}{\partial x}\left\{\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \cdot \frac{e^{2xt}}{1-yt}\right\}$$

$$= \left\{\left(\frac{2t}{\lambda e^t+1}\right)^{\gamma} \cdot \frac{e^{2xt}}{1-yt}\right\} \cdot \left(\frac{t}{1-yt}\right)$$

$$= \sum_{n\geq 0} {}_{\mathrm{e}}G_n^{(\gamma)}(x;y;\lambda)\frac{t^n}{n!} \cdot \sum_{k\geq 0} e_{n-k}^{y}\frac{t^k}{k!} \cdot k$$

$$= n\sum_{k\geq 0}^{n} k\binom{n}{k} {}_{\mathrm{e}}G_{n-k}^{(\gamma)}(x;y;\lambda)e_{k-1}(y)\frac{t^n}{n!} \tag{4.7}$$

By equating both sides, we obtain the result (4.3). □

5. Conclusion

Inspired by mixed polynomial applications, we propose a novel family of truncated exponential-based Apostol-Genocchi polynomials of order γ and examine their various beneficial identities, such as correlation formulas, implicit summation formulae, and derivative formulas.

Based on the findings of this study, we may define other mixed-type polynomials such as truncated exponential based Apostol-Euler polynomials and truncated exponential based Apostol-Bernoulli polynomials in the future.

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