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SUM OF SQUARES AND BELL POLYNOMIALS

H. Kumar[1], T. Kim[2], R. C. Singh Chandel[3], J. López-Bonilla[4]
[1]Department of Mathematics, D. A-V. Postgraduate College, Kanpur, Uttar Pradesh 208 001, India Email: palhemant2007@rediffmail.com
[2]Department of Mathematics, College of Natural Sciences, Kwangwoon University, Seoul 139-704, Republic of Korea
Email: tkkim@kw.ac.kr
[3]Former Head, Department of Mathematics, D. V. Postgraduate College, Orai, Uttar Pradesh 285 001, India; rc_chandel@yahoo.com
[4]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista, CP 07738, CDMX, México
Email: jlopezb@ipn.mx

Abstract

We use known results about sum of squares to obtain identities involving Bell polynomials, Stirling numbers and the sum of divisors function.
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1 Introduction

Li [1, 2] deduced the relation:

$$R_k(n) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} r_j(n), \quad n \geq 1, \tag{1}$$

where $r_j(n)$ and $R_k(n)$ denote the number of representations of n as a sum of j squares [3-5] and k nonvanishing squares [3], respectively.

On the other hand, we know [6-8] that $r_k(n)$ is a polynomial in k of degree n, with the structure:

$$r_k(n) = \sum_{l=1}^{n} a(n,l) k^l, \tag{2}$$

whose application in (2) gives the interesting expression:

$$R_k(n) = k! \sum_{l=1}^{n} a(n,l) S_l^{[k]}, \tag{3}$$

where participate the Stirling numbers of the second kind [9-11] such that:

$$a(n,l) = \frac{2^l}{n!} B_{n,l}(0!A(1), 1!A(2), \ldots, (n-l)!A(n-l+1)), \tag{4}$$

involving the partial Bell polynomials [6, 12-14] and the sum of divisors function $\sigma(m)$[15-17]:

$$A(n) = \begin{cases} -\left(\sigma(n) - \sigma\left(\frac{n}{2}\right)\right) \quad , \\ \quad \sigma \text{ is even} \ , \\ \quad \sigma(n), \qquad n \text{ is odd} \ , \end{cases} \tag{5}$$

besides, the values $A(1) = 1, A(2) = -2, A(3) = 4, A(4) = -4, \ldots$,
give the sequence A186690 [18]:

$$1, -2, 4, -4, 6, -8, 8, -8, 13, -12, 12, -16, 14, -16, 24, -16, 18, -26, 20, -24,$$
$$32, -24, 24, -32, 31, -28, \ldots \tag{6}$$

In Sec. 2 we employ (3), (4) and certain results for $R_k(n)$ [2,3] to obtain identities involving Stirling numbers, sum of divisors function and Bell polynomials.

2. Integers as a sum of nonvanishing squares

We know the following values for the number of representations of an integer as a sum of nonvanishing squares [2]:

$$R_1(n)=r_1(n)=\sum_{l=1}^{n}a(n,l),\quad R_k(1)=2k!S_1^{[k]}=\begin{cases}2, & k=1\\ 0, & k\geq 2\end{cases},R_k(n)=0,\quad k\geq n+1,$$

$$R_k(2)=2k!\left(S_2^{[k]}-S_1^{[k]}\right)=\begin{cases}0, & k=1\\ 4, & k=2\\ 0, & k\geq 3\end{cases},\quad R_{m-1}(m)=0,\quad m\geq 2,R_{n-2}(n)=0,\quad n\geq 3,$$

$$R_k(3)=\frac{4}{3}k!\left(2S_1^{[k]}-3S_2^{[k]}+S_3^{[k]}\right)=\begin{cases}8, & k=3\\ 0, & k\neq 3\end{cases},R_{m-3}(m)=2^{m-3}(m-3),\quad m\geq 4,$$

$$R_n(n)=n!\sum_{l=1}^{n}a(n,l)S_l^{[n]}=n!a(n,n)=2^n,\quad n\geq 1,\ldots \tag{7}$$

and some of them can be applied in (3) and (4) to deduce identities, for example:

$$\sum_{l=n-a}^{n}2^l B_{n,l}(0!A(1),\ldots,(n-l)!A(n-l+1)) \\ \times S_l^{[n-a]}=\begin{cases}0, & a=1,2\\ 3\cdot 2^n\binom{n}{4}, a=3, & n\geq a+1\end{cases} \tag{8}$$

The results indicated by Grosswald [3] for $R_k(n)$ imply via (3) and (4) that:

$$C(n,k):=\sum_{l=k}^{n}2^l B_{n,l}(0!A(1),\ldots,(n-l)!A(n-l+1))S_l^{[k]}=0, \\ n=1,2,\ldots,k-1, \tag{9}$$

and:

$$C(n,k)=\begin{cases}0, \quad k=6,7 \text{ and } n=k+b, \quad b\in Q=\{1,2,4,5,7,10,13\},\\ 0, \quad k=4 \text{ and } n=k+c, \quad c\in QU\{25,37\},\\ 0, \quad k=5 \text{ and } n=k+d, \quad d\in QU\{28\}.\end{cases} \tag{10}$$

Remark. The sequence (5) can be written in terms of the sum of inverses of odd divisors of n :

$$A(n)=(-1)^{n-1}n\sum_{odd} dln\frac{1}{d} \tag{11}$$

such that [2, 19-23]:

$$\sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \binom{n}{j} r_j(n) = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} R_j(n) = \frac{2}{n} A(n) \tag{12}$$

$$\sum_{n=1}^{\infty} A(n) q^n = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} j q^j}{1 - q^{2j}} = -\frac{1}{8} \frac{\vartheta_3''(0,q)}{\vartheta_3(0,q)}, \tag{13}$$

involving a Jacobi theta function [24, 25].

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