Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 1, pp. 105 - 124
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.1.008

ON SOME CONTINUED FRACTIONS OF RAMANUJAN OF ORDER TWELVE

N. A. S. BULKHALI, CHANNABASAVAYYA, D. RANGANATHA AND M. S. SUREKHA

ABSTRACT. In this paper, we define some continued fractions $u(q), v(q), U(q)$, $V(q)$ of order twelve, and we derive some modular relations involving these continued fractions and some of Ramanujan's theta functions. We also establish some dissections for these continued fractions, and we found two integral representations for $U(q)$ and $V(q)$. Moreover, we obtain a relation between $U(q)$, $V(q)$ and Ramanujan's cubic continued fraction $G(q)$.

KEYWORDS AND PHRASES. q-continued fraction, q-series, modular relations, Ramanujan's theta functions.
2010 MATHEMATICS SUBJECT CLASSIFICATION. 11A55, 11F27, 11Y65.

Date of submission: May 9, 2024

1. INTRODUCTION AND PRELIMINARY RESULTS

All through the paper, we use the standard q-product notation

$$(a;q)_0 := 1,$$
$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab|<1. \tag{1.1}$$

The well-known Jacobi triple product identity [5, p. 35, Entry 19] in Ramanujan's notation is

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The three special cases of (1.1) are [5, p. 36, Entry 22]

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.3}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty. \tag{1.4}$$

Define

$$\chi(q) := (-q; q^2)_\infty.$$

The following lemma provides transformation formulas for the function $f(a, b)$:

Lemma 1.1. *[5, p. 46, Entry 30 (ii) & (iii)] We have*

$$\begin{aligned} f(a, b) + f(-a, -b) &= 2f(a^3 b, ab^3), \\ f(a, b) - f(-a, -b) &= 2af(b/a, a^5 b^3). \end{aligned}$$

Adding the above two equations, we obtain

$$f(a, b) = f(a^3 b, ab^3) + af(b/a, a^5 b^3). \tag{1.5}$$

The Rogers-Ramanujan functions are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} \qquad \text{and} \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n}, \tag{1.6}$$

These functions have elegant product forms, now famously known as the Rogers-Ramanujan identities, initially discovered and proved by L. J. Rogers [15] and then rediscovered by S. Ramanujan [14, pp. 214-215]:

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}, \tag{1.7}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}. \tag{1.8}$$

The Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := q^{1/5} \frac{H(q)}{G(q)} = \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \, \cdots, \quad |q| < 1,$$

and first appeared in a manuscript by L. J. Rogers [15] in 1894. This $R(q)$ has many representations, for example it can be written in terms of infinite q-products as follows:

$$R(q) = q^{1/5} \frac{(q^4; q^5)_\infty (q; q^5)_\infty}{(q^3; q^5)_\infty (q^2; q^5)_\infty}. \tag{1.9}$$

The identity (1.9) has been proved by both Rogers [15] and also discovered by S. Ramanujan [12, Vol. II, Chapter 16, Section 15]. On page 50 of [11], Ramanujan established 2- and 5-dissections of $R(q)$ and for $1/R(q)$, see [4, 7].

On page 365 of his 'lost' notebook [13], Ramanujan recorded five identities involving $R(\pm q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$ and $R(q^5)$ and in his first letter to G. H. Hardy [11, p. xxvii], Ramanujan gave the first non-elementary evaluations of $R(q)$, namely $R(e^{-2\pi})$ and $R(-e^{-\pi})$.

Ramanujan [13, p.46] claimed that

$$R(q) = \frac{\sqrt{5}-1}{2} \exp\left((-1/5) \int_q^1 \frac{(1-t)^5(1-t^2)^5 \cdots}{(1-t^5)(1-t^{10}) \cdots} \frac{dt}{t} \right),$$

where $0 < q < 1$ and it was proved by G. E. Andrews [4].

Ramanujan [13, p.366] studied the continued fraction

$$G(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots, \qquad |q| < 1,$$

which is known as Ramanujan's cubic continued fraction. He guaranteed that there are many results of $G(q)$ which are undifferentiated from those of $R(q)$.

Motivated by Ramanujan's theory for these continued fractions, many other continued fractions have been discovered and studied. Ramanujan [12, 13] recorded many fascinating q-continued fractions and some of their explicit values. For instance, we have [5, p. 21, Entry 11].

Entry 11. Suppose that either q, a and b are complex numbers with $|q| < 1$ or q, a and b are complex numbers with $a = bq^m$ for some integer m. Then

$$\frac{(-a;q)_\infty(b;q)_\infty - (a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty + (a;q)_\infty(-b;q)_\infty} = \frac{a-b}{1-q} \underset{+}{} \frac{(a-bq)(aq-b)}{1-q^3} \underset{+}{} \frac{q(a-bq^2)(aq^2-b)}{1-q^5} \underset{+}{} \cdots. \tag{1.10}$$

The above identity provides a continued fraction expansion for the quotient of two specific q-products.

Setting $q \to q^3$, $a = q^2$ and $b = -q$ in (1.10), we obtain

$$u(q) := \frac{q(1+q)}{1-q^3} \underset{+}{} \frac{q^3(1+q^2)(1+q^4)}{1-q^9} \underset{+}{} \frac{q^6(1+q^5)(1+q^7)}{1-q^{15}} \underset{+}{} \cdots,$$

where

$$u(q) = \frac{(-q^2;q^3)_\infty(-q;q^3)_\infty - (q^2;q^3)_\infty(q;q^3)_\infty}{(-q^2;q^3)_\infty(-q;q^3)_\infty + (q^2;q^3)_\infty(q;q^3)_\infty}.$$

Again, setting $q \to q^3$, $a = q^4$ and $b = -q^{-1}$ in (1.10), we obtain

$$v(q) := \frac{q^{-1}(1+q^5)}{1-q^3} {}_{+} \frac{q(1+q^2)(1+q^8)}{1-q^9} {}_{+} \frac{q^6(1+q)(1+q^{11})}{1-q^{15}} {}_{+\cdots},$$

where

$$v(q) = \frac{(-q^4;q^3)_\infty(-q^{-1};q^3)_\infty - (q^4;q^3)_\infty(q^{-1};q^3)_\infty}{(-q^4;q^3)_\infty(-q^{-1};q^3)_\infty + (q^4;q^3)_\infty(q^{-1};q^3)_\infty}.$$

With the help of Lemma 1.1, we can rewrite $u(q)$ and $v(q)$ as follows:

$$u(q) = \frac{f(q,q^2) - f(-q,-q^2)}{f(q,q^2) + f(-q,-q^2)} = q\frac{f(q,q^{11})}{f(q^5,q^7)}, \tag{1.11}$$

and

$$v(q) = \frac{f(q^4,q^{-1}) - f(-q^4,-q^{-1})}{f(q^4,q^{-1}) + f(-q^4,-q^{-1})} = q^{-1}\frac{f(q^5,q^7)}{f(q,q^{11})}. \tag{1.12}$$

In [9], the authors explored the following continued fraction of order 12:

$$H(q) := q\frac{f(-q,-q^{11})}{f(-q^5,-q^7)} = \frac{q(1-q)}{(1-q^3)} + \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} + \frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})} + \cdots.$$

They derived modular relations involving $H(q)$ and $H(q^n)$ for $n = 3, 5$, and $u(q)$ and $u(q^n)$ for $n = 2$, where $u(q) = -H(-q) = q\dfrac{f(q,q^{11})}{f(q^5,q^7)}$. For example, they proved that if $x := u(q)$ and $y := u(q^2)$, then

$$y(y^2+1)(1+2x+x^4) - x^2(y^4+2y^3+6y^2+y+1) + 2x^3y(y^2+1) = 0.$$

They obtained the following integral representation for $u(q)$ as a quotient of definite integrals, along with some explicit valuations of these continued fractions:

$$u(q) = \frac{1 - \exp(-2\int_0^q \psi^2(t)\psi^2(t^3)dt)}{1 + \exp(-2\int_0^q \psi^2(t)\psi^2(t^3)dt)}.$$

The authors in [16] obtained the modular relations involving $H(q)$ and $H(q^n)$ for $n = 2, 3, 5, 7, 9, 11$, and 13 by using modular relations of Rogers-Ramanujan-type functions of order 12 and theta function identities. In [6], modular relations connecting $H(q)$ and $H(q^n)$ for $n = 6, 10, 14$, and 18 were derived. In [2], the researchers obtained the Eisenstein series identities associated with the continued fraction $H(q)$. In [8], properties of the coefficients of the continued fraction $H(q)$ were studied through 2, 3, 4, 6, and 12-dissections of $H(q)$ and its reciprocal.

Define

$$U(q) := \frac{1}{1-u(q)} \qquad \text{and} \qquad V(q) := \frac{-1}{1-v(q)}. \tag{1.13}$$

The motivation for defining $U(q)$ and $V(q)$ by using the continued fractions $u(q)$ and $v(q)$ is that such continued fractions admit nice integral representations and also there are nice relations connecting these two continued fractions $U(q)$ and $V(q)$. From Lemma 1.1, (1.11), (1.12) and (1.13), one can easily verify the following lemma, which expresses the continued fractions in terms of theta functions:

Lemma 1.2. *We have*

$$u(q)v(q) = 1, \tag{1.14}$$

$$U(q) = \frac{f\left(q^5, q^7\right)}{f(-q)}, \tag{1.15}$$

$$V(q) = \frac{q\,f\left(q, q^{11}\right)}{f(-q)}, \tag{1.16}$$

$$U(q)V(q) = \frac{\chi(q) f^3(-q^{12})}{\psi(q^3) f^2(-q)}. \tag{1.17}$$

This paper is organized as follows. In Section 2, we derive some modular relations involving our continued fractions $U(q)$ and $V(q)$, and Ramanujan's theta functions defined above. In Section 3, we establish 2- and 4-dissections for $U(q)$ and 2-dissection for $V(q)$. In Section 4, we prove two integral representations for each $U(q)$ and $V(q)$. In Section 5, we establish a relation between $U(q)$ and $G(q)$. Using this relation and some known values of $G(q)$, we computed $U(q)$ and $V(q)$.

2. Main Results

In this section, we establish modular relations that link the continued fractions $u(q)$, $v(q)$, $U(q)$ and $V(q)$ with the Ramanujan theta functions.

Theorem 2.1. *We have*

$$U(q) - V(q) = 1, \tag{2.1}$$

$$U(q) + V(q) = \frac{f(q, q^2)}{f(-q)}, \tag{2.2}$$

$$U^2(q) - V^2(q) = \frac{f(q, q^2)}{f(-q)}, \tag{2.3}$$

$$U^2(q) + V^2(q) = \frac{\phi(q^3) f(q^2, q^4)}{f^2(-q)}, \tag{2.4}$$

$$U^4(q) - V^4(q) = \frac{\phi(q^3) f(q, q^2) f(q^2, q^4)}{f^3(-q)}. \tag{2.5}$$

Proof. Identities (2.1) and (2.2) follow from (1.15), (1.16) and Lemma 1.1. Identity (2.3) follows from (2.1) and (2.2). Now we proceed to prove (2.4). We have

$$\phi\left(q^3\right) f\left(q^2, q^4\right) = \sum_{k,\ell=-\infty}^{\infty} q^{3k^2+3\ell^2+\ell}.$$

Here we set

$$k + \ell = 2K + a \text{ and } -k + \ell = 2L + b,$$

where $a, b \in \{0, 1\}$ and $K, L \in \mathbb{Z}$. Then

$$k = K - L - (b - a)/2 \text{ and } \ell = K + L + (b + a)/2.$$

Then we have $a = b$, and so $k = K - L$ and $\ell = K + L + a$, where $a \in \{0, 1\}$. Thus, there is 1-1 correspondence between the set $\{(k, \ell)/ - \infty < k, \ell < \infty\}$ and $\{(K, L, a)/ - \infty < K, L < \infty,\ a \in \{0, 1\}\}$. Thus, we find that

$$\begin{aligned} \phi\left(q^3\right) f\left(q^2, q^4\right) &= \sum_{a=0}^{1} q^{3a^2+a} \left(\sum_{K=-\infty}^{\infty} q^{6K^2+(1+6a)K} \right)^2 \\ &= f^2\left(q^5, q^7\right) + q^2 f^2\left(q, q^{11}\right). \end{aligned}$$

Using (1.15) and (1.16), we obtain (2.4). Identity (2.5) follows easily from (2.3) and (2.4). □

Theorem 2.2. *We have*

$$\frac{u\left(q^2\right)u\left(q\right)+1}{\left(1-u\left(q\right)\right)\left(1-u\left(q^2\right)\right)}=\frac{\psi\left(q\right)\phi\left(q^4\right)-q\psi\left(q^3\right)\psi\left(q^6\right)}{f\left(-q\right)f\left(-q^2\right)}, \tag{2.6}$$

$$\frac{u\left(q\right)u\left(q^5\right)+1}{\left(1-u\left(q^5\right)\right)\left(1-u\left(q\right)\right)}=\frac{\phi\left(q^5\right)\psi\left(q^2\right)-q^2\psi\left(q^{15}\right)\psi\left(q^3\right)}{f\left(-q\right)f\left(-q^5\right)}, \tag{2.7}$$

$$\begin{aligned}\frac{u\left(q\right)u\left(q^3\right)+1}{\left(1-u\left(q^3\right)\right)\left(1-u\left(q\right)\right)}&=\frac{\phi\left(-q^9\right)\phi\left(-q^3\right)}{f\left(-q\right)f\left(-q^3\right)\chi\left(-q\right)\chi\left(-q^3\right)}\\ -q\frac{\chi\left(q^2\right)\psi\left(-q^6\right)\phi\left(-q^{36}\right)}{f\left(-q\right)f\left(-q^3\right)\chi\left(-q^{12}\right)}&-q^2\frac{\chi\left(q^6\right)\psi\left(-q^{18}\right)\phi\left(-q^{12}\right)}{f\left(-q\right)f\left(-q^3\right)\chi\left(-q^4\right)},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\frac{u\left(q^3\right)+u\left(q\right)}{\left(1-u\left(q^3\right)\right)\left(1-u\left(q\right)\right)}&=2\,\frac{q\psi\left(q^9\right)\phi\left(-q^3\right)}{f\left(-q\right)f\left(-q^3\right)\chi\left(-q\right)}\\ -q\frac{\phi\left(-q^{36}\right)\chi\left(q^2\right)\psi\left(-q^6\right)}{f\left(-q\right)f\left(-q^3\right)\chi\left(-q^{12}\right)}&-q^2\frac{\chi\left(q^6\right)\psi\left(-q^{18}\right)\phi\left(-q^{12}\right)}{f\left(-q\right)f\left(-q^3\right)\chi\left(-q^4\right)}\end{aligned} \tag{2.9}$$

Proof. Identity (2.6) can be written in the form

$$\begin{aligned}\frac{2}{\left(1-u\left(q\right)\right)\left(1-u\left(q^2\right)\right)}-\frac{1}{\left(1-u\left(q\right)\right)}-\frac{1}{\left(1-u\left(q^2\right)\right)}+1\\ =\frac{\phi\left(q^4\right)\psi\left(q\right)-q\psi\left(q^6\right)\psi\left(q^3\right)}{f\left(-q\right)f\left(-q^2\right)}.\end{aligned}$$

Using (1.13), we may rewrite the above identity in the form

$$U(q)U(q^2)+\left(U(q)-1\right)\left(U(q^2)-1\right)=\frac{\phi(q^4)\psi(q)-q\psi(q^6)\psi(q^3)}{f(-q)f(-q^2)}. \tag{2.10}$$

By applying (1.15), (1.16) and (2.1) in the identity (2.10), we obtain

$$f(q^{14},q^{10})f(q^7,q^5)+q^3f(q^{22},q^2)f(q^{11},q)=\phi(q^4)\psi(q)-q\psi(q^6)\psi(q^3). \tag{2.11}$$

Thus, it is enough to prove (2.11). We have

$$\phi\left(q^4\right)\psi\left(q\right)=\sum_{k,\ell=-\infty}^{\infty}q^{4\,k^2+2\,\ell^2+\ell}. \tag{2.12}$$

Here we set

$$k+\ell=3K+a \text{ and } -2k+\ell=3L+b,$$

where $a,b\in\{-1,0,1\}$. Then

$$k=K-L-(b-a)/3 \text{ and } \ell=2K+L+(b+2a)/3.$$

Then we have $a=b$, and so $k=K-L$ and $\ell=2K+L+a$, where $-1<a<1$. Thus, there is 1-1 correspondence between the set $\{(k,\ell)/-\infty<k,\ell<\infty\}$ and $\{(K,L,a)/-\infty<K,L<\infty,\ -1<a<1\}$. Thus, we find that

$$\begin{aligned}\phi\left(q^{4}\right)\psi\left(q\right)&=\sum_{a=-1}^{1}q^{2\,a^{2}+a}\sum_{K,L=-\infty}^{\infty}q^{12\,K^{2}+2(1+4\,a)\,K+6\,L^{2}+(1+4\,a)L}\\&=\sum_{a=-1}^{1}q^{2\,a^{2}+a}f\left(q^{2(7+4\,a)},q^{2(5-4\,a)}\right)f\left(q^{7+4\,a},q^{5-4\,a}\right)\\&=qf\left(q^{6},q^{18}\right)f\left(q^{3},q^{9}\right)+f\left(q^{14},q^{10}\right)f\left(q^{7},q^{5}\right)\\&\quad+q^{3}f\left(q^{22},q^{2}\right)f\left(q^{11},q\right),\end{aligned}$$

which is same as (2.11). This completes the proof of (2.6). The proof of (2.7)–(2.9) follows in a similar way, with a suitable choice of change of the indices. □

Theorem 2.3. *We have*

$$\frac{u\left(q^{5}\right)u\left(q^{7}\right)+1}{\left(1-u\left(q^{5}\right)\right)\left(1-u\left(q^{7}\right)\right)} \tag{2.13}$$

$$=\frac{\phi\left(q^{70}\right)\psi\left(q^{4}\right)+q^{17}\phi\left(q^{2}\right)\psi\left(q^{140}\right)-q^{4}\psi\left(q^{21}\right)\psi\left(q^{15}\right)}{f\left(-q^{5}\right)f\left(-q^{7}\right)}, \tag{2.14}$$

$$\frac{u(q)u\left(q^{11}\right)+1}{\left(1-u\left(q\right)\right)\left(1-u\left(q^{11}\right)\right)} \tag{2.15}$$

$$=\frac{\phi\left(q^{22}\right)\psi\left(q^{4}\right)+q^{5}\phi\left(q^{2}\right)\psi\left(q^{44}\right)-q^{4}\psi\left(q^{33}\right)\psi\left(q^{3}\right)}{f\left(-q\right)f\left(-q^{11}\right)}. \tag{2.16}$$

Proof. Applying (1.13), (1.15), (1.16) and (2.1) in (2.13), we see that

$$\begin{aligned}&f\left(q^{35},q^{49}\right)f\left(q^{25},q^{35}\right)+q^{12}f\left(q^{7},q^{77}\right)f\left(q^{5},q^{55}\right)\\&\quad=\phi\left(q^{70}\right)\psi\left(q^{4}\right)+q^{17}\phi\left(q^{2}\right)\psi\left(q^{140}\right)-q^{4}\psi\left(q^{21}\right)\psi\left(q^{15}\right).\end{aligned} \tag{2.17}$$

Changing q to q^{2} in (2.17) and then multiplying the resulting identity by $4q$, we obtain

$$\begin{aligned}&4\,qf\left(q^{70},q^{50}\right)f\left(q^{98},q^{70}\right)+4\,q^{25}f\left(q^{10},q^{110}\right)f\left(q^{14},q^{154}\right)\\&=4\,q\left\{\phi\left(q^{140}\right)\psi\left(q^{8}\right)+q^{34}\phi\left(q^{4}\right)\psi\left(q^{280}\right)\right\}-4\,q^{9}\psi\left(q^{42}\right)\psi\left(q^{30}\right).\end{aligned} \tag{2.18}$$

Setting $a = b = q$ and $a = b = q^{35}$ in (1.5), we find that

$$\begin{aligned} \phi(q) &= \phi\left(q^4\right) + 2\,q\psi\left(q^8\right), \\ \phi\left(q^{35}\right) &= \phi\left(q^{140}\right) + 2\,q^{35}\psi\left(q^{280}\right). \end{aligned}$$

Using the above two identities, we obtain

$$\phi\left(q^{35}\right)\phi(q) - \phi\left(-q^{35}\right)\phi(-q) = 4\,q\left\{\phi\left(q^{140}\right)\psi\left(q^{8}\right) + q^{34}\phi\left(q^{4}\right)\psi\left(q^{280}\right)\right\}. \tag{2.19}$$

From (2.18) and (2.19), we conclude that

$$\begin{aligned} &4\,qf\left(q^{70}, q^{50}\right)f\left(q^{98}, q^{70}\right) + 4\,q^{25}f\left(q^{10}, q^{110}\right)f\left(q^{14}, q^{154}\right) \\ &\qquad = \phi(q)\,\phi\left(q^{35}\right) - \phi(-q)\,\phi\left(-q^{35}\right) - 4\,q^{9}\psi\left(q^{42}\right)\psi\left(q^{30}\right). \end{aligned} \tag{2.20}$$

Thus, we need to prove (2.20).
Using (1.2), we have

$$\phi\left(q^{35}\right)\phi(q) = f\left(q^{35}, q^{35}\right)f(q, q) = \sum_{k,\ell=-\infty}^{\infty} q^{35\,k^2+\ell^2}. \tag{2.21}$$

Here we set

$$5k + \ell = 12K + a \qquad \text{and} \qquad -7k + \ell = 12L + b,$$

where $a, b \in \{0, \pm1, \pm2, \pm3, \pm4, \pm5, 6\}$. Then

$$k = K - L + (a - b)/12 \qquad \text{and} \qquad \ell = 7K + 5L + (7a + 5b)/12.$$

Since k, ℓ, K and L are all integers, we have $a = b$, and so $k = K - L$ and $\ell = 7K + 5L + a$, where $-5 \le a \le 6$. Thus, there is 1-1 correspondence between the set $\{(k, \ell)/-\infty < k, \ell < \infty\}$ and $\{(K, L, a)/-\infty < K, L < \infty,\ -5 \le a \le 6\}$.
From (2.21), we find that

$$\begin{aligned}\phi\left(q^{35}\right)\phi\left(q\right) &= \sum_{a=-5}^{6} q^{a^2} \sum_{K,L=-\infty}^{\infty} q^{84\,K^2+14\,Ka+60\,L^2+10\,La}\\ &= \sum_{a=-5}^{6} q^{a^2} f\left(q^{14(6+a)}, q^{14(6-a)}\right) f\left(q^{10(6+a)}, q^{10(6-a)}\right)\\ &= 2\,q^{25} f\left(q^{14}, q^{154}\right) f\left(q^{10}, q^{110}\right) + 2\,q^{16} f\left(q^{28}, q^{140}\right) f\left(q^{20}, q^{100}\right)\\ &\quad + 2\,q^{9} f\left(q^{42}, q^{126}\right) f\left(q^{30}, q^{90}\right) + 2\,q^{4} f\left(q^{56}, q^{112}\right) f\left(q^{40}, q^{80}\right)\\ &\quad + 2\,q f\left(q^{70}, q^{98}\right) f\left(q^{50}, q^{70}\right) + f\left(q^{84}, q^{84}\right) f\left(q^{60}, q^{60}\right)\\ &\quad + q^{36} f\left(q^{168}, 1\right) f\left(q^{120}, 1\right). \qquad (2.22)\end{aligned}$$

Replacing q by $-q$ in (2.22) and then subtracting the resulting identity from (2.22), we obtain (2.20). This completes the proof of (2.13). In a similar way, one can establish (2.15). □

Theorem 2.4. *We have*

$$4U^3(q) - 4U^2(q)V(q) - 3U^2(q) + V^2(q) - 2U(q)V(q) - U(q) + V(q) = 0, \qquad (2.23)$$

$$4V^2(q)U(q) - 4V^3(q) - 3V^2(q) + U^2(q) - 2U(q)V(q) - U(q) + V(q) = 0. \qquad (2.24)$$

Proof. We have

$$\begin{aligned}4U(q) - \frac{1}{U(q)} &= 4\frac{f(q^7,q^5)}{f(-q)} - \frac{f(-q)}{f(q^5,q^7)}\\ &= \frac{4f^2(q^7,q^5) - f^2(-q)}{f(q^5,q^7)f(-q)}. \qquad (2.25)\end{aligned}$$

Recall that we have by [10, Eq. 26],

$$\phi(q) + \phi(q^3) = 2f(-q^7,-q^5)\chi(q) \qquad (2.26)$$

and by [5, Entry 24(iii)],

$$f(-q) = \frac{\phi(-q)}{\chi(-q)}. \qquad (2.27)$$

Changing q to $-q$ in (2.26), then employing resulting identity and (2.27) to (2.25), we get

$$4U(q)-\frac{1}{U(q)}=\frac{2\{\phi^2(-q^3)+2\phi(-q)\phi(-q^3)\}}{\phi^2(-q)+\phi(-q)\phi(-q^3)}$$
$$=\frac{2\left(\dfrac{\phi(-q^3)}{\phi(-q)}+2\right)}{\left(\dfrac{\phi(-q)}{\phi(-q^3)}+1\right)}. \tag{2.28}$$

From (2.1) and (2.2), we have

$$\frac{\phi(-q)}{\phi(-q^3)}=\frac{U(q)-V(q)}{V(q)+U(q)}. \tag{2.29}$$

Employing (2.29) in (2.28), we obtain (2.23).

The proof of (2.24) follows similarly, and we omit the details. □

Theorem 2.5. *We have*

$$\frac{1}{\sqrt{2U(q)}}\left(\frac{1}{\sqrt{2V(q)}}+\sqrt{2V(q)}\right)=\frac{1}{2}\sqrt{\frac{\psi(q^2)\psi^3(-q^3)}{q\psi(-q)\psi^3(q^6)}}. \tag{2.30}$$

Proof. By definition $V(q)$, we have

$$\frac{1}{\sqrt{2V(q)}}+\sqrt{2V(q)}=\frac{f(-q)+2qf(q,q^{11})}{\sqrt{2qf(q,q^{11})f(-q)}}. \tag{2.31}$$

From Lemma 3.1 of [10, Eq. 27], we have

$$\phi(q)-\phi(q^3)=2q\chi(q)f(-q,-q^{11}). \tag{2.32}$$

Changing q to $-q$ in (2.32), we deduce that

$$\phi(-q)-\phi(-q^3)=-2q\chi(-q)f(q,q^{11}). \tag{2.33}$$

Employing (2.27) and (2.33) to (2.31), we get

$$\frac{1}{\sqrt{2V(q)}}+\sqrt{2V(q)}=\frac{\phi(-q^3)}{\chi(-q)\sqrt{\dfrac{-2f(-q)}{2\chi(-q)}(\phi(-q)-\phi(-q^3))}}$$
$$=\frac{\phi(-q^3)}{\sqrt{-\left(\dfrac{(\phi^2(-q)-\phi^2(-q^3))\chi(-q)f(-q)}{\phi(-q)+\phi(-q^3)}\right)}}$$
$$=\frac{1}{\sqrt{-\left(\dfrac{\phi^2(-q)}{\phi^2(-q^3)}-1\right)\dfrac{1}{2U(q)}}}. \tag{2.34}$$

In [3], we have

$$\frac{\phi^2(q)}{\phi^2(q^3)} - 1 = \frac{4q\psi(q)\psi^3(q^6)}{\psi(q^2)\psi^3(q^3)}, \quad \text{where} \quad \psi(q) = f(q, q^3). \tag{2.35}$$

Employing (2.35) in (2.34), we obtain (2.30). □

Theorem 2.6. *We have*

$$\frac{1}{\sqrt{2U(q)}} - \sqrt{2U(q)} = -\frac{\phi(-q^3)}{\phi(-q)} \frac{1}{\sqrt{\dfrac{\phi(-q^3)}{\phi(-q)} + 1}}. \tag{2.36}$$

Proof. By the definition of $U(q)$, we have

$$\frac{1}{\sqrt{2U(q)}} - \sqrt{2U(q)} = \frac{f(-q) - 2f(q^5, q^7)}{\sqrt{2f(q^5, q^7)f(-q)}}. \tag{2.37}$$

Employing (2.27) and (2.26) in (2.37) and recall $\phi(q) = f(q, q)$, we get

$$\begin{aligned} \frac{1}{\sqrt{2U(q)}} - \sqrt{2U(q)} &= \frac{-\phi(-q^3)}{\sqrt{\phi(-q)(\phi(-q) + \phi(-q^3))}} \\ &= \frac{-\sqrt{\dfrac{\phi(-q^3)}{\phi(-q)}}}{\sqrt{\dfrac{\phi(-q)}{\phi(-q^3)} + 1}} \\ &= -\frac{\phi(-q^3)}{\phi(-q)} \frac{1}{\sqrt{\dfrac{\phi(-q^3)}{\phi(-q)} + 1}}. \end{aligned}$$

□

3. 2- AND 4-DISSECTIONS FOR $U(q)$ AND 2-DISSECTION FOR $V(q)$

In this section we obtain 2 and 4-dissections for $U(q)$ and $V(q)$.

Theorem 3.1. *If* $U(q)=\sum_{n=0}^{\infty}a_nq^n$ *and* $V(q)=\sum_{n=0}^{\infty}b_nq^n$, *then*

$$\sum_{n=0}^{\infty}a_{2n}q^n=\frac{1}{f^2(-q)}\left\{f(q^{11},q^{13})f(q^3,q^5)+q^4f(q,q^{23})f(q,q^7)\right\}, \tag{3.1}$$

$$\sum_{n=0}^{\infty}a_{2n+1}q^n=\frac{1}{f^2(-q)}\left\{q^2f(q,q^{23})f(q^3,q^5)+qf(q^{11},q^{13})f(q,q^7)\right\}, \tag{3.2}$$

$$\sum_{n=0}^{\infty}b_{2n}q^n=\frac{q}{f^2(-q)}\left\{f(q^7,q^{17})f(q,q^7)+f(q^5,q^{19})f(q^3,q^5)\right\}, \tag{3.3}$$

$$\sum_{n=0}^{\infty}b_{2n+1}q^n=\frac{1}{f^2(-q)}\left\{f(q^7,q^{17})f(q^3,q^5)+qf(q^5,q^{19})f(q,q^7)\right\}. \tag{3.4}$$

Proof. In view of (1.3), we have

$$\begin{aligned}U(q)&=\frac{f(q^7,q^5)}{f(-q^2,-q)}\\&=\frac{f(q^7,q^5)f(q,q^3)}{f^2(-q^2)}.\end{aligned} \tag{3.5}$$

Setting $(a,b)=(q^5,q^7)$ in (1.5), we obtain

$$f(q^7,q^5)=f(q^{26},q^{22})+q^5f(q^{46},q^2). \tag{3.6}$$

Putting $a=q$ and $b=q^3$ in (1.5), we obtain

$$f(q,q^3)=f(q^6,q^{10})+q^3f(q^2,q^{14}). \tag{3.7}$$

Employing (3.6) and (3.7) in (3.5), then comparing coefficients of q^{2n} and q^{2n+1} respectively, we deduce that

$$\sum_{n=0}^{\infty}a_{2n}q^{2n}=\frac{1}{f^2(-q^2)}\left\{f(q^6,q^{10})f(q^{22},q^{26})+q^8f(q^2,q^{14})f(q^2,q^{46})\right\}, \tag{3.8}$$

$$\sum_{n=0}^{\infty}a_{2n+1}q^{2n+1}=\frac{1}{f^2(-q^2)}\left\{q^5f(q^2,q^{46})f(q^6,q^{10})+q^3f(q^{22},q^{26})f(q^2,q^{14})\right\}. \tag{3.9}$$

Changing q to $q^{1/2}$ in (3.8) and (3.9), we deduce (3.1) and (3.2), respectively. Proof of (3.3) and (3.4) are similar and we omit the details. □

We define $P := f(q^{23}, q^{25})$, $Q := f(q, q^{47})$, $R := f(q^7, q^9)$, $S := f(q, q^{15})$, $A := f(q^{13}, q^{35})$, $B := f(q^{11}, q^{37})$, $C := f(q^5, q^{11})$, $D := f(q^3, q^{13})$, $M := f(q^3, q^5)$ and $N := f(q, q^7)$.

Theorem 3.2. *If* $U(q) = \sum_{n=0}^{\infty} a_n q^n$, *then*

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4n} q^n \equiv & \frac{1}{f^4(-q)\phi^2(-q)} \big\{(N^2q^8 + M^2q^7)QS - 2\,MNQRq^6 \\ & + BDN^2q^4 + \big(ACN^2 - 2\,(AD + BC)MN + BDM^2\big)\, q^3 \\ & + \big(ACM^2 - 2\,MNPS\big)\, q^2 + (N^2q + M^2)PR\big\} \pmod 4,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4n+1} q^n \equiv & \frac{1}{f^4(-q)\phi^2(-q)} \big\{\big(CN^2Q - 2\,DMNQ\big)\, q^7 + QCM^2q^6 + BSN^2q^4 \\ & + \big(-2\,AMNS + BM^2S\big)\, q^3 + \big((AR + DP)\,N^2 - 2\,BRMN\big)\, q^2 \\ & + \big(-2\,PCMN + (AR + DP)\,M^2\big)\, q\big\} \pmod 4,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4n+2} q^n \equiv & \frac{1}{f^4(-q)\phi^2(-q)} \big\{-2\,MNQSq^7 + N^2QRq^6 + M^2QRq^5 \\ & + \big((AD + BC)N^2 - 2\,BDMN\big)\, q^3 + \big(-2\,ACMN + (AD \\ & + BC)M^2 + N^2PS\big)\, q^2 + M^2PSq - 2\,MNPRq\big\} \pmod 4,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4n+3} q^n \equiv & \frac{1}{f^4(-q)\phi^2(-q)} \big\{QDN^2q^7 + \big(-2\,CMNQ + DM^2Q\big)\, q^6 + \big(AN^2S \\ & -2\,BMNS\big)\, q^3 + \big(AM^2S + BN^2R\big)\, q^2 + \big(PCN^2 - 2\,(AR + DP \\ & \times MN + BRM^2\big)\, q + PCM^2\big\} \pmod 4.\end{aligned} \tag{3.13}$$

Proof. we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n}q^n &= \frac{1}{f^2(-q)}\left\{f(q^{11},q^{13})f(q^3,q^5)+q^4f(q,q^{23})f(q,q^7)\right\}\\ &= \frac{\chi^2(q)}{\phi^2(-q^2)}\left\{f(q^{11},q^{13})f(q^3,q^5)+q^4f(q,q^{23})f(q^7,q)\right\}\\ &= \frac{\phi^2(q)}{\phi^2(-q^2)f^2(q)}\left\{f(q^{11},q^{13})f(q^3,q^5)+q^4f(q,q^{23})f(q^7,q)\right\}\\ &= \frac{\phi^2(q)\psi^2(-q)}{f^4(-q)\phi^2(-q^2)}\left\{f(q^{11},q^{13})f(q^3,q^5)+q^4f(q,q^{23})f(q,q^7)\right\}.\end{aligned} \tag{3.14}$$

From (1.2), we have

$$\phi^{2k}(q) = \left(1+2\sum_{n\geq 1} q^{n^2}\right)^{2k} \equiv 1 (\text{mod } 4). \tag{3.15}$$

In view of (3.15), identity (3.14) follows as

$$\sum_{n=0}^{\infty} a_{2n}q^n \equiv \frac{\psi^2(-q)}{f^4(-q)\phi^2(-q^2)}\left\{f(q^{11},q^{13})f(q^3,q^5)+q^4f(q,q^{23})f(q,q^7)\right\}(\text{mod } 4). \tag{3.16}$$

We can easily find that

$$f(q^{11},q^{13}) = f(q^{46},q^{50})+q^{11}f(q^2,q^{94}), \tag{3.17}$$

$$f(q^3,q^5) = f(q^{14},q^{18})+q^3f(q^2,q^{30}), \tag{3.18}$$

$$f(q,q^{23}) = f(q^{26},q^{70})+qf(q^{22},q^{74}), \tag{3.19}$$

$$f(q,q^7) = f(q^{10},q^{22})+qf(q^6,q^{26}), \tag{3.20}$$

$$\psi(-q) = f(-q,-q^3) = f(q^6,q^{10})-qf(q^2,q^{14}). \tag{3.21}$$

Employing (3.17)-(3.21) to (3.16) and then comparing the coefficients q^{2n} and q^{2n+1} powers of resulting identity and changing q to $q^{1/2}$, we obtain (3.10) and (3.12).

The Proofs of (3.11) and (3.13) are similar to the proofs of (3.10) and (3.12), respectively. □

4. Integral representations for $U(q)$ and $V(q)$

In this section, we derive integral representations for the functions $U(q)$ and $V(q)$.

Theorem 4.1. *For* $0 < q < 1$,

$$U(q) = \frac{1}{2} + \mathcal{C}_1 \exp\left(2 \int \psi^2(q)\psi^2\left(q^3\right) dq\right), \tag{4.1}$$

$$U(q) = \frac{1}{2} + \mathcal{C}_2 \exp\left(\frac{1}{8} \int \left\{2\,\phi^2(q)\phi^2\left(q^3\right) + \phi^4(q) - 3\,\phi^4\left(q^3\right)\right\} \frac{dq}{q}\right), \tag{4.2}$$

$$V(q) = -\frac{1}{2} + \mathcal{C}_1 \exp\left(2 \int \psi^2(q)\psi^2\left(q^3\right) dq\right), \tag{4.3}$$

and

$$V(q) = -\frac{1}{2} + \mathcal{C}_2 \exp\left(\frac{1}{8} \int \left\{2\,\phi^2(q)\phi^2\left(q^3\right) + \phi^4(q) - 3\,\phi^4\left(q^3\right)\right\} \frac{dq}{q}\right), \tag{4.4}$$

where $\phi(q)$ *and* $\psi(q)$ *are defined as in* (1.2) *and* (1.3)*, respectively, and* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are some constants.*

Proof. To prove (4.1), Using (2.1) and (2.2), we find that

$$2U(q) - 1 = \frac{\phi(-q^3)}{\phi(-q)} = \frac{(-q;q)_\infty \left(q^3;q^3\right)_\infty}{(q;q)_\infty \left(-q^3;q^3\right)_\infty}. \tag{4.5}$$

Taking logarithm on both sides of (4.5), we obtain

$$\log\left(2\,U(q) - 1\right) = \sum_{\ell=1}^{\infty} \log\left(1 + q^{\ell}\right) + \sum_{\ell=1}^{\infty} \log\left(1 - q^{3\,\ell}\right) - \sum_{\ell=1}^{\infty} \log\left(1 - q^{\ell}\right) - \sum_{\ell=1}^{\infty} \log\left(1 + q^{3\,\ell}\right). \tag{4.6}$$

Taking derivative on both sides of (4.6), after some simplifications, we deduce that

$$\frac{d}{dq}\left[\log\left(2\,U(q) - 1\right)\right] = \frac{2}{q}\left[\sum_{\ell=1}^{\infty} \frac{\ell\,q^{\ell}}{1 - q^{2\,\ell}} - 3\sum_{\ell=1}^{\infty} \frac{\ell\,q^{3\,\ell}}{1 - q^{6\,\ell}}\right]. \tag{4.7}$$

Using identity found in Entry 1 (iii) in Chapter 19 of Ramanujan's notebooks [5, p.225], we deduce that

$$\frac{d}{dq}\left[\log\left(2\,U(q) - 1\right)\right] = 2\,\psi^2(q)\psi^2\left(q^3\right).$$

Integrating both sides of the above identity and then exponentiating, we obtain (4.1).

To prove (4.2), we use (4.5) and (1.2), thus we have

$$2U(q) - 1 = \frac{(q^6;q^6)_\infty (q^3;q^6)_\infty (-q^2;q^2)_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty (-q^3;q^6)_\infty (q;q^2)_\infty (-q^6;q^6)_\infty}. \tag{4.8}$$

In a similar way, we obtain

$$\begin{aligned}\frac{d}{dq}\left[\log\left(2\,U(q)-1\right)\right]=&-\sum_{\ell=1}^{\infty}\frac{(6\ell-3)q^{6\ell-4}}{1-q^{6\ell-3}}-\sum_{\ell=1}^{\infty}\frac{6\ell\,q^{6\ell-1}}{1-q^{6\ell}}-\sum_{\ell=1}^{\infty}\frac{(6\ell-3)q^{6\ell-4}}{1+q^{6\ell-3}}\\&-\sum_{\ell=1}^{\infty}\frac{6\ell q^{6\ell-1}}{1+q^{6\ell}}+\sum_{\ell=1}^{\infty}\frac{(2\ell-1)q^{2\ell-2}}{1+q^{2\ell-1}}+\sum_{\ell=1}^{\infty}\frac{2\ell\,q^{2\ell-1}}{1+q^{2\,\ell}}\\&+\sum_{\ell=1}^{\infty}\frac{(2\ell-1)q^{2\ell-2}}{1-q^{2\ell-1}}+\sum_{\ell=1}^{\infty}\frac{2\ell q^{2\ell-1}}{1-q^{2\,\ell}}\\=&\frac{1}{q}\left[\sum_{\ell=1}^{\infty}\frac{\ell\,q^{\ell}}{1-(-q)^{\ell}}-3\sum_{\ell=1}^{\infty}\frac{\ell\,q^{3\ell}}{1-(-q)^{3\ell}}\right]\\&+\frac{1}{q}\left[\sum_{\ell=1}^{\infty}\frac{\ell\,q^{\ell}}{1+(-q)^{\ell}}-3\sum_{\ell=1}^{\infty}\frac{\ell\,q^{3\ell}}{1+(-q)^{3\ell}}\right].\end{aligned}\tag{4.9}$$

Recall that we have by [5, p. 226],

$$\frac{1}{4}\left(\phi^2(q)\phi^2\left(q^3\right)-1\right)=\sum_{\ell=1}^{\infty}\frac{\ell\,q^{\ell}}{1-(-q)^{\ell}}-3\sum_{\ell=1}^{\infty}\frac{\ell\,q^{3\ell}}{1-(-q)^{3\ell}}.\tag{4.10}$$

Also, we have [5, p. 114, Entry 8(ii)]

$$\frac{1}{8}\left(\phi^4(q)-1\right)=\sum_{\ell=1}^{\infty}\frac{\ell\,q^{\ell}}{1+(-q)^{\ell}}.\tag{4.11}$$

From (4.9), (4.10) and (4.11), we deduce that

$$\frac{d}{dq}\left[\log\left(2\,U(q)-1\right)\right]=\frac{1}{8q}\left(2\,\phi^2(q)\phi^2\left(q^3\right)+\phi^4(q)-3\,\phi^4\left(q^3\right)\right),$$

which leads us to (4.2). Substituting (4.1) and (4.2) in (2.1), we obtain (4.3) and (4.4), respectively. This completes the proof of the theorem. □

5. Explicit Evaluations of $U(q)$ and $V(q)$

We know that $G(q)$ satisfy the following interesting identity:

$$1-8G^3(q)=\frac{\phi^4(-q)}{\phi^4(-q^3)},\tag{5.1}$$

which is due to Ramanujan and can be found in [5, p. 347]. We observed that there is a remarkable relation between $U(q)$ and $G(q)$, which can be found using the first equality of (4.5) and (5.1) and given by

$$U(q)=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{1-8G^3(q)}\right)^{1/4}.$$

Substituting above relation in (2.1), we obtain the relation between $V(q)$ and $G(q)$ and is given by

$$V(q) = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{1-8G^3(q)}\right)^{1/4}.$$

It is known that, many mathematician have computed several explicit evaluations for the $G(q)$ for example see [1, Theorem 5.6]. Thus from the above identities, by using the known values of $G(q)$ we can compute $U(q)$ and $V(q)$. For example, we have

$$\begin{aligned}
U(-e^{-\pi/\sqrt{3}}) =& \frac{1}{2}\left(1+\frac{1}{\sqrt[4]{3}}\right),\\
U(-e^{-\pi\sqrt{5}}) =& \frac{1}{2} + \frac{1}{2\left(\sqrt[4]{1-\frac{(3-\sqrt{5})^3(\sqrt{3}-\sqrt{5})^3}{8}}\right)}\\
U(-e^{-\pi}) =& \frac{1+\sqrt[4]{(6\sqrt{3}-9)}}{2\sqrt[4]{(6\sqrt{3}-9)}},\\
V(-e^{-\pi\sqrt{5}}) =& -\frac{1}{2} + \frac{1}{2\left(\sqrt[4]{1-\frac{(3-\sqrt{5})^3(\sqrt{3}-\sqrt{5})^3}{8}}\right)}\\
V(-e^{-\pi/\sqrt{3}}) =& \frac{1}{2}\left(\frac{1}{\sqrt[4]{3}}-1\right) \quad \text{and} \quad V(-e^{-\pi}) = \frac{1-\sqrt[4]{(6\sqrt{3}-9)}}{2\sqrt[4]{(6\sqrt{3}-9)}}.
\end{aligned}$$

Funding: The second author wants to thank KSTePS for awarding the Karnataka DST-Ph.D. Fellowship (No. DST/KSTePS/Ph.D. Fellowship/MAT-01:2021-22/1027). The third author wishes to thank UGC for the Start-Up Research Grant No.F.30-489/2019(BSR).

Acknowledgments: The authors extend their sincere gratitude to the anonymous referees for their thorough review and valuable feedback, which have significantly enhanced the quality of the manuscript.

Conflict of interest: The authors have no conflicts of interest to declare that are relevant to the content of this article.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

REFERENCES

[1] C. ADIGA, T. KIM, M. S. M. NAIKA, and H. S. MADHUSUDHAN, *On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions*, Indian J. Pure Appl. Math., 2004, 35(9), pp. 1047–1062.

[2] C. ADIGA, Z.-G. LIU, and A. VANITHA, *On a continued fraction of order twelve and new Eisenstein series identities*, Journal of Number Theory, 2014, 145, pp. 554–571.

[3] C. ADIGA, K. R. VASUKI, and N. BHASKAR, *Some new modular relations for the cubic functions*, South East Asian Bull. Math., 2012, 36, pp. 1–19.

[4] G. E. ANDREWS and B. C. BERNDT, *Ramanujan's Lost Notebook-III*, volume 1, Springer, 2005.

[5] B. BERNDT, *Ramanujan's Notebooks, Part III*, Springer–Verlag, New York, 1991.

[6] B. N. DHARMENDRA, M. R. R. KANNA, and R. JAGADEESH, *On continued fraction of order twelve*, Pure Math. Sci, 2012, 1, pp. 197–205.

[7] M. D. HIRSCHHORN, *On the expansion of Ramanujan's continued fraction*, The Ramanujan Journal, 1998, 2, pp. 521–527.

[8] B. L. S. LIN, *On the expansion of a continued fraction of order twelve*, International Journal of Number Theory, 2013, 9(08), pp. 2019–2031.

[9] M. S. MAHADEVA NAIKA, B. N. DHARMENDRA, and K. SHIVASHANKARA, *A continued fraction of order twelve*, Central European Journal of Mathematics, 2008, 6, pp. 393–404.

[10] M. S. M. NAIKA, B. N. DHARMENDRE, and K. SHIVASHANKAR, *A continued fraction of order twelve*, Centr. Eur. J. Math., 2008, 6 (3), pp. 393–404.

[11] S. RAMANUJAN, *Collected Papers*, Cambridge University Press, Cambridge, 1927, reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

[12] ——, *Notebooks (2 Volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.

[13] ——, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[14] S. RAMANUJAN AIYANGAR and G. E. ANDREWS, *The lost notebook and other unpublished papers*, (No Title), 1988.

[15] L. J. ROGERS, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc., 1894, 25, pp. 318–343.

[16] K. R. VASUKI, A. A. A. KAHTAN, G. SHARATH, and C. S. KUMAR, *On a continued fraction of order 12*, Ukrainian Mathematical Journal, 2011, 62,

pp. 1866–1878.

(N. A. S. Bulkhali) Department of Mathematics, Faculty of Education-Zabid, Hodeidah University, Hodeidah, Yemen.
Email address: nassbull@hotmail.com

(D. Ranganatha) Department of Mathematics, Central University of Karnataka, Kalaburagi-585367, Karnataka, India.
Email address: ddranganatha@gmail.com

(Channabasavayya) Department of Mathematics, Govt. First Grade College, Shiralakoppa-577 428, Karnataka, India.
Email address: cshmatics@gmail.com

(M. S. Surekha) Department of Mathematics, Vidya Vikas Institute of Engineering and Technology, Mysuru-570028, Karnataka, India.
Email address: surekhams82@gmail.com