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Some notes on Calculus of Variations and Optimal Control

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ABSTRACT. Dynamic optimization plays a central role in many scientific and engineering disciplines, where the objective is to determine a control strategy that optimizes a given performance criterion over some constraints, in particular time. Two basic approaches – Calculus of Variations and Optimal Control – are considered in their comparison. Euler-Lagrange equation is derived from the Pontryagin's Maximum Principle for the simplest problem of Calculus of Variations. It is discussed effectiveness and applicability of Euler-Lagrange equation and Pontryagin's Maximum Principle for solving a wide range of dynamic optimization problems.

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1. Introduction

Calculus of Variations and Optimal Control theory are two fundamental mathematical frameworks used to determine optimal solutions in dynamic systems. While they share many conceptual foundations, they serve different applications and utilize distinct methods. The Calculus of Variations is primarily concerned with optimizing functional — mappings from functions to real numbers — and optimal control extends these ideas to systems governed by differential equations with control inputs [2,4,6,7,9,10]. Together, these fields play a fundamental role in physics, engineering, economics, and modern control theory.

Optimal Control theory is regarded the modern branch of Calculus of Variations and includes some additional requirements to seeking process and/or the model [1,3,5,8,9,10,11].

This article provides a comparative analysis of these two disciplines, emphasizing their connections, key principles, and the relationship between Pontryagin's Maximum Principle and the Euler-Lagrange equations.

2. Calculus of Variations and Optimal Control

The Calculus of Variations studies problems of the form: find a function that minimizes (or maximizes) a given functional. A classic example is finding the curve of shortest length between two points, which leads to the concept of a geodesic. In general, a variational problem can be written as

$$J(x) = \int_a^b L(t, x(t), \dot{x}(t))dt \,,$$

where L is called the Lagrangian.

A key result in the calculus of variations is the Euler–Lagrange equation, which provides a necessary condition for optimality. Solutions to variational problems are functions that satisfy this differential equation. The calculus of variations has been historically important in classical mechanics, particularly in the principle of least action.

While the Calculus of Variations deals with optimizing over functions directly, many real-world systems evolve according to dynamic laws and can be influenced by external inputs. This leads naturally to Optimal Control theory, where the goal is to determine a control function that steers a system from an initial state to a desired final state while optimizing a performance criterion.

In this sense, Optimal Control can be viewed as a generalization of the Calculus of Variations, where constraints are imposed in the form of differential equations describing system dynamics.

An Optimal Control problem typically consists of:

- A state equation (system dynamics),
- A control variable,

- A cost functional to be minimized or maximized.

A standard formulation is:

$$\min_{u(t)} J = \int_{t_0}^{t_1} L(x(t),u(t),t)dt$$

subject to

$$\dot{x}(t) = f(x(t),u(t),t).$$

One of the central results in Optimal Control is Pontryagin's Maximum Principle, which provides necessary conditions for optimality. It introduces the Hamiltonian function and conjugate (costate) variables, drawing a strong parallel with the Euler–Lagrange equations in the calculus of variations.

A formulation of the problem of optimal control includes a control objective, a mathematical model of the controlled object, constraints and a description of a class of controls [1].

The control objective is a request expressed in a formal form for the behavior of a controlled object. An objective of the control can be, for example, a transfer of the controlled object from one position to another in a finite amount of time or to keep the trajectory of motion within given limits, etc. Often the objective of control is to optimize (maximize or minimize) an objective functional, that is, a numerical parameter specified on a set of processes. The values of the objective functional characterize a "quality" of processes. For the optimization of a functional procedure, we allocate the best quality processes from various ones.

A *mathematical model of a controlled object* is some law of transformation of controls into trajectories of an object. It can be set by a system of ordinary differential equations, partial differential equations, integral equations, recurrence relations, or in other ways.

Constraints are additional conditions for processes that arise from the physical meaning of the statement of a control problem. The requirements related with the safe operation of a controlled object lead to *phase constraints* on a state vector or to *mixed constraints* on state vectors and controls simultaneously. In particular, the initial conditions for differential equations can be regarded as the simplest phase constraints.

The class of controls is defined by specifying the analytical properties and the range of control variables. For example, we can use class of controls $K(R \to U)$ consists of piecewise continuous functions $u(t): R \to R^r$ with values in a compact $U \subset R^r$. But optimal control can use more general classes of summarizing or measurable controls that are dictated by the physical meaning of the problem or by the wish to ensure the solvability of the problem. A wider a class of controls allows for greater possibility for the optimal control to exist. However, the expansion of the class of controls requires using a more sophisticated mathematical apparatus and details of the theory of functions, functional analysis and differential equations.

Another important approach is dynamic programming, developed by Richard Bellman, which leads to the Hamilton–Jacobi–Bellman (HJB) equation. Consideration of this approach is beyond of this survey.

Principles of Optimality

- **Euler-Lagrange Equation** in Calculus of Variations: Provides necessary conditions purely based on the function $x(t)$.
- **Pontryagin's Maximum Principle**: Extends the local optimality condition to control systems by introducing the Hamiltonian

$$H(\psi, x, u, t) = \psi^T f(x, u, t),$$

and conjugate variables $\psi(t)$. The principle states that optimal controls maximize (or minimize) the Hamiltonian at each instant, along with the system dynamics and costate equations:

$$\dot{x}(t) = \frac{\partial H}{\partial \psi}, \ \dot{\psi}(t) = \frac{\partial H}{\partial x}, \ u(t) \text{ maximizes } H.$$

Comparison and Connection

- **Similarity in Variational Principles:** Both frameworks seek stationary points of integral functionals; variational calculus does so directly for functions, while optimal control incorporates control variables and system dynamics.
- **Mathematical Formulation:** Variational calculus's Euler-Lagrange equations appear as a special case of the more general Hamiltonian system underlying optimal control—specifically, the conditions derived from Pontryagin's Maximum Principle.
- **Linkage via the Principle of Maximum:** The principle of maximum in Pontryagin's theory generalizes the stationarity condition of calculus of variations

by incorporating constraints on control and dynamic system equations, allowing for more complex and realistic models.

- **Reduction of Optimal Control to Variational Problems:** When controls are explicitly defined as functions of state variables or are unconstrained, the control problem reduces to a classical variational problem, and the necessary conditions become the Euler-Lagrange equations.

3. Maximum Principle and Euler-Lagrange Equation

Consider the simplest problem of calculus of variations

$$J = \int_{t_0}^{t_1} F(x(t), \dot{x}(t), t)dt \to \min, \; x(t_0) = x^0, \; x(t_1) = x^1, \tag{1}$$

in which the minimum of the integral is sought on a set of functions $x(t)$ from the class $C_2([t_0, t_1] \to R)$ with fixed ends. The numbers t_0, t_1, x^0, x^1 and the function $F(x, \dot{x}, t)$ from the class $C_2(R \times R \times [t_0, t_1] \to R)$ are regarded as given. We assume that the problem (1) has a solution $x(t)$ and that there exists a bounded interval $V \subset R$ containing all values of the derivative $\dot{x}(t)$, $t_0 \le t \le t_1$.

We can write the problem (1) as a General problem (*G*-problem) if we put control $u = \dot{x}$ and phase variables $x_1 = x, \; x_2 = \int_{t_0}^{t} F(x(\tau), \dot{x}(\tau), \tau)d\tau$ for function $x = x(t)$. According to [1], General Optimal Control Problem (*G-problem*) is the problem that has mobile ends of an integral curve:

$$J_0 = \Phi_0(x(t_0), x(t_1), t_0, t_1) \to \min,$$
$$J_i = \Phi_i(x(t_0), x(t_1), t_0, t_1) \begin{cases} \le 0, \; i = 1, \ldots, m_0, \\ = 0, \; i = m_0 + 1, \ldots, m, \end{cases}$$
$$\dot{x} = f(x, u, t), \; u \in U, \; t_0 \le t_1.$$

Here $\Phi_0, \ldots, \Phi_m$ are the given functions of the class $C_1(R^n \times R^n \times R \times R \to R)$, m_0 is an integer nonnegative number, and m is a natural number. If $m_0 = 0$ or $m_0 = m$, then the *G*-problem only has constraints-equalities $J_i = 0, \; i = 1, \ldots, m$, or only constraints-inequalities $J_i \le 0,\ i = 1,\dots,m$, respectively. The *process* is said to be a quaternion $x(t), u(t), t_0, t_1$ that satisfies all conditions of the *G*-problem except, possibly, the first condition. A process $x(t), u(t), t_0, t_1$ is regarded to be *optimal* if for any other process $\tilde{x}(t), \tilde{u}(t), \tilde{t}_0, \tilde{t}_1$, the following inequality is true

$$\Phi_0(x(t_0), x(t_1), t_0, t_1) \le \Phi_0(\tilde{x}(\tilde{t}_0), \tilde{x}(\tilde{t}_1), \tilde{t}_0, \tilde{t}_1) \cdot$$

The *G*-problem consists of determining the optimal process.

Using this notation, the problem (1) takes the form

$$J = x_2(t_1) \to \min,\ x_1(t_0) - x^0 = 0,\ x_2(t_0) = 0,\ x_1(t_1) - x^1 = 0, \qquad (2)$$
$$\dot{x}_1 = u,\ \dot{x}_2 = F(x_1, u, t),\ u \in U,$$

where *U* is a closure V . Obviously, the triple of functions

$$x_1(t) = x(t),\ x_2(t) = \int_{t_0}^{t} F(x(\tau), \dot{x}(\tau), \tau)\, d\tau,\ u(t) = \dot{x}(t) \qquad (3)$$

is a process of the *G*-problem (2), and we write the necessary conditions of optimality for it. We form the functions

$$H(\psi, x, u) = \psi_1 u + \psi_2 F(x_1, u, t),$$

$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \lambda_0 x_2(t_1) + \lambda_1(x_1(t_0) - x^0) + \lambda_2 x_2(t_0) + \lambda_3(x_1(t_1) - x^1).$$

Basic theoretical result for solution of G-problem is obtained in [1].

Theorem (maximum principle for *G*-problem) *Let* $x(t), u(t), t_0, t_1$ *be an optimal process of the G-problem. Then there exists a vector* $\lambda = (\lambda_0, \dots, \lambda_m)$ *and a continuous solution* $\psi(t)$ *of a conjugate system of differential equations*

$$\dot{\psi} = -H_x(\psi, x(t), u(t), t),$$

satisfying conditions:

1) non-triviality, non-negativity and complementary slackness

$$\lambda \ne 0,\ \lambda_i \ge 0,\ i = 0,\dots,m_0,\ \lambda_i \Phi_i(x(\mathbf{t}), \mathbf{t}) = 0,\ i = 1,\dots,m_0;$$

2) transversality

$$\psi(t_0) = L_{x^0}(\lambda, x(\mathbf{t}), \mathbf{t}),\ \psi(t_1) = -L_{x^1}(\lambda, x(\mathbf{t}), \mathbf{t}),$$

$$\dot{L}_{t_0}(\lambda, x(\mathbf{t}), \mathbf{t}) = 0,\ \dot{L}_{t_1}(\lambda, x(\mathbf{t}), \mathbf{t}) = 0;$$

3) maximum of Hamiltonian

$$H(\psi(t), x(t), u(t), t) = \max_{u \in U} H(\psi(t), x(t), u, t)\ ,\ t \in [t_0, t_1]$$

with functions

$$L(\lambda, x^0, x^1, t_0, t_1) = \sum_{i=0}^{m} \lambda_i \Phi_i(x^0, x^1, t_0, t_1)\ ,\ H(\psi, x, u, t) = \sum_{j=1}^{n} \psi_j f_j(x, u, t) \cdot$$

By this Theorem, to ensure the optimality of a process (3), the existence of a vector $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \neq 0$, $\lambda_0 \geq 0$ is required, and a continuous solution $\psi(t) = (\psi_1(t), \psi_2(t))$ of a conjugate system

$$\dot{\psi}_1 = -F_{x_1}(x_1(t), u(t), t)\psi_2,\ \dot{\psi}_2 = 0,$$

satisfying the transversality conditions

$$\psi_1(t_0) = \lambda_1,\ \psi_2(t_0) = \lambda_2,\ \psi_1(t_1) = -\lambda_3,\ \psi_2(t_1) = -\lambda_0$$

and the condition of the stationarity of the Hamiltonian

$$H_u(\psi(t), x(t), u(t)) = \psi_1(t) + \psi_2(t) F_{\dot{x}}(x(t), \dot{x}(t), t) = 0,\ t \in [t_0, t_1].$$

Since left-hand side of trajectory, initial and terminal time are fixed corresponding variations $\Delta x_0 = 0$, $\Delta t_0 = 0$, $\Delta t_1 = 0$. By [1], we can omit transversality conditions on the left-hand side of conjugate function $\psi_1(t_0) = \lambda_1,\ \psi_2(t_0) = \lambda_2$ and derivative of Lagrange function $\frac{dL}{dt_0} = 0$, $\frac{dL}{dt_1} = 0$. This means that Lagrange function simplifies for

$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \lambda_0 x_2(t_1) + \lambda_1 (x_1(t_1) - x_1)$$

and Lagrange vector becomes $\lambda = (\lambda_0, \lambda_1)$. We relabeled index λ_3 by λ_1 in the previous form of Lagrange function. Thus, by maximum principle [1], process $x(t), u(t)$ is optimal if there exist conjugate function $\psi(t) = (\psi_1(t), \psi_2(t))$ and Lagrange vector $\lambda = (\lambda_0, \lambda_1)$ such that

1. $|\lambda_0| + |\lambda_1| > 0$, $\lambda_0 \geq 0$.
2. $\psi_1(t) = -\lambda_1$, $\psi_2(t) = -\lambda_0$.
3. $\frac{\partial H(\psi, x, u, t)}{\partial u} = 0$.

We use the conjugate equations and the transversality conditions to obtain

$$\psi_1(t) = -\lambda_1 + \lambda_0 \int_{t_1}^{t} F_{x_1}(x_1(\tau), u(\tau), \tau) d\tau, \ \ \psi_2(t) = -\lambda_0 .$$

Condition 3 of maximum of Hamiltonian gives

$$-\lambda_1 + \lambda_0 \int_{t_1}^{t} F_{x_1}(x_1(\tau), u(\tau), \tau) d\tau - \lambda_0 F_u(x_1(t), u(t), t) = 0 . \qquad (4)$$

If $\lambda_0 = 0$, then from (4) we get $\lambda_1 = 0$. The latter contradicts to non-triviality of Lagrange vector in condition 1. Therefore $\lambda_0 > 0$. Without a loss of generality, we can put $\lambda_0 = 1$ and (4) becomes

$$-\lambda_1 + \int_{t_1}^{t} F_{x_1}(x_1(\tau), u(\tau), \tau) d\tau - F_u(x_1(t), u(t), t) = 0 . \qquad (5)$$

Differentiation (5) by t arrives us at

$$F_{x_1}(x_1(t), u(t), t) - \frac{d}{dt} F_u(x_1(t), u(t), t) = 0 .$$

Replacing x_1 by x and u by $\dot{x}$ we get well-known *equation of Euler-Lagrange*

$$F_x(x(t), \dot{x}(t), t) - \frac{d}{dt} F_{\dot{x}}(x(t), \dot{x}(t), t) = 0 \qquad (6)$$

for the sought-for function $x(t)$. So, *in order for function $x(t)$ to be a solution of the simplest problem of calculus of variations (1), it is necessary for it to satisfy the Euler-Lagrange equation.*

The Euler-Lagrange equation is derived from the maximum principle with the assumption that all values of a derivative for the sought-for function are located in the interior of the range of control *U*. For control problems, this situation is not typical – that is, the values of the optimal control may belong to the boundary of *U*. For this reason, the maximum principle is in a general a more necessary optimality condition.

4. Illustrating example

Consider smooth curves $x = x(t)$ passing through the points (0,0), (1,1) of the coordinate plane. Find out which of these has the shortest length S, and write down the requirements in the form

$$S = \int_0^1 \left(1 + \dot{x}^2(t)\right)^{1/2} dt \to \min,\ x(0) = 0,\ x(1) = 1, \qquad (7)$$

we obtain the simplest problem of calculus of variations (1) with the function

$$F(x, \dot{x}, t) = \left(1 + \dot{x}^2\right)^{1/2}.$$

Compute the derivatives

$$F_x = 0,\ F_{\dot{x}} = \dot{x}\left(1 + \dot{x}^2\right)^{-1/2},\ \frac{d}{dt}F_{\dot{x}} = \ddot{x}(1 + \dot{x}^2)^{-3/2},$$

and write the Euler-Lagrange equation

$$-\ddot{x}(1 + \dot{x}^2)^{-3/2} = 0 \Leftrightarrow \ddot{x} = 0.$$

Its general solution is $x = c_1 t + c_2$, where c_1, c_2 are arbitrary constants. Then the boundary conditions of (7) are satisfied to obtain $c_1 = 1,\ c_2 = 0$ and, as a consequence, the particular solution is $x(t) = t$. Therefore, the function $x(t) = t,\ 0 \le t \le 1$ meets the necessary conditions for the extremum. This graph is a straight line with ends (0,0) and (1,1).

We show that the necessary condition of the extremum for problem (7) is a sufficient condition simultaneously. Indeed, by analogy with (2), problem (7) can be represented as a linearly-convex G-problem

$$\begin{gathered} J = x_2(1) \to \min,\ x_1(0) = 0,\ x_2(0) = 0,\ x_1(1) - 1 = 0, \\ \dot{x}_1 = u,\ \dot{x}_2 = (1 + u^2)^{1/2},\ u \in U. \end{gathered} \qquad (8)$$

According to [1], a process satisfying maximum principle is optimal solution of linearly-convex G-problem with $\lambda_0 > 0$.

Let us show that the triple of functions

$$x_1(t) = x(t) = t,\ x_2(t) = \int_0^t (1 + \dot{x}^2(t))^{1/2} dt = 2^{1/2} t,\ u(t) = \dot{x}(t) = 1 \qquad (9)$$

forms a process of the problem (8) and satisfies the maximum principle with factor $\lambda_0 = 1$. We have

$$H(\psi, x, u, t) = \psi_1 u + \psi_2 (1 + u^2)^{1/2} \text{ and}$$

$$L(\lambda, x(t_0), x(t_1), t_0, t_1) = \lambda_0 x_2(1) + \lambda_1 (x_1(1) - 1).$$

Write conjugate system

$$\begin{cases} \dot{\psi}_1 = 0 \\ \dot{\psi}_2 = 0 \end{cases}.$$

Its solution is $\psi_1(t) = C_1$, $\psi_2(t) = C_2$. By maximum principle [1], process $x(t), u(t)$ is optimal if there exist conjugate function $\psi(t) = (\psi_1(t), \psi_2(t))$ and Lagrange vector $\lambda = (\lambda_0, \lambda_1)$ such that

1. $|\lambda_0| + |\lambda_1| > 0$, $\lambda_0 \geq 0$.
2. $\psi_1(t) = -\lambda_1$, $\psi_2(t) = -\lambda_0$.
3. $\dfrac{\partial H(\psi, x, u, t)}{\partial u} = 0$.

Transversality conditions 2 give $\psi_1(t) = -\lambda_1$ and $\psi_2(t) = -\lambda_0$. Condition 3 is equivalent to

$$\frac{\partial H(\psi, x, u, t)}{\partial u} = -\lambda_1 - \lambda_0 u (1 + u^2)^{-1/2} = 0.$$

If $\lambda_0 = 0$ then $\lambda_1 = 0$ and we get trivial Lagrange vector (contradiction with 1). Thus, $\lambda_0 > 0$. In particular, $\lambda_0 = 1$ and $\lambda_1 = -\dfrac{u}{(1+u^2)^{1/2}}$. From here if $u(t) = 1$, then $\lambda_1 = -\dfrac{1}{\sqrt{2}}$. Therefore, the pair $u(t) = 1$ and $x_1(t) = x(t) = t$, $0 \leq t \leq 1$ satisfies maximum principle of linearly-convex G-problem (8) and by the Theorem, the process (9) is optimal. Thus, among all curves coupling two given points in the plane, the shortest length has a straight line. Of course, this conclusion holds for the Euclidean metric embedded in the formula used to calculate the length of the curve.

Conclusion

Calculus of Variations and Optimal Control theory are interconnected mathematical disciplines that address the problem of finding extremal solutions in dynamic systems. The Euler-Lagrange equations form the cornerstone of Calculus of Variations, offering a direct approach to problems involving fixed functions. In contrast, Pontryagin’s Maximum Principle provides a broader framework accommodating system dynamics, constraints, and control variables, with the principle of maximum serving as an extension of the stationarity condition.

Understanding this relationship not only deepens the theoretical foundation of optimal decision-making in engineering and economics but also enhances the ability to model and solve real-world problems involving complex dynamic systems.

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