

CENTERFREE CONNECTED TOPOLOGICAL GROUPS WITH COMPACT NORMAL SUBGROUP

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February 16, 2026

ABSTRACT. The centerfree connected topological groups with compact connected normal subgroups are described.

§ 1. INTRODUCTION

As is well known, a centerfree connected Lie group is linear. It is less known that every group of this kind is the direct product of Iwasawa's largest compact connected centerfree normal subgroup N ([1], Theorem 14, and [5], Proposition 1), and a centerfree Lie subgroup of elements commuting with N elementwise. These assertions were generalized in [5] to centerfree connected locally compact groups.

We extend both results to centerfree connected topological groups admitting a compact connected normal subgroup. The main tool is Iwasawa's decomposition theorem ([1], Theorem 1'; see also [2], Theorem 9.82, (i)).

2020 *Mathematics Subject Classification*. Primary 43A07, Secondary 20C15, 20C99, 22C05.

Key words and phrases. Topological group, connected group, compact connected normal subgroup, Iwasawa decomposition, center.

Supported by NRC "Kurchatov Institute" - SRISA according to the project FNEF-2024-0001.

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§ 2. PRELIMINARIES

Recall the details of Theorem 2 in [1]. This theorem claims that, if G is a connected topological group and N a compact normal subgroup of G , then the set H of all elements in G commuting with every element of N is a closed normal subgroup of G and $G = HN$. Therefore, the centers Z_H of H and Z_N of N are closed subgroups of the center Z of G .

Note that an element $z \in Z$ can be represented as the product $z = h_z n_z$. It follows from

$$z = aza^{-1} = ah_z a^{-1} n_z = h_z n_z$$

for any $a \in H$ that

$$ah_z a^{-1} = h_z$$

for any $a \in H$, and hence $h_z \in Z_H$. Similarly, $n_z \in Z_N$. Next, it is clear that

$$H \cap N \subset Z_N$$

and, similarly,

$$H \cap N \subset Z_H.$$

However, by (1.17) in [1], the intersection $H \cap N$ is Z_N . Similarly, $H \cap K = Z_H$. Hence $Z_N = Z_H$. Conversely, it is clear that any element of Z_N commutes with N and H , and thus commutes with G , which means that $Z_N \subset Z$. Similarly, $Z_H \subset Z$. Thus,

$$Z \subset Z_H Z_N = Z_N = Z_H,$$

Therefore,

$$z \subset Z_H Z_N \subset Z$$

and, finally,

$$Z_N = Z_H = N \cap H = Z.$$

This completes the proof of the following assertion.

Proposition 1. *Let G be a connected topological group, let N be a compact normal subgroup of G , let H be the set of all elements in G commuting with every element of N is a closed normal subgroup of G , and thus $G = HN$ by Iwasawa's decomposition theorem (Theorem 2 in [1]). Then*

$$Z = H \cap N = Z_N = Z_H.$$

In particular, G is centerfree if and only if $N \cap H = \{e\}$, and then N and H are centerfree. In this case, G is the direct product of N and H ([1], the text after (1.17)).

Iwasawa himself noted that $H \cap N$ is contained in the center of G (see [the end of the page 515]1)

§ 3. MAIN RESULTS

Theorem 1. *Let G be a connected topological group with trivial center. Let N be a compact normal subgroup such that G/N is a Lie group. Then N is connected.*

Proof. Let there be a compact normal subgroup N such that G/N is a Lie group and $N \neq N_0$, where N_0 stands for the connected component of the identity element in N . Then the quotient group G/N_0 has nontrivial center. Indeed, N/N_0 is a totally disconnected normal subgroup of the connected group G/N_0 , and thus N/N_0 is central in G/N_0 . Therefore, the quotient group G/N_0 is not isomorphic to $H(N_0)$. On the other hand,

$$G/N_0 = (H(N_0)N_0)/N_0 = H(N_0)/(H(N_0) \cap N_0)$$

(recall that N_0 is compact and use § 3 of [3]). Therefore, the center of N_0 is nontrivial, and thus the center of G is nontrivial. A contradiction. This means that the assumption $N \neq N_0$ cannot hold.

Theorem 2. *Let G be a centerfree connected topological group with a compact normal subgroup K . Let N be the normal subgroup formed by the elements of G that commute with every element of K . Then K and N are centerfree and G is the direct product of K and N .*

Proof. The assertion immediately follows from Proposition 1.

Corollary 1. *Every centerfree connected Lie-projective topological group is a direct product of the greatest connected compact normal subgroup which contains all other connected compact normal subgroups and is a centerfree connected compact group (and thus a product of (possibly infinitely many) centerfree simple Lie groups) and the centerfree normal subgroup elementwise commuting with the compact normal subgroup.*

The proof follows immediately from the above considerations.

Corollary 2. *Every centerfree connected Lie-projective topological group admits sufficiently many continuous (not necessarily unitary) finite-dimensional representations, which means that, for every nonidentity element g of such a group, there is a continuous finite-dimensional representation of the group for which the image of the element g differs from the identity operator on the representation space.*

Proof. Choose a neighborhood O of the identity element e in G that does not contain the element g . Let N be a normal subgroup of G such that $N \subset O$ and G/N is a Lie group. Then the canonical image $\pi_N(g)$ of the element g in G/N differs from the identity element of G/N . Since the Lie group G/N is centerfree, its adjoint representation is faithful. Therefore, the composition of the canonical homomorphism π_N and the adjoint representation of G/N takes the element g to a nonidentity linear operator on the Lie algebra of G/N (the representation space of the adjoint representation of G/N).

§ 5. COMMENTS ON LIE QUOTIENTS BY CONNECTED NORMAL SUBGROUPS

There is a well-known characterization of connected finite-dimensional locally compact groups that are Lie groups. Namely, by Corollary 2.4 in [4], a finite-dimensional connected locally compact group is a Lie group if and only if the center of the group is a Lie group. This makes it possible to give a simple criterion for a locally compact group to have “sufficiently many” connected compact normal subgroups for the group to be the projective limit of quotient Lie groups by connected compact normal subgroups.

Proposition 2. *A connected locally compact group G is a projective limit of its quotient Lie groups by connected compact normal subgroups if every neighborhood O of the identity element contains a connected compact normal subgroup N_O for which the quotient group $Z/(Z \cap N_O)$, where Z stands for the center of G , is a Lie group.*

Proof. The condition concerning the quotient group of the center is necessary by Corollary 2.4 in [4]. Conversely, let N be a connected compact normal subgroup of G for which $Z/(Z \cap N)$ is a Lie group. Then the group $(ZN)/N$, which is canonically isomorphic to $Z/(Z \cap N)$, is also a Lie group. If gN , $g \in G$, commutes with every element of G/N for any N , which gives $gNhN = hNgN$ for every $h \in G$, then

$$gh = hgn(g.h)$$

and

$$g^{-1}h^{-1}gh = n(g, h)$$

for some $n = n(g, h) \in N$. Therefore, for any $h \in G$ there is an $n(g, h) \in N$ such that $h^{-1}gh = gn(g, h)$. Since, for any neighborhood O of the zero element, there is a connected normal subgroup $N = N_O$ with this property, it follows that one can choose an $n(g, h) \in O$ for any O , and thus $g^{-1}h^{-1}gh = e$ in G , which means that $g \in Z$. In turn, this means that $gN \in ZN$. We have thus proved that the center of G/N is ZN/N , which is a Lie group, as was noted above. It remains to use Corollary 2.4 in [4] in the other direction. This completes the proof of the proposition.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

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