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Refinement of Ky Fan Inequalities involving two positive arguments

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Abstract: The mathematical proof for establishing some new Ky-Fan inequalities involving centroidal mean and invariant centroidal mean, including a few well-known means for the arguments lying on the paths of triangular wave function (linear), new parabolic function (curved) and a remark on the well-known parabola $y = -2(x-1/2)^2 + 1/2$ over the interval (0, 1) are discussed. The results represent an extension as well as strengthening of Ky-Fan inequality.

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1. Introduction

The Handbook of Means and Their Inequalities, by Bullen [1], gave tremendous work on mathematical means and the corresponding inequalities involving famous means. The authors in [3, 4, 5] discussed about the relations between the well-known means and series as well as their extensions. The generalization of the means is discussed in [3, 12]. Relevant to this paper, the authors in [9, 11, 13] established the Ky Fan inequalities. In [6] authors introduced new homogeneous function; as an application, inequalities involving means are obtained. In [7, 8, 10] authors provided the simple way of establishing inequalities and their improvements. The set of arbitrary non-negative real numbers $x_i \in (0, \frac{1}{2}]$ and $x_i' = (1-x_i) \in [\frac{1}{2}, 1)$ is represented as a function in the form given by [1].

$$y = \begin{cases} x_i, & 0 < x_i \le \frac{1}{2} \\ 1 - x_i, & \frac{1}{2} \le x_i < 1 \end{cases}$$

The following are the few definitions of means extracted from the above survey papers. For given n arbitrary nonnegative real numbers $x_1, x_2, \cdots, x_n \in (0, \frac{1}{2}]$, the standard notation for the unweighted arithmetic, geometric, and harmonic means are represented by A_n, G_n and H_n are respectively, given by

$$A_n = \frac{1}{n}\sum_{i=1}^{n} y_i, \qquad G_n = \prod_{i=1}^{n} \sqrt[n]{y_i} \qquad \text{and} \qquad H_n = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{y_i}}$$

Also, the arithmetic, geometric and harmonic means of the set of elements $1-y_1, 1-y_2, \cdots, 1-y_n$ represented by A'_n, G'_n and H'_n are respectively given by

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1-y_i), \qquad G'_n = \prod_{i=1}^{n} \sqrt[n]{1-y_i} \qquad \text{and} \qquad H'_n = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{(1-y_i)}}$$

Ky-Fan initiated the popular inequalities that affected them and later was strengthened by several authors, namely Rooin [11], Sandor and Trif [13]. The inequalities obtained in this paper have numerous applications in the study of majorization problems [2]. This work motivates us to develop two double inequalities in this paper.

For two positive arguments e and f, the following means respectively called the arithmetic, geometric, harmonic, centroidal, and invariant centroidal means.

For all $e, f \in (0, \frac{1}{2}]$

$$A = \frac{e+f}{2} \qquad \text{and} \qquad A' = \frac{(1-e)+(1-f)}{2} \tag{1.1}$$

$$G = \sqrt{ef} \qquad \text{and} \qquad G' = \sqrt{(1-e)(1-f)} \tag{1.2}$$

$$H = \frac{2ef}{e+f} \qquad \text{and} \qquad H' = \frac{2(1-e)(1-f)}{(1-e)+(1-f)} \tag{1.3}$$

$$C = \frac{2(e^2+ef+f^2)}{3(e+f)} \quad \text{and} \quad C' = \frac{2[(1-e)^2+(1-e)(1-f)+(1-f)^2]}{3[(1-e)+(1-f)]} \tag{1.4}$$

$$C^d = \frac{3ef(e+f)}{2(e^2+ef+f^2)} \quad \text{and} \quad (C^d)' = \frac{3(1-e)(1-f)[(1-e)+(1-f)]}{2[(1-e)^2+(1-e)(1-f)+(1-f)^2]} \tag{1.5}$$

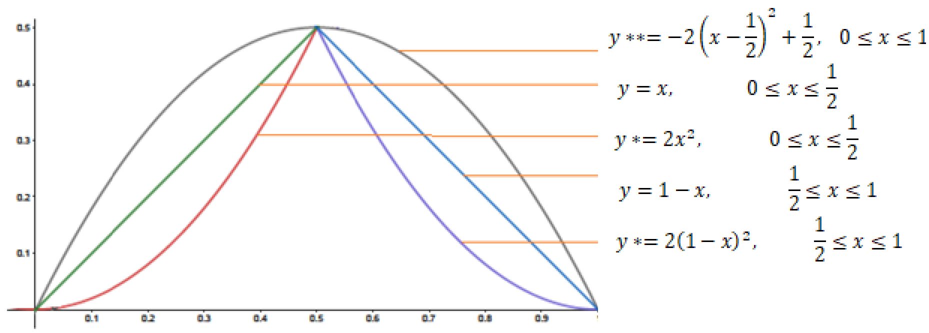
The motivation of the work carried out by the eminent researchers and discussion with experts results in the study of a function that is symmetric about the point $x = \frac{1}{2}$ and which is similar to a parabolic curve in nature, given as follows:

$$y^* = \begin{cases} 2x^2, & 0 < x \le \frac{1}{2} \\ 2(1-x)^2, & \frac{1}{2} \le x < 1 \end{cases}$$

and

$$\left\{ y^{**} = -2(x-1/2)^2 + 1/2, \qquad 0 < x < 1 \right.$$

The functions y, y^* and y^{**} are graphically represented as shown in the following figure:



The corresponding arithmetic mean A^*, geometric mean, mean G^* and harmonic mean H^* are considered for the arguments lying on the curved path of $f^*(y)$ and some important inequality chains involving them are established.

$$A_n^* = \frac{1}{n}\sum_{i=1}^{n} 2e_i^2, \qquad G_n^* = \prod_{i=1}^{n} \sqrt[n]{2^n(e_i^2)} \qquad \text{and} \qquad H_n^* = \frac{n}{\sum_{i=1}^{n} \frac{1}{2e_i^2}}$$

and

$$(A_n^*)' = \frac{1}{n}\sum_{i=1}^{n} 2(1-e_i)^2, \quad (G_n^*)' = \prod_{i=1}^{n} \sqrt[n]{2^n(1-e_i)^2} \quad \text{and} \quad (H_n^*)' = \frac{n}{\sum_{i=1}^{n} \frac{1}{2(1-e_i)^2}}$$

For two positive arguments e and f, the following means respectively called the arithmetic, geometric, harmonic, centroidal, and invariant centroidal means.

For all $e, f \in (0, \frac{1}{2}]$

$$A^* = \frac{2e^2 + 2f^2}{2} \qquad \text{and} \qquad (A^*)' = \frac{2(1-e)^2 + 2(1-f)^2}{2} \tag{1.6}$$

$$G^* = \sqrt{[2e^2][2f^2]} \qquad \text{and} \qquad (G^*)' = \sqrt{[2(1-e)^2][2(1-f)^2]} \tag{1.7}$$

$$H^* = \frac{4e^2 f^2}{e^2 + f^2} \qquad \text{and} \qquad (H^*)' = \frac{4(1-e)^2(1-f)^2}{(1-e)^2 + (1-f)^2} \tag{1.8}$$

$$C^* = \frac{4(e^4 + e^2 f^2 + f^4)}{3(e^2 + f^2)} \quad \text{and} \quad (C')^* = \frac{4[(1-e)^4 + (1-e)^2(1-f)^2 + (1-f)^4]}{3[(1-e)^2 + (1-f)^2]} \tag{1.9}$$

$$\left(C^d\right)^* = \frac{3(e^2 f^2)(e^2 + f^2)}{4(e^4 + e^2 f^2 + f^4)} \text{and} \left((C^d)'\right)^* = \frac{3[(1-e)^2(1-f)^2][(1-e)^2 + (1-f)^2]}{4[(1-e)^4 + (1-e)^2(1-f)^2 + (1-f)^4]} \tag{1.10}$$

2. Some inequalities for the arguments lie on the curve y

This section provides the analytical proof to develop a new Ky-Fan inequality chain.

Theorem 2.1. *For all $e, f \in (0, \frac{1}{2}]$ and $e^{'} = 1 - e, f^{'} = 1 - f \in [\frac{1}{2}, 1)$, then*

$$\frac{G}{G'} \leq \frac{A}{A'}.$$

Proof. From eqs. (1.1) and (1.2), one can write easily:

$$(AG^{'})^2 - (GA^{'})^2 = \frac{(e+f)^2}{4}(1 + ef - e - f) - \frac{(2-e-f)^2}{4}ef$$

which is equivalently,

$$(AG^{'})^2 - (GA^{'})^2 = \frac{-1}{4}\left[(e-f)^2(e+f-1)\right] \tag{2.1}$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$,
Equivalently, $0 < e + f < 1$, that is $e + f - 1 < 0$
Thus,

$$(AG^{'})^2 - (GA^{'})^2 = \frac{-1}{4}\left[(e-f)^2(e+f-1)\right] \geq 0 \tag{2.2}$$

That is,

$$(AG^{'}) - (GA^{'}) \geq 0 \tag{2.3}$$

Hence,

$$A^{'}G^{'}\left(\frac{G}{G'} - \frac{A}{A'}\right) \leq 0 \quad or \quad \left(\frac{G}{G'} \leq \frac{A}{A'}\right) \tag{2.4}$$

□

Theorem 2.2. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$\frac{G}{G'} \leq \frac{C}{C'}.$$

Proof. From eqs. (1.2) and (1.4), one can write easily:

$$(GC^{'})^2 - (CG^{'})^2 = (ef)\left[\frac{2[(1-e)^2 + (1-e)(1-f) + (1-f)^2]}{3(2-e-f)}\right]^2 - \left[\frac{2[e^2 + ef + f^2]}{3(e+f)}\right]^2 (1 + ef - e - f) \tag{2.5}$$

On simplification, eqn. (2.5) takes the form:

$$(GC^{'})^2 - (CG^{'})^2 = \frac{4(e-f)^2(e+f-1)}{9(e+f)^2(2-e-f)^2} \quad [\Delta] \tag{2.6}$$

where,

$$\Delta = \left[e^4 + f^4 - 4e^3 + 5f^2f^2 - 11e^2f + 4e^2 + 4ef^3 + 4e^3f - 11ef^2 + 7ef + 4f^2 - 4f^3\right]$$

on regrouping, the terms Δ can be rewritten as:

$$\Delta = \left[(e^2 - f^2)^2 + 4e^2(f-1)(e-1) + +f^2(e-1) + 7ef(1-e)(1-f)\right]$$

Again, for all $e, f \in (0, \frac{1}{2}]$, implies that $1-e>0, 1-f>0$, $e-1<0, f-1<0$, $e+f-1<0$ and $(e-f)^2>0$, based on this the value of $\Delta>0$.
Thus,

$$(GC')^2-(CG')^2 \geq 0 \quad or \quad GC'-CG' \geq 0 \tag{2.7}$$

Hence,

$$G'G'\left(\frac{G}{G'}-\frac{C}{C'}\right) \leq 0 \quad or \quad \left(\frac{G}{G'} \leq \frac{C}{C'}\right) \tag{2.8}$$

□

Theorem 2.3. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0<e<\frac{1}{2}$ and $0<f<\frac{1}{2}$, then*

$$\frac{A}{A'} \leq \frac{C}{C'}.$$

Proof. From eqs. (1.1) and (1.4), one can write easily:

$$AC'-CA' = \left(\frac{e+f}{2}\right)\left[\frac{2[(1-e)^2+(1-e)(1-f)+(1-f)^2]}{3(2-e-f)}\right] - \left[\frac{2[e^2+ef+f^2]}{3(e+f)}\right]\left(\frac{2-e-f}{2}\right) \tag{2.9}$$

On simplification, eqn. (2.9) takes the form:

$$AC'-CA' = \frac{(e-f)^2(e+f-1)}{3(e+f)(2-e-f)} \tag{2.10}$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e+f-1<0$ and $(e-f)^2>0$, based on this the value of $AC'-CA'<0$.
Hence,

$$A'C'\left(\frac{A}{A'}-\frac{C}{C'}\right) \leq 0 \quad or \quad \left(\frac{A}{A'} \leq \frac{C}{C'}\right) \tag{2.11}$$

□

Theorem 2.4. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0<e<\frac{1}{2}$ and $0<f<\frac{1}{2}$, then*

$$\frac{A}{A'} \leq \frac{C^d}{(C^d)'}.$$

Proof. From eqs. (1.1) and (1.5), one can write easily:

$$A(C^d)'-(C^d)A' = \left(\frac{e+f}{2}\right)\left[\frac{3(1-e)(1-f)(2-e-f)}{2[(1-e)^2+(1-e)(1-f)+(1-f)^2]}\right] - \left[\frac{3ef(e+f)}{2[e^2+ef+f^2]}\right]\left(\frac{2-e-f}{2}\right) \tag{2.12}$$

On simplification, eqn. (2.12) takes the form:

$$A(C^d)'-(C^d)A' = \frac{-3(e+f)(2-e-f)(e-f)^2(e+f-1)}{4(e^2+ef+f^2)[(1-e)^2+(1-e)(1-f)+(1-f)^2]} \tag{2.13}$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e+f-1<0$, $2-e-f>0$, $(1-e)(1-f)>0$ and $(e-f)^2>0$, based on this the value of $A(C^d)'-(C^d)A'>0$.
Hence,

$$A'(C^d)'\left(\frac{A}{A'}-\frac{C^d}{(C^d)'}\right) \geq 0 \quad or \quad \left(\frac{C}{(C^d)'} \leq \frac{A}{A'}\right) \tag{2.14}$$

□

Theorem 2.5. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$\frac{C^d}{(C^d)'} \leq \frac{G}{G'}.$$

Proof. From eqs. (1.2) and (1.5), one can write easily:

$$(G(C^d)')^2 - (C^dG')^2 = (ef)\left[\frac{3(1-e)(1-f)(2-e-f)}{2[(1-e)^2+(1-e)(1-f)+(1-f)^2]}\right]^2 - \left[\frac{3ef(e+f)}{2[e^2+ef+f^2]}\right]^2 (1+ef-e-f) \tag{2.15}$$

On simplification, eqn. (2.15) takes the form:

$$(G(C^d)')^2 - (C^dG')^2 = \frac{9(ef)(1+ef-e-f)}{4\left[(1-e)^2+(1-e)(1-f)+(1-f)^2\right]^2\left[e^2+ef+f^2\right]^2} \quad [\Delta_1] \tag{2.16}$$

where,

$$\Delta_1 = -(e-f)^2(e+f-1)[\Delta] \quad (\Delta \quad \text{is given in eqn.(2.5)})$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e + f - 1 < 0$ and $(e - f)^2 > 0$, based on this the value of $\Delta_1 > 0$.
Thus,

$$(G(C^d)')^2 - (C^dG')^2 \geq 0 \quad or \quad (G(C^d)') - (C^dG') \geq 0 \tag{2.17}$$

Hence,

$$G'(C^d)'\left(\frac{G}{G'} - \frac{C^d}{(C^d)'}\right) \geq 0 \quad or \quad \left(\frac{C}{(C^d)'} \leq \frac{G}{G'}\right) \tag{2.18}$$

□

Theorem 2.6. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$\frac{C^d}{(C^d)'} \leq \frac{H}{H'}.$$

Proof. From eqs. (1.3) and (1.5), one can write easily:

$$H(C^d)' - (C^d)H' = \left(\frac{2ef}{e+f}\right)\left[\frac{3(1-e)(1-f)(2-e-f)}{2[(1-e)^2+(1-e)(1-f)+(1-f)^2]}\right] - \left[\frac{3ef(e+f)}{2[e^2+ef+f^2]}\right]\left(\frac{2(1-e)(1-f)}{2-e-f}\right) \tag{2.19}$$

On simplification, eqn. (2.19) takes the form:

$$H(C^d)' - (C^d)H' = \frac{3ef(1+ef-e-f)\left(-(e-f)^2(e+f-1)\right)}{\nabla} \tag{2.20}$$

where,

$$\nabla = (e+f)[(1-e)^2+(1-e)(1-f)+(1-f)^2][e^2+ef+f^2](2-e-f) > 0$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e + f - 1 < 0$, $1 + ef - e - f > 0$ and $(e - f)^2 > 0$, based on this the value of $H(C^d)' - (C^d)H' > 0$.
Hence,

$$H'(C^d)'\left(\frac{H}{H'} - \frac{C^d}{(C^d)'}\right) \geq 0 \quad or \quad \left(\frac{C^d}{(C^d)'} \leq \frac{H}{H'}\right) \tag{2.21}$$

□

Theorem 2.7. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$\frac{C^d}{(C^d)'} \leq \frac{H}{H'} \leq \frac{G}{G'} \leq \frac{A}{A'} \leq \frac{C}{C'}.$$

Proof. The proof of Theorem (2.7) is established by combining the results of the Theorems (2.1) to (2.6). □

3. Some inequalities for the arguments lie on the curve y^*

In this section, the analytical proof of the inequality chain of the form $\frac{C^d}{(C^d)'} \leq \frac{H}{H'} \leq \frac{G}{G'} \leq \frac{A}{A'} \leq \frac{C}{C'}$ is provided.

Theorem 3.1. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$\frac{A}{A'} \leq \frac{C}{C'}.$$

Proof. From eqs. (1.6) and (1.9), one can write easily:

$$AC' - CA' = \left(e^2 + f^2\right)\left[\frac{4[(1-e)^4 + (1-e)^2(1-f)^2 + (1-f)^4]}{3(1-e)^2 + (1-f)^2}\right] - \left[\frac{2[e^4 + e^2 f^2 + f^4]}{3(e^2 + f^2)}\right]\left((1-e)^2 + (1-f)^2\right) \tag{3.1}$$

On simplification, eqn. (3.1) takes the form:

$$AC' - CA' = \frac{4}{3}\frac{[\Delta_2]}{\left((1-e)^2 + (1-f)^2\right)^2 [e^4 + e^2 f^2 + f^4]}$$

where,

$$\Delta_2 = \left(e^2 + f^2\right)^2 [(1-e)^4 + (1-e)^2(1-f)^2 + (1-f)^4] - \left((1-e)^2 + (1-f)^2\right)^2 \left[e^4 + e^2 f^2 + f^4\right]$$

$$\Delta_2 = (e - f)^2 (e + f - 1)(2ef - e - f)(e^2 - e + f^2 - f)$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e + f - 1 < 0$, $e^2 - e + f^2 - f = e(e-1) + f(f-1) < 0$, $2ef - e - f = ef - e + ef - f = e(f-1) + f(e-1) < 0$ and $(e - f)^2 > 0$, based on this the value of $AC' - CA' < 0$.
Hence,

$$A'C'\left(\frac{A}{A'} - \frac{C}{C'}\right) \leq 0 \quad or \quad \left(\frac{A}{A'} \leq \frac{C}{C'}\right) \tag{3.2}$$

□

Theorem 3.2. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$\frac{C^d}{(C^d)'} \leq \frac{H}{H'}.$$

Proof. From eqs. (1.8) and (1.9), one can write easily:

$$H(C^d)^{'} - (C^d)H^{'} = \left(\frac{4e^2f^2}{e^2+f^2}\right)\left[\frac{3(1-e)^2(1-f)^2[(1-e)^2+(1-f)^2]}{4[(1-e)^4+(1-e)^2(1-f)^2+(1-f)^4]}\right] - \left(\frac{3e^2f^2(e^2+f^2)}{4(e^4+e^2f^2+f^4)}\right)\left[\frac{4(1-e)^2(1-f)^2}{(1-e)^2+(1-f)^2}\right] \tag{3.3}$$

On simplification, eqn. (3.3) takes the form:

$$H(C^d)^{'} - (C^d)H^{'} = \frac{3e^2f^2(1-e)^2(1-f)^2}{\nabla_1}[\Delta_3] \tag{3.4}$$

where,

$$\nabla_1 = (e^2+f^2)((1-e)^4+(1-e)^2(1-f)^2+(1-f)^4)(e^4+e^2f^2+f^4)((1-e)^2+(1-f)^2) > 0$$

$$\Delta_3 = \left((1-e)^2+(1-f)^2\right)^2(e^4+e^2f^2+f^4) - \left(e^2+f^2\right)^2\left[(1-e)^4+(1-e)^2(1-f)^2+(1-f)^4\right]$$

$$\Delta_3 = -(e-f)^2(e+f-1)(2ef-e-f)(e^2-e+f^2-f)$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e+f-1 < 0$, $e^2-e+f^2-f = e(e-1)+f(f-1) < 0$, $2ef-e-f = ef-e+ef-f = e(f-1)+f(e-1) < 0$ and $(e-f)^2 > 0$, based on this the value of $H(C^d)^{'} - (C^d)H^{'} > 0$.

Hence,

$$H^{'}(C^d)^{'}\left(\frac{H}{H'} - \frac{C^d}{(C^d)'}\right) \geq 0 \quad or \quad \left(\frac{C^d}{(C^d)'} \leq \frac{H}{H'}\right) \tag{3.5}$$

□

Theorem 3.3. *For all $e, f \in (0, \frac{1}{2}]$ and $e^{'} = 1-e, f^{'} = 1-f \in [\frac{1}{2}, 1)$, then*

$$\frac{G}{G'} \leq \frac{A}{A'}.$$

Proof. From eqs. (1.6) and (1.7), one can write easily:

$$(AG^{'})^2 - (GA^{'})^2 = (e^2+f^2)^2[4(1-e)^2(1-f)^2] - 4e^2f^2[(1-e)^2+(1-f)^2]^2$$

Equivalently,

$$(AG^{'})^2 - (GA^{'})^2 = -4\,(e-f)^2\,(e+f-1)(2ef-e-f)(e^2-e+f^2-f)$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e+f-1 < 0$, $e^2-e+f^2-f = e(e-1)+f(f-1) < 0$, $2ef-e-f = ef-e+ef-f = e(f-1)+f(e-1) < 0$ and $(e-f)^2 > 0$.

Thus,

$$(AG^{'})^2 - (GA^{'})^2 \geq 0 \tag{3.6}$$

or

$$(AG^{'}) - (GA^{'}) \geq 0 \tag{3.7}$$

Hence,

$$A^{'}G^{'}\left(\frac{G}{G'} - \frac{A}{A'}\right) \leq 0 \quad or \quad \left(\frac{G}{G'} \leq \frac{A}{A'}\right) \tag{3.8}$$

□

Theorem 3.4. *For all $e, f \in (0, \frac{1}{2}]$ and $e^{'} = 1-e, f^{'} = 1-f \in [\frac{1}{2}, 1)$, then*

$$\frac{H}{H'} \leq \frac{G}{G'}.$$

Proof. From eqs. (1.6) and (1.7), one can write easily:

$$(HG^{'})^2 - (GH^{'})^2 = \left(\frac{4e^2f^2}{e^2+f^2}\right)^2 4(1-e)^2(1-f)^2 - 4e^2f^2\left[\frac{4(1-e)^2(1-f)^2}{(1-e)^2+(1-f)^2}\right]^2$$

Equivalently,

$$(HG^{'})^2 - (GH^{'})^2 = \frac{-64\,(e-f)^2\,(e+f-1)(2ef-e-f)(e^2-e+f^2-f)}{(e^2+f^2)^2\,[(1-e)^2+(1-f)^2]^2}$$

For all $e, f \in (0, \frac{1}{2}]$, implies that $e+f-1 < 0$, $e^2-e+f^2-f = e(e-1)+f(f-1) < 0$, $2ef-e-f = ef-e+ef-f = e(f-1)+f(e-1) < 0$ and $(e-f)^2 > 0$.
Thus,

$$(HG^{'})^2 - (GH^{'})^2 \leq 0 \tag{3.9}$$

or

$$(HG^{'}) - (GH^{'}) \leq 0 \tag{3.10}$$

Hence,

$$H^{'}G^{'}\left(\frac{G}{G^{'}} - \frac{H}{H^{'}}\right) \geq 0 \quad or \quad \left(\frac{G}{G^{'}} \geq \frac{H}{H^{'}}\right) \tag{3.11}$$

□

Theorem 3.5. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$\frac{C^d}{(C^d)^{'}} \leq \frac{H}{H^{'}} \leq \frac{G}{G^{'}} \leq \frac{A}{A^{'}} \leq \frac{C}{C^{'}}.$$

Proof. The proof of Theorem (3.5) is established by combining the results of the Theorems (3.1) to (3.4). □

Theorem 3.6. *For all $e, f \in (0, \frac{1}{2}]$ implies that $0 < e < \frac{1}{2}$ and $0 < f < \frac{1}{2}$, then*

$$G^3 \leq \frac{(G^{'})^3}{C(C^d)^{'}} \leq \frac{HG^{'}A^{'}}{C(C^d)} \leq \frac{G(G^{'})^2}{C(C^d)} \leq \frac{H^{'}G^{'}A}{C(C^d)} \leq \frac{(G^{'})^3}{C^{'}(C^d)} \leq (G^{'})^3.$$

Proof. The proof of Theorem (3.6) is established by combining the results of Theorems (3.1) to (3.4), as follows. It is well known fact and easy to prove that:

$$\frac{A}{A^{'}} \leq 1; \qquad \frac{G}{G^{'}} \leq 1; \qquad \frac{H}{H^{'}} \leq 1; \qquad \frac{C}{C^{'}} \leq 1; \qquad \frac{C^d}{(C^d)^{'}} \leq 1 \tag{3.12}$$

and the well-known identity states that:

$$G^2 = AH \tag{3.13}$$

Based on eqs (3.12) and (3.13), the following are holds:

$$G^3 = AGH; \quad and \quad (G^{'})^3 = A^{'}G^{'}H^{'} \quad or \quad G^3 \leq (G^{'})^3 \quad and \quad AGH \leq A^{'}G^{'}H^{'} \tag{3.14}$$

Using the eqn (3.12), the inequality of the Theorem (3.5) becomes;

$$0 \leq \frac{C^d}{(C^d)^{'}} \leq \frac{H}{H^{'}} \leq \frac{G}{G^{'}} \leq \frac{A}{A^{'}} \leq \frac{C}{C^{'}} \leq 1$$

Further, simplification provides;

$$(C^d)H^{'}G^{'}A^{'}C^{'} \leq (C^d)^{'}HG^{'}A^{'}C^{'} \leq (C^d)^{'}H^{'}GA^{'}C^{'}$$
$$\leq (C^d)^{'}H^{'}G^{'}AC^{'} \leq (C^d)^{'}H^{'}G^{'}A^{'}C \leq (C^d)^{'}H^{'}G^{'}A^{'}C^{'}$$

Using the eqn (3.14) leads to the following:

$$(C^d)^{'} HGAC^{'} \leq (C^d)' H^{'} GA^{'} C^{'}$$

Hence,

$$G^3 \leq \frac{(G^{'})^3}{C(C^d)'} \leq \frac{HG^{'}A^{'}}{C(C^d)} \leq \frac{G(G^{'})^2}{C(C^d)} \leq \frac{H^{'}G^{'}A}{C(C^d)} \leq \frac{(G^{'})^3}{C'(C^d)} \leq (G^{'})^3$$

□

Remark 3.7. For the well-known parabola $y = -2(x - 1/2)^2 + 1/2$ over the interval $(0, 1)$, the ratio of means A to A'; G to G'; H to H'; C to C'; and C^d to $(C^d)'$ for the arguments $e, f \in (0, \frac{1}{2}]$ and $e^{'} = 1 - e, f^{'} = 1 - f \in [\frac{1}{2}, 1)$ are equal to 1. Hence, no interesting results were obtained. It is also observed that y and y^* are continuous functions but not differentiable, however the y^{**} is continuous and differentiable function.

4. Conclusion

In this paper, the Ky-Fan inequalities involving invariant centroidal mean, harmonic mean, geometric mean, arithmetic mean and centroidal mean are established by providing the analytical proof. The results are verified by using the software Wolfram Alpha.

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