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On ϕ-n-Coherent Rings and ϕ-n-Semihereditary Rings

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Abstract

This paper is devoted to the study of a generalization of coherence in ring theory. The concept of n-coherence, where $n \in \mathbb{N}^* \cup \{\infty\}$, was originally introduced as an extension of the classical notion of coherence for rings. Building upon this, we introduce the ϕ-version of n-coherence, referred to as *ϕ-n-coherence*, which unifies and generalizes several existing approaches. We further explore and characterize various important classes of rings: the notion of nonnil-coherent rings, first proposed to generalize coherence in the context of rings with nonzero nilradicals; the concept of strongly nonnil-coherent rings, recently defined to capture stronger finiteness conditions; and the class of nonnil-semihereditary rings, which extend semihereditary properties to the ϕ-torsion setting. In addition, a new characterization of Prüfer domains is provided using these generalized coherence conditions.

Keywords: ϕ-n-coherence, strongly ϕ-n-coherence, nonnil-coherence, strongly nonnil-coherence, ϕ-n-flatness, ϕ-n-absolutely pureness, (ϕ-)Prüfer ring, nonnil-semihereditary rings, ϕ-n-semihereditary rings.

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1 Introduction

In this introductory paragraph, we outline certain conventions and provide a review of standard background material. The set of nilpotent elements of a ring R is denoted by $\mathrm{Nil}(R)$, while $Z(R)$ represents the set of zero-divisors of R. A ring is termed a *ϕ-ring* if its nilradical $\mathrm{Nil}(R)$ is divided prime, meaning that $\mathrm{Nil}(R) \subset xR$ for every $x \in R \setminus \mathrm{Nil}(R)$. An ideal I of R is called *nonnil* if $I \not\subseteq \mathrm{Nil}(R)$. The notation $\mathcal{H}$ (resp., $\overline{\mathcal{H}}$) refers to the set of all rings with a divided prime nilradical (resp., those with a divided prime but not maximal nilradical). A ring R is called a *strongly ϕ-ring* if $R \in \mathcal{H}$ and $Z(R) = \mathrm{Nil}(R)$.

For a ring R and an R-module M, we define

$$\phi\text{-}\operatorname{tor}(M) = \{x \in M \mid sx = 0 \text{ for some } s \in R \setminus \mathrm{Nil}(R)\}.$$

An R-module M is called a *ϕ-torsion module* (resp., a *ϕ-torsion-free module*) if $\phi\text{-}\operatorname{tor}(M) = M$ (resp., $\phi\text{-}\operatorname{tor}(M) = 0$). An R-module M is said to be *uniformly ϕ-torsion* (or *u-ϕ-torsion*) if $sM = 0$ for some $s \in R \setminus \mathrm{Nil}(R)$, and it is said to be *$\phi$-divisible* if $M = sM$ for every $s \in R \setminus \mathrm{Nil}(R)$. The classical projective dimension and flat dimension of an R-module M are denoted by $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$, respectively.

A submodule N of an R-module M is called a *ϕ-submodule* if M/N is a ϕ-torsion module [6, Definition 2.1]. Similarly, N is called a *uniformly ϕ-submodule* (or *u-ϕ-submodule*) if M/N is a u-ϕ-torsion R-module. We also recall from [13] that a submodule N of an R-module M is said to be *pure* if the sequence $0 \to F \otimes_R N \to F \otimes_R M$ is exact for every R-module F. Moreover, as introduced in [9], N is said to be *nonnil-pure* if this sequence is exact for every finitely presented ϕ-torsion R-module F.

Let R be a ring and n a non-negative integer. An R-module M is said to be *n-presented* if there exists an exact sequence

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

such that each F_i is a finitely generated free R-module, equivalently, a finitely generated projective R-module. If M is a ϕ-torsion R-module that is n-presented, then M is called a *ϕ-n-presented* module. A finite n-presentation of a ϕ-torsion R-module is called a *ϕ-n-presentation*. Obviously, every finitely generated projective module is n-presented for every n. A module is 0-presented (resp., 1-presented) if and only if it is finitely generated (resp., finitely presented), and every m-presented module is n-presented for all $m \geq n$.

In 2002, the author of [14] defined and studied a new class of rings that generalizes the concept of classical coherent rings. Specifically, a ring R is said to be *n-coherent* if every finitely generated submodule N of a free module F with $\mathrm{pd}_R(N) \leq n$ is finitely presented, for any positive integer n, possibly infinite.

In [17], G. H. Tang, F. G. Wang, and W. Zhao introduced the notion of ϕ-von Neumann regular rings. An R-module M is said to be *ϕ-flat* if, for every R-monomorphism $f : A \to B$ with ϕ-torsion cokernel, the induced map $f \otimes 1 : A \otimes_R M \to B \otimes_R M$ is an R-monomorphism [17, Definition 3.1]. An R-module M is ϕ-flat if and only if $M_{\mathfrak{p}}$ is ϕ-flat for every prime ideal $\mathfrak{p}$ of R, or equivalently, if $M_{\mathfrak{m}}$ is ϕ-flat for every maximal ideal $\mathfrak{m}$ of R [17, Theorem 3.5]. A ϕ-ring R is called a *ϕ-von Neumann regular ring* if every R-module is ϕ-flat, which is equivalent to the condition that $R/\operatorname{Nil}(R)$ is a von Neumann regular ring [17, Theorem 4.1].

Next, the authors of [16] introduced the concept of *strongly ϕ-flat* modules, defined as follows: an R-module F is said to be strongly ϕ-flat if $\operatorname{Tor}_k^R(F, K) = 0$ for every ϕ-torsion R-module K and every $k \in \mathbb{N}^*$.

In [3], K. Bacem and A. Benhissi introduced two new classes of ϕ-rings that further generalize the concept of coherence in rings. A ϕ-ring R is called *ϕ-coherent* (resp., *nonnil-coherent*) if $R/\operatorname{Nil}(R)$ is a coherent domain [3, Corollary 3.1] (resp., every finitely generated nonnil ideal of R is finitely presented).

According to [1], a ϕ-ring R is said to be *strongly nonnil-coherent* if every ϕ-1-presented R-module is ϕ-n-presented for every $n \in \mathbb{N}^*$ [1, Definition 2.2]. It is shown in [1, Theorem 2.4] that a strongly nonnil-coherent ring is characterized by the property that every direct product of strongly ϕ-flat R-modules is strongly ϕ-flat.

D. F. Anderson and A. Badawi, in [2], introduced the concept of *ϕ-Prüfer* rings. A ϕ-ring R is ϕ-Prüfer if $R/\operatorname{Nil}(R)$ is a Prüfer domain [2, Theorem 2.6]. It is noted that every ϕ-Prüfer ring is a Prüfer ring [2, Theorem 2.14], and if $Z(R) = \operatorname{Nil}(R)$, then every Prüfer ring is ϕ-Prüfer [2, Theorem 2.16].

The authors of [20] introduced the notion of *nonnil-FP-injective modules* over rings with prime nilradical, called NP-rings. An R-module M is called nonnil-FP-injective if $\operatorname{Ext}_R^1(T, M) = 0$ for every finitely presented ϕ-torsion module T.

The authors of [9] introduced and defined the concept of the *ϕ-(weak) global dimension* for rings with prime nilradicals. An R-module P is said to be *ϕ-u-projective* if $\operatorname{Ext}_R^1(P, N) = 0$ for every u-ϕ-torsion R-module N. The *ϕ-projective dimension* of a module M over R, denoted by ϕ-$\operatorname{pd}_R M$, is said to be at most $n \geq 1$ (where $n \in \mathbb{N}$) if either $M = 0$, or M is a nonzero module that is not ϕ-u-projective but satisfies $\operatorname{Ext}_R^{n+1}(M, N) = 0$ for all u-ϕ-torsion R-modules N. If n is the smallest non-negative integer such that $\operatorname{Ext}_R^{n+1}(M, N) = 0$ for all such modules N, then we set ϕ-$\operatorname{pd}_R M = n$. If no such n exists, we define ϕ-$\operatorname{pd}_R M = \infty$.

For a ring R, the *ϕ-global dimension* is either 0 or the supremum of all ϕ-$\operatorname{pd}_R(R/I)$, where I ranges over the nonnil ideals of R such that R/I is not ϕ-u-projective. In particular, if R is a ring with $Z(R) = \operatorname{Nil}(R)$, then the ϕ-global dimension of R is the supremum of ϕ-$\operatorname{pd}_R(R/I)$ over all nonnil ideals I of R.

Similarly, the *ϕ-flat dimension* of a module M over R, denoted by ϕ-$\operatorname{fd}_R M$, is said to be at most $n \geq 1$ (where $n \in \mathbb{N}$) if either $M = 0$, or M is a nonzero module that is not ϕ-flat but satisfies $\operatorname{Tor}_{n+1}^R(M, N) = 0$ for every u-ϕ-torsion R-module N. If n is the least non-negative integer such that $\operatorname{Tor}_{n+1}^R(M, N) = 0$ for every such module N, then we set ϕ-$\operatorname{fd}_R M = n$. If no such n exists, then we define ϕ-$\operatorname{fd}_R M = \infty$.

For rings R with $Z(R) = \operatorname{Nil}(R)$, the *$\phi$-weak global dimension* of R is defined by:

$$\begin{aligned}
\phi\text{-}\operatorname{w.gl.dim}(R) &= \sup\{\phi\text{-}\operatorname{fd}_R M \mid M \text{ is a } \phi\text{-torsion module}\} \\
&= \sup\{\phi\text{-}\operatorname{fd}_R(R/I) \mid I \text{ is a nonnil ideal of } R\} \\
&= \sup\{\phi\text{-}\operatorname{fd}_R(R/I) \mid I \text{ is a finitely generated nonnil ideal of } R\}.
\end{aligned}$$

Thus, the ϕ-weak global dimension of a ring R is either 0 or the supremum of all ϕ-$\mathrm{fd}_R(R/I)$, where I is a nonnil ideal of R such that R/I is not ϕ-flat.

The authors of [20] introduced the notion of *nonnil-FP-injective modules* as follows: an R-module E is said to be nonnil-FP-injective if $\mathrm{Ext}^1_R(F,E) = 0$ for every finitely presented ϕ-torsion R-module F.

In [5], the authors introduced a new class of ϕ-rings, called *nonnil-semihereditary rings*. A ϕ-ring R is said to be nonnil-semihereditary if every finitely generated nonnil ideal is u-ϕ-projective. It was shown that nonnil-semihereditary rings coincide with ϕ-Prüfer strongly ϕ-rings. The authors also introduced the notion of *nonnil-FP-projective modules*. An R-module F is said to be nonnil-FP-projective if $\mathrm{Ext}^1_R(F,E) = 0$ for every nonnil-FP-injective R-module E.

Given a class $\mathscr{L}$ of R-modules, we denote by

$$\mathscr{L}^{\perp} = \{M \mid \mathrm{Ext}^1_R(L,M) = 0 \text{ for all } L \in \mathscr{L}\}$$

the *orthogonal class* of $\mathscr{L}$, and by

$$^{\perp}\mathscr{L} = \{M \mid \mathrm{Ext}^1_R(M,L) = 0 \text{ for all } L \in \mathscr{L}\}$$

the *dual orthogonal class* of $\mathscr{L}$.

Let $\mathcal{F}$ be a class of R-modules and M an R-module. Following [11], a homomorphism $\varphi : M \to F$ with $F \in \mathcal{F}$ is called an *$\mathcal{F}$-preenvelope* of M if for every homomorphism $f : M \to F'$ with $F' \in \mathcal{F}$, there exists $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is called an *$\mathcal{F}$-envelope* if every endomorphism $g : F \to F$ satisfying $g\varphi = \varphi$ is an isomorphism. Dually, one defines the notions of *$\mathcal{F}$-precovers* and *$\mathcal{F}$-covers*. $\mathcal{F}$-envelopes (resp., $\mathcal{F}$-covers), if they exist, are unique up to isomorphism. It is easy to see that every $\mathscr{C}$-injective preenvelope is monic, and every $\mathscr{C}$-projective precover is epic.

Following [11], a pair $(\mathscr{A},\mathscr{B})$ of classes of R-modules is called a *cotorsion pair* if $\mathscr{A}^{\perp} = \mathscr{B}$ and $^{\perp}\mathscr{B} = \mathscr{A}$. A cotorsion pair $(\mathscr{A},\mathscr{B})$ is called *hereditary* [12, Definition 1.1] if whenever $0 \to A' \to A \to A'' \to 0$ is exact with $A, A'' \in \mathscr{A}$, then $A' \in \mathscr{A}$. Equivalently, by [12, Definition 1.2], the cotorsion pair is hereditary if whenever $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathscr{B}$, then $B'' \in \mathscr{B}$.

A cotorsion pair $(\mathscr{A},\mathscr{B})$ is called *perfect* [12] if every R-module has an $\mathscr{A}$-cover and a $\mathscr{B}$-envelope. It is called *complete* (see [11, Definition 7.16] and [18, Lemma 1.13]) if for every R-module M, there exist exact sequences

$$0 \to M \to B \to A \to 0 \quad \text{with } A \in \mathscr{A},\ B \in \mathscr{B},$$

and

$$0 \to B' \to A' \to M \to 0 \quad \text{with } A' \in \mathscr{A},\ B' \in \mathscr{B}.$$

In our paper, we denote by ϕ-$\mathscr{T}_n$, where $n \in \mathbb{N}^* \cup \{\infty\}$, the class of all finitely presented ϕ-torsion R-modules whose ϕ-projective dimension is less than or equal to n.

This paper is organized into three sections, including the introduction. The second section introduces the concept of *ϕ-n-coherent rings*, which generalizes the classical notion of n-coherent rings as presented in [14]. We extend the classical concepts of n-flat and n-absolutely pure modules to define *ϕ-n-flat* and *ϕ-n-absolutely pure* modules. These notions are formally introduced in Definition 2.1: an R-module M is said to be ϕ-n-flat (resp., ϕ-n-absolutely pure) for $n \in \mathbb{N}^* \cup \{\infty\}$ if $\operatorname{Tor}_1^R(M, N) = 0$ (resp., $\operatorname{Ext}_R^1(N, M) = 0$) for all $N \in \phi$-$\mathscr{T}_n$.

In Theorem 2.2, we characterize ϕ-n-absolutely pure modules, while Propositions 2.3 and 2.4 present additional results concerning families of modules. Theorem 2.5 establishes a connection between ϕ-n-flat and ϕ-n-absolutely pure modules.

Next, we define the concept of a *ϕ-n-coherent ring* in Definition 2.9, generalizing the classical notion of an n-coherent ring by incorporating ϕ-torsion submodules into the coherence condition. Theorem 2.11 provides a series of equivalences characterizing ϕ-n-coherent rings, demonstrating that such rings satisfy several properties related to extension preservation, direct products, and conditions involving flat and absolutely pure modules. Lemma 2.12 and Corollary 2.13 offer further characterizations for nonnil-coherent rings and their associated modules in the context of ϕ-coherence.

We also explore results concerning *ϕ-n-FI-injective modules*, which generalize the concept of FI-injectivity introduced in [15]. Several characterizations of strongly nonnil-coherent rings and their modules are presented. Specifically, in Corollary 2.25, we show that a ϕ-ring R is strongly nonnil-coherent if and only if certain conditions on exact sequences of R-modules are satisfied, including the nonnil-FP-injectivity of specific modules.

Proposition 2.26 characterizes when an R-module over a ϕ-n-strongly coherent ring is injective, and Corollary 2.29 extends these results by providing equivalent conditions for an R-module to be injective in the strongly nonnil-coherent case. Theorem 2.28 offers a second characterization of ϕ-n-strongly coherent rings, while additional equivalences involving injective covers and pure quotients of modules are also discussed. Theorem 2.33 and its corollaries characterize when every ϕ-n-absolutely pure R-module is injective, yielding further insights into the structure of ϕ-n-strongly coherent rings.

Finally, Corollary 2.38 shows that strongly nonnil-coherent rings are characterized by the fact that nonnil-FP-injectivity (resp., ϕ-flatness) forms a co(resolving) class.

In this section, we define the notion of *ϕ-n-semihereditary rings* for $n \in \mathbb{N}^* \cup \{\infty\}$, as introduced in Definition 3.1. A ϕ-ring R is said to be *ϕ-n-semihereditary* if, for every exact sequence

$$0 \to K \to P \to C \to 0,$$

where $C \in \phi$-$\mathscr{T}_n$ and P is a finitely generated projective R-module, the module K is also projective. This definition is followed by a remark noting that a ϕ-ring is nonnil-semihereditary if and only if it is ϕ-(∞)-semihereditary, according to [5, Theorem 3.8].

The characterization of ϕ-n-semihereditary rings is provided in Theorem 3.3, where several equivalent conditions are established. These include ϕ-n-coherence and various homological properties involving ϕ-n-flat and ϕ-n-absolutely pure modules. Corollary 3.8 extends these results to the context of nonnil-semihereditary rings, offering deeper insight into the interplay between these rings and their associated module categories.

Next, we introduce the concept of *weakly ϕ-n-semihereditary rings.* A ϕ-ring R is said to be weakly ϕ-n-semihereditary if, for every short exact sequence

$$0 \to K \to P \to Q \to 0,$$

where $Q \in \phi$-$\mathscr{T}_n$ and P is a finitely generated projective R-module, the module K is flat. This notion is formally defined in Definition 3.6, and additional characterizations are provided in Theorem 3.7 and Corollary 3.8, showing that submodules of flat modules preserve ϕ-n-flatness in this context.

Finally, Theorem 3.9 presents a more detailed characterization of ϕ-n-semihereditary rings. These developments culminate in a comprehensive view of the structure and properties of nonnil-semihereditary and ϕ-Prüfer rings, as synthesized in Corollary 3.5 of [5].

2 On ϕ-n-coherent rings

Refer to [14] for the definitions of n-flat and n-absolutely pure R-modules, where $n \in \mathbb{N}^* \cup \{+\infty\}$. An R-module M is said to be *n-flat* (resp., *n-absolutely pure*) if $\mathrm{Tor}_1^R(X, M) = 0$ (resp., $\mathrm{Ext}_R^1(X, M) = 0$) for every finitely presented R-module X with $\mathrm{pd}_R(X) \leq n$.

The forthcoming Definition 2.1 extends these classical notions in the setting of ϕ-torsion modules. Throughout this paper, we denote by ϕ-$\mathscr{T}_n$ the class of all finitely presented ϕ-torsion R-modules whose ϕ-projective dimension is less than or equal to n, where $n \in \mathbb{N}^* \cup \{\infty\}$.

Definition 2.1. Let R be a ring and let $n \in \mathbb{N}^* \cup \{+\infty\}$. An R-module X is said to be:

1. *ϕ-n-flat* if $\mathrm{Tor}_1^R(M, X) = 0$ for every $M \in \phi$-$\mathscr{T}_n$.
2. *ϕ-n-absolutely pure* if $\mathrm{Ext}_R^1(M, X) = 0$ for every $M \in \phi$-$\mathscr{T}_n$.

The following Theorem 2.2 provides a characterization of ϕ-n-absolutely pure R-modules.

Theorem 2.2. *Let M be an R-module. Then the following statements are equivalent:*

1. *M is ϕ-n-absolutely pure.*
2. *M is injective with respect to every exact sequence*

$$0 \to K \to P \to C \to 0$$

of R-modules, where $C \in \phi$-$\mathscr{T}_n$, P is finitely generated projective, and K is finitely generated.

3. *There exists a pure exact sequence*

$$0 \to M \to M' \to M'' \to 0$$

of R-modules such that M' is ϕ-n-absolutely pure.

Proof. This follows immediately from [21, Theorem 2.4]. □

Proposition 2.3. *Let $\{M_i \mid i \in I\}$ be a family of R-modules. Then the following statements are equivalent:*

1. *Each M_i is ϕ-n-absolutely pure.*
2. *$\prod_{i\in I} M_i$ is ϕ-n-absolutely pure.*
3. *$\bigoplus_{i\in I} M_i$ is ϕ-n-absolutely pure.*

Proof. This follows immediately from [21, Proposition 2.5]. □

Proposition 2.4. *Let $\{M_i \mid i \in I\}$ be a family of R-modules. Then the following conditions are equivalent:*

1. *Each M_i is ϕ-n-flat.*
2. *$\bigoplus_{i\in I} M_i$ is ϕ-n-flat.*

Proof. This follows immediately from [21, Proposition 2.6]. □

Theorem 2.5. *Let M be an R-module. Then M is ϕ-n-flat if and only if M^+ is ϕ-n-absolutely pure.*

Proof. This follows immediately from [21, Theorem 2.7]. □

Corollary 2.6. *Every pure submodule of a ϕ-n-flat module is also ϕ-n-flat.*

Proof. This follows immediately from [21, Corollary 2.8]. □

Recall from the Introduction the definition of an n-coherent ring, where $n \in \mathbb{N}^* \cup \{+\infty\}$. A ring is said to be *$n$-coherent* if every finitely generated submodule N of a free module F with $\mathrm{pd}_R(N) \leq n-1$ is also finitely presented.

Definition 2.7. A ϕ-ring R is called *ϕ-n-coherent*, where $n \in \mathbb{N}^* \cup \{+\infty\}$, if every finitely generated ϕ-submodule N of a finitely generated free R-module F satisfying ϕ-$\mathrm{pd}_R(N) \leq n-1$ is finitely presented.

Remark 2.8. *From [7, Proposition 3.2], when R is considered as a ϕ-ring, it follows that R is nonnil-coherent if and only if it is ϕ-(∞)-coherent.*

As shown in [14], there are various characterizations of n-coherent rings. To draw parallels with those, we introduce the following auxiliary definition:

Definition 2.9. Let $n \in \mathbb{N}^* \cup \{+\infty\}$. A submodule N of an R-module E is called *ϕ-n-pure* if the short exact sequence

$$0 \to N \to E \to E/N \to 0$$

is preserved by $\mathrm{Hom}_R(F,-)$ for every finitely presented ϕ-torsion R-module F with ϕ-$\mathrm{pd}_R(F) \leq n$. In this case, the sequence is referred to as a *ϕ-n-pure short exact sequence.*

Remark 2.10. *By [8, Theorem 2.9], an R-submodule N of an R-module M is nonnil-pure if and only if it is ϕ-(∞)-pure.*

We now arrive at the main theorem of this section. To proceed, recall that for any R-module M, we define its character module by $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Theorem 2.11. *The following statements are equivalent for a ring R and $n \in \mathbb{N}^* \cup \{\infty\}$:*

1. *R is ϕ-n-coherent.*
2. *For any $C \in \phi$-$\mathscr{T}_n$ and any direct system $(M_\alpha)_{\alpha\in A}$ of R-modules, we have*

$$\varinjlim \mathrm{Ext}^1_R(C, M_\alpha) \cong \mathrm{Ext}^1_R\left(C, \varinjlim M_\alpha\right).$$

3. *For any family $\{N_\alpha\}$ of R-modules and any $C \in \phi$-$\mathscr{T}_n$, we have*

$$\mathrm{Tor}^R_1\left(\prod N_\alpha, C\right) \cong \prod \mathrm{Tor}^R_1(N_\alpha, C).$$

4. *Any direct product of copies of R is ϕ-n-flat.*
5. *Any direct product of ϕ-n-flat R-modules is ϕ-n-flat.*
6. *Any direct limit of ϕ-n-absolutely pure R-modules is ϕ-n-absolutely pure.*
7. *Any direct limit of injective R-modules is ϕ-n-absolutely pure.*
8. *An R-module M is ϕ-n-absolutely pure if and only if M^+ is ϕ-n-flat.*
9. *An R-module M is ϕ-n-absolutely pure if and only if M^{++} is ϕ-n-absolutely pure.*
10. *An R-module M is ϕ-n-flat if and only if M^{++} is ϕ-n-flat.*
11. *For any ring S, and for all $C \in \phi$-$\mathscr{T}_n$, B a left R- and right S-module, and E an injective right S-module, we have*

$$\mathrm{Tor}^R_1\left(\mathrm{Hom}_S(B, E), C\right) \cong \mathrm{Hom}_S\left(\mathrm{Ext}^1_R(C, B), E\right).$$

12. *For every injective R-module E and every ϕ-n-pure submodule N of E, the quotient E/N is ϕ-n-absolutely pure.*
13. *$\mathrm{Ext}^2_R(C, N) = 0$ for every $C \in \phi$-$\mathscr{T}_n$ and every FP-injective R-module N.*
14. *If N is a ϕ-n-absolutely pure R-module and N_1 is an FP-injective submodule of N, then the quotient N/N_1 is ϕ-n-absolutely pure.*
15. *For every FP-injective R-module N, the quotient $E(N)/N$ is ϕ-n-absolutely pure, where $E(N)$ denotes the injective envelope of N.*
16. *The dual of any projective (or free) R-module is ϕ-n-flat.*

If, in addition, R is a strongly ϕ-ring, then the above statements are also equivalent to the following:

17. *Every R-module has a ϕ-n-flat preenvelope.*
18. *For every ϕ-n-absolutely pure R-module E and every ϕ-n-pure submodule N of E, the quotient E/N is ϕ-n-absolutely pure.*

The proof of Theorem 2.11 requires the following lemma.

Lemma 2.12. *Let $n \in \mathbb{N}^*$. For a strongly ϕ-ring R, the following conditions are equivalent:*

1. *R is a ϕ-n-coherent ring.*
2. *$\mathrm{Ext}^2_R(M, N) = 0$ for every finitely presented ϕ-torsion R-module M with ϕ-$\mathrm{pd}_R(M) \leq n$, and for every ϕ-n-absolutely pure R-module N.*

Proof. Assume that R is a ϕ-n-coherent ring, and let M be a finitely presented ϕ-torsion R-module with $\phi\text{-pd}_R(M) \leq n$. Then there exists a short exact sequence

$$0 \to H \to F \to M \to 0,$$

where F is a finitely generated free R-module and H is a finitely generated ϕ-submodule of F. Since R is a strongly ϕ-ring, it follows that $\phi\text{-pd}_R(H) \leq n-1$. Therefore, for any ϕ-n-absolutely pure R-module N, the above sequence induces the exact sequence

$$\text{Ext}^1_R(H,N) \to \text{Ext}^2_R(M,N) \to 0.$$

Because N is ϕ-n-absolutely pure, we have $\text{Ext}^1_R(H,N) = 0$, and thus $\text{Ext}^2_R(M,N) = 0$.

Conversely, assume that $\text{Ext}^2_R(M,N) = 0$ for all finitely presented ϕ-torsion R-modules M with $\phi\text{-pd}_R(M) \leq n$, and all ϕ-n-absolutely pure R-modules N. Let M be a finitely generated ϕ-submodule of a finitely generated free R-module F such that $\phi\text{-pd}_R(M) \leq n-1$. Since $Z(R) = \text{Nil}(R)$, it follows that $\phi\text{-pd}_R(F/M) \leq n$, so $\text{Ext}^2_R(F/M,N) = 0$ for every ϕ-n-absolutely pure R-module N. From the long exact sequence of Ext, we get $\text{Ext}^1_R(M,N) = 0$ for each such N, and in particular, for every absolutely pure R-module N. By [10], this implies that M is finitely presented. Hence, R is a ϕ-n-coherent ring. □

Proof of Theorem 2.11. The equivalences between statements (1) through (12) follow immediately from [21, Theorem 3.3].

The equivalences between statements (1), (13), (14), and (15) follow from [21, Theorem 3.5].

The equivalence between statements (1) and (16) follows directly from [22, Theorem 2.1].

(1) ⇒ (17): Assume that R is a ϕ-n-coherent ring, and let N be a ϕ-n-pure submodule of a ϕ-n-absolutely pure R-module E. Then, for every finitely presented ϕ-torsion R-module M with $\phi\text{-pd}_R(M) \leq n$, the long exact sequence of Ext gives

$$\text{Ext}^1_R(M,E) \longrightarrow \text{Ext}^1_R(M,E/N) \longrightarrow \text{Ext}^2_R(M,N).$$

Since N is ϕ-n-pure in E, the following commutative diagram with exact rows arises for every such M:

$$\begin{array}{ccccccc}
\text{Hom}_R(M,E) & \longrightarrow & \text{Hom}_R(M,E/N) & \longrightarrow & \text{Ext}^1_R(M,N) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
\text{Hom}_R(M,E) & \longrightarrow & \text{Hom}_R(M,E/N) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

From this diagram, it follows that $\text{Ext}^1_R(M,N) = 0$, so N is ϕ-n-absolutely pure. By Lemma 2.12, we have $\text{Ext}^2_R(M,N) = 0$. Moreover, since E is ϕ-n-absolutely pure, $\text{Ext}^1_R(M,E) = 0$. Hence, the exact sequence above yields $\text{Ext}^1_R(M,E/N) = 0$, showing that E/N is also ϕ-n-absolutely pure.

(17) $\Rightarrow$ (18): This implication is straightforward.

(18) $\Rightarrow$ (1): Let N be a ϕ-n-absolutely pure R-module, and let M be a finitely presented ϕ-torsion R-module with $\phi\text{-pd}_R(M) \leq n$. To show that R is ϕ-n-coherent, by Lemma 2.12 it suffices to show that $\operatorname{Ext}_R^2(M, N) = 0$.

Let E be an injective R-module containing N. Since N is a ϕ-n-absolutely pure submodule of E, the sequence

$$0 \to N \to E \to E/N \to 0$$

is a ϕ-n-pure exact sequence. By assumption, E/N is also ϕ-n-absolutely pure. Hence, $\operatorname{Ext}_R^1(M, E/N) = 0$. From the associated long exact sequence of Ext, it follows that

$$\operatorname{Ext}_R^1(M, E/N) \cong \operatorname{Ext}_R^2(M, N),$$

and so $\operatorname{Ext}_R^2(M, N) = 0$, as desired. □

The following Corollary 2.13 provides a complete characterization of nonnil-coherent rings. It extends the well-known result [7, Corollary 3.13].

Corollary 2.13. *The following statements are equivalent for a ϕ-ring R:*

1. *R is nonnil-coherent.*
2. *For any finitely presented ϕ-torsion R-module C and any direct system $(M_\alpha)_{\alpha \in A}$ of R-modules, we have*

$$\varinjlim \operatorname{Ext}_R^1(C, M_\alpha) \cong \operatorname{Ext}_R^1\left(C, \varinjlim M_\alpha\right).$$

3. *For any family $\{N_\alpha\}$ of R-modules and any finitely presented ϕ-torsion R-module C, we have*
$$\operatorname{Tor}_1^R\left(\prod N_\alpha, C\right) \cong \prod \operatorname{Tor}_1^R(N_\alpha, C).$$
4. *Any direct product of copies of R is ϕ-flat.*
5. *Any direct product of ϕ-flat R-modules is ϕ-flat.*
6. *Any direct limit of nonnil-FP-injective R-modules is nonnil-FP-injective.*
7. *Any direct limit of injective R-modules is nonnil-FP-injective.*
8. *An R-module M is nonnil-FP-injective if and only if M^+ is ϕ-flat.*
9. *An R-module M is nonnil-FP-injective if and only if M^{++} is nonnil-FP-injective.*
10. *An R-module M is ϕ-flat if and only if M^{++} is ϕ-flat.*
11. *For any ring S, and any finitely presented ϕ-torsion R-module C, R-S-bimodule B, and injective S-module E, we have*

$$\operatorname{Tor}_1^R\left(\operatorname{Hom}_S(B, E), C\right) \cong \operatorname{Hom}_S\left(\operatorname{Ext}_R^1(C, B), E\right).$$

12. *For every injective R-module E and every nonnil-FP-injective submodule N of E, the quotient E/N is nonnil-FP-injective.*
13. *$\operatorname{Ext}_R^2(C, N) = 0$ for every finitely presented ϕ-torsion R-module C and every FP-injective R-module N.*

14. *If N is a nonnil-FP-injective R-module and N_1 is an FP-injective submodule of N, then N/N_1 is nonnil-FP-injective.*
15. *For every FP-injective R-module N, the quotient $E(N)/N$ is nonnil-FP-injective, where $E(N)$ denotes the injective envelope of N.*
16. *The dual of any projective (or free) R-module is ϕ-flat.*

If, in addition, R is a strongly ϕ-ring, then the above are also equivalent to:

17. *Every R-module has a ϕ-flat preenvelope.*
18. *For every nonnil-FP-injective R-module E and every nonnil-pure submodule N of E, the quotient E/N is nonnil-FP-injective.*

Proof. This follows immediately from Remark 2.8 and Theorem 2.11 by taking $n = \infty$. □

To explore further properties of nonnil-coherence, we now present a strengthened analogue of the well-known behavior of FI-injective modules, as introduced and studied by the authors in [4].

Definition 2.14. *Let $R \in \mathcal{H}$ and $n \in \mathbb{N}^* \cup \{\infty\}$. An R-module E is said to be ϕ-n-*FI-injective *if* $\mathrm{Ext}^1_R(F, E) = 0$ *for every ϕ-n-absolutely pure R-module F. When $n = \infty$, the ϕ-n-FI-injective modules are simply called* ϕ-FI-injective.

Remark 2.15. *It is clear that every ϕ-FI-injective R-module is also FI-injective. Moreover, the ϕ-n-FI-injective modules correspond to the $\mathscr{C}$I-injective modules in [22], where the class $\mathscr{C}$ is taken to be ϕ-$\mathscr{T}_n$.*

Proposition 2.16. *Let R be an integral domain and E an R-module. Then E is ϕ-FI-injective if and only if E is FI-injective.*

Proof. This follows immediately from [20, Theorem 1.6] and the definition of FI-injectivity. □

The following proposition (Proposition 2.17) establishes the equivalence of several conditions for an R-module M, including its ϕ-n-FI-injectivity and properties related to exact sequences and injective precovers associated with this class of modules.

Proposition 2.17. *Let $n \in \mathbb{N}^* \cup \{\infty\}$ and let $R \in \mathcal{H}$. The following statements are equivalent for an R-module M:*

1. *M is ϕ-n-FI-injective.*
2. *For every exact sequence*

$$0 \to M \to E \to L \to 0,$$

where E is ϕ-n-FI-injective, the map $E \to L$ is a ϕ-n-FI-injective precover of L.

3. *M is the kernel of a ϕ-n-FI-injective precover $f : E \to L$ with E injective.*
4. *M is injective with respect to every exact sequence*

$$0 \to A \to B \to C \to 0,$$

where C is ϕ-n-FI-injective.

Proof. This follows immediately from Remark 2.15 and [22, Proposition 2.3]. □

Recall that an R-module M is said to be *pure injective* [11, Definition 5.3.6] if it is injective with respect to every pure exact sequence of R-modules. The following definition generalizes this concept to the context of nonnil-purity.

Definition 2.18. *Let $n \in \mathbb{N}^* \cup \{\infty\}$ and let $R \in \mathcal{H}$. An R-module M is said to be* ϕ-n-pure injective *if it is injective with respect to every nonnil-pure exact sequence of R-modules. When $n = \infty$, such modules are simply called* nonnil-pure injective.

Remark 2.19. *Obviously, every ϕ-n-pure injective module is pure injective. Moreover, the class of ϕ-n-pure injective modules corresponds to the class of $\mathscr{C}$-pure injective modules defined in [22], by taking $\mathscr{C} = \phi$-$\mathscr{T}_n$.*

Proposition 2.20. *Let $R \in \mathcal{H}$ and $n \in \mathbb{N}^* \cup \{\infty\}$. If R is a ϕ-n-coherent ring, then every ϕ-n-pure injective R-module M admits a ϕ-n-FP-injective cover $f : N \to M$, where N is injective. Moreover, $\ker(f)$ is a ϕ-n-FI-injective module that contains a nonzero injective submodule.*

Proof. This follows immediately from Remark 2.19 and [22, Proposition 2.7]. □

Corollary 2.21. *Let R be a nonnil-coherent ring. Then every nonnil-pure injective R-module M admits a nonnil-FP-injective cover $f : N \to M$, where N is injective. Moreover, $\ker(f)$ is a ϕ-FI-injective module that contains a nonzero injective submodule.*

Proof. This follows immediately from Remark 2.19 and Proposition 2.20. □

The following theorem characterizes ϕ-n-FI-injective modules over ϕ-n-coherent rings.

Theorem 2.22. *Let $R \in \mathcal{H}$ and $n \in \mathbb{N}^* \cup \{\infty\}$. If R is a ϕ-n-coherent ring, then an R-module M is ϕ-n-FI-injective if and only if M is the direct sum of an injective R-module and a ϕ-n-FI-injective R-module that contains a nonzero injective submodule.*

Proof. This follows immediately from Remark 2.19 and [22, Theorem 2.8]. □

Next, the authors of [1] introduced the notion of *strongly nonnil-coherent rings* as ϕ-rings for which every finitely presented ϕ-torsion R-module is n-presented for all $n \in \mathbb{N}^*$; see [1, Definition 2.2]. By taking $\mathscr{C} = \phi$-$\mathscr{T}_\infty$, this notion coincides with the concept of *strongly $\mathscr{C}$-coherent rings* as defined in [22], as shown in [22, Lemma 3.7]. Building upon the work of [22], one can derive several characterizations of strongly nonnil-coherent rings in the sense of [1].

Definition 2.23. *Let $n \in \mathbb{N}^* \cup \{\infty\}$ and let $R \in \mathcal{H}$. Then R is said to be* ϕ-n-strongly coherent *if every $F \in \phi$-$\mathscr{T}_n$ is n-presented. In particular, every ϕ-n-strongly coherent ring is ϕ-n-coherent.*

The class of ϕ-n-strongly coherent rings can be characterized as follows.

Theorem 2.24. *The following statements are equivalent for a ϕ-ring R and $n \in \mathbb{N}^* \cup \{\infty\}$:*

1. *R is a ϕ-n-strongly coherent ring.*
2. *If $0 \to K \to E \to L \to 0$ is an exact sequence of R-modules with K ϕ-n-absolutely pure and E FP-injective, then L is ϕ-n-absolutely pure.*
3. *If $0 \to K \to E \to L \to 0$ is an exact sequence of R-modules with K ϕ-n-absolutely pure and E injective, then L is ϕ-n-absolutely pure.*
4. *R is ϕ-n-coherent, and if*
$$0 \to N \to M \to Q \to 0$$
is an exact sequence of R-modules with M and Q ϕ-n-flat, then N is ϕ-n-flat.
5. *R is ϕ-n-coherent, and if*
$$0 \to N \to M \to Q \to 0$$
is an exact sequence of R-modules with M flat and Q ϕ-n-flat, then N is ϕ-n-flat.
6. *R is ϕ-n-coherent, and if*
$$0 \to N \to P \to Q \to 0$$
is an exact sequence of R-modules with P projective and Q ϕ-n-flat, then N is ϕ-n-flat.

Proof. This follows immediately from [22, Theorem 3.1]. □

The following Corollary 2.25 provides a characterization of strongly nonnil-coherent rings.

Corollary 2.25. *The following statements are equivalent for a ϕ-ring R and $n \in \mathbb{N}^* \cup \{\infty\}$:*

1. *R is a strongly nonnil-coherent ring.*
2. *If*
$$0 \to K \to E \to L \to 0$$
is an exact sequence of R-modules with K nonnil-FP-injective and E FP-injective, then L is nonnil-FP-injective.
3. *If*
$$0 \to K \to E \to L \to 0$$
is an exact sequence with K nonnil-FP-injective and E injective, then L is nonnil-FP-injective.
4. *R is nonnil-coherent, and if*
$$0 \to N \to M \to Q \to 0$$
is an exact sequence with M and Q ϕ-flat, then N is ϕ-flat.

5. *R is nonnil-coherent, and if*

$$0 \to N \to M \to Q \to 0$$

is exact with M flat and Q ϕ-flat, then N is ϕ-flat.

6. *R is nonnil-coherent, and if*

$$0 \to N \to P \to Q \to 0$$

is exact with P projective and Q ϕ-flat, then N is ϕ-flat.

Proof. This follows immediately from Theorem 2.24 by setting $n = \infty$. □

Next, Proposition 2.26 characterizes when an R-module over a ϕ-n-strongly coherent ring is injective.

Proposition 2.26. *Let $R \in \mathcal{H}$ and $n \in \mathbb{N}^* \cup \{\infty\}$. If R is a ϕ-n-strongly coherent ring, then the following statements are equivalent for an R-module M:*

1. *M is injective.*
2. *M is both ϕ-n-injective and ϕ-n-FI-injective.*
3. *There exists a ϕ-n-injective cover $f : M \to N$, where N is ϕ-n-FI-injective.*

Proof. This follows immediately from [22, Proposition 3.2]. □

Corollary 2.27. *Let R be a strongly nonnil-coherent ring. Then the following statements are equivalent for an R-module M:*

1. *M is injective.*
2. *M is both nonnil-FP-injective and ϕ-FI-injective.*
3. *There exists a ϕ-injective cover $f : M \to N$ with N being ϕ-FI-injective.*

Proof. Follows directly from Proposition 2.26 by setting $n = \infty$. □

Our second characterization of ϕ-n-strong coherence is given below.

Theorem 2.28. *Let $n \in \mathbb{N}^* \cup \{\infty\}$. The following statements are equivalent for a ϕ-ring R:*

1. *R is a ϕ-n-strongly coherent ring.*
2. *R is ϕ-n-coherent, and every ϕ-n-injective R-module that is also ϕ-n-FI-injective is injective.*
3. *Every R-module has a ϕ-n-injective cover, and every ϕ-n-injective ϕ-n-FI-injective R-module is injective.*
4. *R is ϕ-n-coherent, and for every ϕ-n-FI-injective R-module L, there exists a ϕ-n-injective cover $E \to L$ with E injective.*
5. *Every R-module has a ϕ-n-injective cover, and for every ϕ-n-FI-injective R-module L, there exists a ϕ-n-injective cover $E \to L$ with E injective.*
6. *Every ϕ-n-pure quotient of a ϕ-n-injective R-module has a ϕ-n-injective cover, and for every ϕ-n-FI-injective R-module L, there exists a ϕ-n-injective cover $E \to L$ with E injective.*

7. *Every ϕ-n-pure quotient of a ϕ-n-injective R-module has a ϕ-n-injective cover, and every ϕ-n-injective ϕ-n-FI-injective R-module is injective.*

Proof. This follows immediately from [22, Theorem 3.3]. □

The following Corollary 2.29 provides another characterization of strongly nonnil-coherent rings.

Corollary 2.29. *The following statements are equivalent for a ϕ-ring R:*

1. *R is a strongly nonnil-coherent ring.*
2. *R is nonnil-coherent, and every nonnil-FP-injective R-module that is also ϕ-FI-injective is injective.*
3. *Every R-module has a nonnil-FP-injective cover, and every nonnil-FP-injective ϕ-FI-injective R-module is injective.*
4. *R is nonnil-coherent, and for every ϕ-FI-injective R-module L, there exists a nonnil-FP-injective cover $E \to L$ with E injective.*
5. *Every R-module has a nonnil-FP-injective cover, and for every ϕ-FI-injective R-module L, there exists a nonnil-FP-injective cover $E \to L$ with E injective.*
6. *Every nonnil-pure quotient of a nonnil-FP-injective R-module has a nonnil-FP-injective cover, and for every ϕ-FI-injective R-module L, there exists a nonnil-FP-injective cover $E \to L$ with E injective.*
7. *Every nonnil-pure quotient of a nonnil-FP-injective R-module has a nonnil-FP-injective cover, and every nonnil-FP-injective ϕ-FI-injective R-module is injective.*

Proof. This follows immediately from Theorem 2.28 by taking $n = \infty$. □

Recall from [5, Definition 4.1] that an R-module M is said to be *nonnil-FP-projective* if $\mathrm{Ext}_R^1(M, N) = 0$ for every nonnil-FP-injective R-module N.

Definition 2.30. *Let $n \in \mathbb{N}^* \cup \{\infty\}$ and let $R \in \mathcal{H}$. An R-module P is said to be ϕ-n-*FP-projective *if $\mathrm{Ext}_R^1(P, E) = 0$ for every ϕ-n-absolutely pure R-module E. When $n = \infty$, ϕ-n-FP-projective modules coincide with nonnil-FP-projective modules.*

Remark 2.31. *The concept of ϕ-n-FP-projectivity corresponds to the notion of $\mathscr{C}$-projectivity defined in [22], where $\mathscr{C} = \phi$-$\mathscr{T}_n$.*

Proposition 2.32. *Let R be an integral domain and M an R-module. Then M is nonnil-FP-projective if and only if M is FP-projective.*

Proof. If R is an integral domain, then by [20, Theorem 1.6], an R-module N is nonnil-FP-injective if and only if $\mathrm{Ext}_R^1(R/I, N) = 0$ for every finitely generated ideal I of R. This implies that any nonnil-FP-projective module over a coherent domain is FP-projective. □

The following theorem establishes when every ϕ-n-absolutely pure R-module is injective.

Theorem 2.33. *Let $n \in \mathbb{N}^* \cup \{\infty\}$. The following statements are equivalent for a ϕ-ring R:*

1. *Every R-module is ϕ-n-FP-projective.*
2. *Every ϕ-n-absolutely pure R-module is injective.*
3. *Every nonzero R-module has a nonzero ϕ-n-FP-projective submodule.*
4. *R is ϕ-n-strongly coherent, and every ϕ-n-absolutely pure R-module has a ϕ-n-FP-projective cover with the unique mapping property.*

Proof. This follows immediately from [22, Theorem 3.4]. □

Corollary 2.34. *The following statements are equivalent for a ϕ-ring R:*

1. *Every R-module is nonnil-FP-projective.*
2. *Every nonnil-FP-injective R-module is injective.*
3. *Every nonzero R-module has a nonzero nonnil-FP-projective submodule.*
4. *R is strongly nonnil-coherent, and every nonnil-FP-injective R-module has a nonnil-FP-projective cover with the unique mapping property.*

Proof. This follows immediately from Theorem 2.33 by taking $n = \infty$. □

From [23], a ring R is said to be *strongly $\mathscr{C}$-coherent*, where $\mathscr{C}$ is a class of finitely presented R-modules, if every short exact sequence of R-modules

$$0 \to K \to P \to Q \to 0,$$

with $Q \in \mathscr{C}$ and P finitely generated projective, implies that K is $\mathscr{C}$-projective; see [23, Definition 2]. Our goal is to justify that, for $\mathscr{C} = \phi$-$\mathscr{T}_\infty$, the notion of strongly $\mathscr{C}$-coherence in [23] coincides with the notion of strongly $\mathscr{C}$-coherence in [1].

Definition 2.35. *Let $n \in \mathbb{N}^* \cup \{\infty\}$ and let $R \in \mathcal{H}$. Then R is said to be* strongly nonnil-n-coherent *if it is strongly $\mathscr{C}$-coherent in the sense of [23], where $\mathscr{C} = \phi$-$\mathscr{T}_n$. When $n = \infty$, we refer to this as* strongly ϕ-coherent *instead of strongly nonnil-∞-coherent.*

The following theorem characterizes strongly nonnil-n-coherent rings.

Theorem 2.36. *The following statements are equivalent for a ring R:*

1. *R is strongly nonnil-n-coherent.*
2. *For every $C \in \phi$-$\mathscr{T}_n$, there exists an exact sequence*

$$0 \to K \to P \to C \to 0,$$

where P is finitely generated projective and K is ϕ-n-FP-projective.

3. *For every $C \in \phi$-$\mathscr{T}_n$, there exists an exact sequence*

$$0 \to K \to P \to C \to 0,$$

where P is projective and K is ϕ-n-FP-projective.

4. *If*

$$0 \to K \to P \to C \to 0$$

is exact with $C \in \phi$-$\mathscr{T}_n$ and P projective, then K is ϕ-n-FP-projective.

5. $\mathrm{Ext}_R^{k+1}(C,N)=0$ *for every* $k\in\mathbb{N}$*, every* $C\in\phi$-$\mathscr{T}_n$*, and every* ϕ-n-*absolutely pure* R-*module* N.
6. $\mathrm{Ext}_R^2(C,N)=0$ *for every* $C\in\phi$-$\mathscr{T}_n$ *and every* ϕ-n-*absolutely pure* R-*module* N.
7. *If* N *is a* ϕ-n-*absolutely pure* R-*module and* $N_1\subseteq N$ *is a* ϕ-n-*absolutely pure submodule, then* N/N_1 *is* ϕ-n-*absolutely pure.*
8. *For every* ϕ-n-*absolutely pure* R-*module* N*, the quotient* $E(N)/N$ *is* ϕ-n-*absolutely pure, where* $E(N)$ *denotes the injective hull of* N.
9. *The pair* $(\phi$-$\mathscr{T}_n\mathscr{P},\phi$-$\mathscr{T}_n\mathcal{I})$ *forms a hereditary cotorsion pair.*
10. R *is* ϕ-n-*coherent, and* $\mathrm{Tor}_{k+1}^R(M,C)=0$ *for every* $k\in\mathbb{N}$*, every* $C\in\phi$-$\mathscr{T}_n$*, and every* ϕ-n-*flat* R-*module* M.
11. R *is* ϕ-n-*coherent, and* $\mathrm{Tor}_2^R(M,C)=0$ *for every* $C\in\phi$-$\mathscr{T}_n$ *and every* ϕ-n-*flat* R-*module* M.
12. $\mathrm{Ext}_R^{k+1}(P,N)=0$ *for every* $k\in\mathbb{N}$*, every* ϕ-n-*FP-projective* R-*module* P*, and every* ϕ-n-*absolutely pure* R-*module* N.

Proof. This follows immediately from [23, Theorem 1]. □

Before concluding with the next characterization of strongly nonnil-coherent rings, we first present the following corollary.

Corollary 2.37. *Let* $R\in\mathcal{H}$*. Then* R *is strongly nonnil-coherent if and only if* R *is strongly* ϕ-*coherent.*

Proof. If R is a strongly nonnil-coherent ring, then R is nonnil-coherent by [1, Corollary 2.7], and every ϕ-flat R-module F satisfies $\mathrm{Tor}_k^R(F,C)=0$ for all $k>0$, by [1, Theorem 2.4]. It then follows from Theorem 2.36 (10) that R is strongly ϕ-coherent.

Conversely, if R is a strongly ϕ-coherent ring, then R is nonnil-coherent by Theorem 2.36 (10), and so every direct product of copies of R is ϕ-flat by [6, Theorem 2.6]. Again, by Theorem 2.36 (10), every direct product of copies of R is strongly ϕ-flat. Therefore, R is strongly nonnil-coherent by [6, Theorem 2.6]. □

As a consequence of Theorem 2.36 and Corollary 2.37, we obtain the following result, which characterizes strongly nonnil-coherent rings by the fact that the nonnil-FP-projective and ϕ-flat modules form resolving classes, while the nonnil-FP-injective modules form a coresolving class.

Corollary 2.38. *The following statements are equivalent for a ring* R:

1. R *is strongly nonnil-coherent.*
2. *For every finitely presented* ϕ-*torsion* R-*module* C*, there exists an exact sequence*
$$0\to K\to P\to C\to 0,$$
where P *is finitely generated projective and* K *is nonnil-FP-projective.*
3. *For every finitely presented* ϕ-*torsion* R-*module* C*, there exists an exact sequence*
$$0\to K\to P\to C\to 0,$$
where P *is projective and* K *is nonnil-FP-projective.*

4. *If*
$$0 \to K \to P \to C \to 0$$
is exact with C a finitely presented ϕ-torsion module and P projective, then K is nonnil-FP-projective.
5. $\mathrm{Ext}_R^{n+1}(C, N) = 0$ *for all $n \in \mathbb{N}$, all finitely presented ϕ-torsion R-modules C, and all nonnil-FP-injective R-modules N.*
6. $\mathrm{Ext}_R^2(C, N) = 0$ *for every finitely presented ϕ-torsion R-module C and every nonnil-FP-injective R-module N.*
7. *If N is a nonnil-FP-injective module and $N_1 \subseteq N$ is a nonnil-FP-injective submodule, then N/N_1 is nonnil-FP-injective.*
8. *For every nonnil-FP-injective module N, the quotient $E(N)/N$ is nonnil-FP-injective.*
9. *The pair $(\phi\text{-}\mathscr{T}_\infty\mathscr{P}, \phi\text{-}\mathscr{T}_\infty\mathcal{I})$ is a hereditary cotorsion pair.*
10. *R is nonnil-coherent, and $\mathrm{Tor}_{n+1}^R(M, C) = 0$ for all $n \in \mathbb{N}$, all finitely presented ϕ-torsion R-modules C, and all ϕ-flat R-modules M.*
11. *R is nonnil-coherent, and $\mathrm{Tor}_2^R(M, C) = 0$ for all finitely presented ϕ-torsion R-modules C and all ϕ-flat R-modules M.*
12. $\mathrm{Ext}_R^{n+1}(P, N) = 0$ *for all $n \in \mathbb{N}$, all nonnil-FP-projective R-modules P, and all nonnil-FP-injective R-modules N.*

3 On ϕ-n-semihereditary rings

Our aim in this section is to provide new characterizations of Prüfer domains by studying ϕ-n-semihereditary rings, where $n \in \mathbb{N}^* \cup \{\infty\}$.

Definition 3.1. *Let $n \in \mathbb{N}^* \cup \{\infty\}$. A ϕ-ring R is called* ϕ-n-semihereditary *if for every exact sequence*
$$0 \to K \to P \to C \to 0,$$
where $C \in \phi\text{-}\mathscr{T}_n$ and P is a finitely generated projective module, it follows that K is projective.

Remark 3.2. *From [5, Theorem 3.8], it follows that a ϕ-ring R is nonnil-semihereditary if and only if it is ϕ-(∞)-semihereditary.*

The following theorem provides several equivalent characterizations of ϕ-n-semihereditary rings.

Theorem 3.3. *The following statements are equivalent for a ring R and $n \in \mathbb{N}^* \cup \{\infty\}$:*
1. *R is a ϕ-n-semihereditary ring.*
2. *R is ϕ-n-coherent, and every submodule of a ϕ-n-flat module is ϕ-n-flat.*
3. *R is ϕ-n-coherent, and every ideal is ϕ-n-flat.*
4. *R is ϕ-n-coherent, and every finitely generated ideal is ϕ-n-flat.*
5. *Every quotient module of a ϕ-n-absolutely pure module is ϕ-n-absolutely pure.*
6. *Every quotient module of an injective module is ϕ-n-absolutely pure.*
7. *Every module has a monic ϕ-n-absolutely pure cover.*
8. *Every module has an epic ϕ-n-flat envelope.*
9. *For every module A, the sum of an arbitrary family of ϕ-n-absolutely pure submodules of A is ϕ-n-absolutely pure.*

Proof. This follows immediately from [21, Theorem 4.3]. □

Corollary 3.4. *The following statements are equivalent for a ϕ-ring R:*

1. *R is a ϕ-Prüfer ring with $Z(R) = \mathrm{Nil}(R)$,*
2. *R is ϕ-coherent and every submodule of a ϕ-flat module is ϕ-flat,*
3. *R is ϕ-coherent and every ideal is ϕ-flat,*
4. *R is ϕ-coherent and every finitely generated ideal is ϕ-flat,*
5. *every quotient module of a nonnil-FP-injective R-module is nonnil-FP-injective,*
6. *every quotient module of an injective module is nonnil-FP-injective,*
7. *every module has a monic nonnil-FP-injective cover,*
8. *every module has an epic ϕ-flat envelope,*
9. *for every module A, the sum of an arbitrary family of nonnil-FP-injective submodules of A is nonnil-FP-injective.*

Next, the author of [22] introduced the concept of weakly $\mathscr{C}$-semihereditary rings, where $\mathscr{C}$ is a class of finitely presented R-modules. A ring R is said to be *weakly $\mathscr{C}$-semihereditary* if, whenever there is a short exact sequence

$$0 \to K \to P \to Q \to 0,$$

with $Q \in \mathscr{C}$ and P a finitely generated projective R-module, it follows that K is a flat R-module.

Definition 3.5. *Let $R \in \mathcal{H}$ and $n \in \mathbb{N}^* \cup \{\infty\}$. Then R is said to be* weakly ϕ-n-semihereditary *if R is a weakly ϕ-$\mathscr{T}_n$-semihereditary ring. When $n = \infty$, we refer to R as* weakly nonnil-semihereditary *instead of weakly ϕ-∞-semihereditary.*

Proposition 3.6. *Let $R \in \mathcal{H}$. Then R is weakly nonnil-semihereditary if and only if R is nonnil-semihereditary.*

Proof. Clearly, every nonnil-semihereditary ring is weakly nonnil-semihereditary by [5, Theorem 3.8]. Conversely, assume R is weakly nonnil-semihereditary. If $R \notin \overline{\mathcal{H}}$, then R is a ϕ-von Neumann regular ring, and hence nonnil-semihereditary. Suppose now that $R \in \overline{\mathcal{H}}$, and let I be a finitely generated nonnil ideal of R. Then I is ϕ-flat by Corollary 3.8, which implies that ϕ-w.gl.dim$(R) = 1$. Therefore, R is nonnil-semihereditary by [5, Theorem 3.8]. □

The weakly ϕ-n-semihereditary rings can be characterized as follows.

Theorem 3.7. *Let $n \in \mathbb{N}^* \cup \{\infty\}$. The following statements are equivalent for a ϕ-ring R:*

1. *R is a weakly ϕ-n-semihereditary ring.*
2. *Every submodule of a ϕ-n-flat R-module is ϕ-n-flat.*
3. *Every submodule of a flat R-module is ϕ-n-flat.*
4. *Every submodule of a projective right R-module is ϕ-n-flat.*
5. *Every submodule of a free right R-module is ϕ-n-flat.*
6. *Every finitely generated right ideal of R is ϕ-n-flat.*

Proof. This follows immediately from [22, Theorem 4.3] and Proposition 3.6. □

In particular, we obtain the following result, which characterizes nonnil-semihereditary rings.

Corollary 3.8. *The following statements are equivalent for a ϕ-ring R:*

1. *R is a nonnil-semihereditary ring.*
2. *Every submodule of a ϕ-flat R-module is ϕ-flat.*
3. *Every submodule of a flat R-module is ϕ-flat.*
4. *Every submodule of a projective right R-module is ϕ-flat.*
5. *Every submodule of a free right R-module is ϕ-flat.*
6. *Every finitely generated ideal of R is ϕ-flat.*

Proof. This follows immediately from Theorem 3.7 by taking $n = \infty$. □

Another characterization of ϕ-n-semihereditary rings is given below.

Theorem 3.9. *The following statements are equivalent for a ϕ-ring R:*

1. *R is ϕ-n-semihereditary.*
2. *R is ϕ-n-coherent and weakly ϕ-n-semihereditary.*
3. *R is strongly ϕ-n-coherent and every ϕ-n-FP-projective R-module has a monic ϕ-n-FI-injective cover.*
4. *Every ϕ-n-FP-projective R-module has projective dimension at most 1.*
5. *R is ϕ-n-coherent and every ϕ-n-FI-injective module is injective.*
6. *Every R-module has a ϕ-n-FI-injective cover, and every ϕ-n-FI-injective module is injective.*
7. *Every ϕ-n-pure quotient of a ϕ-n-injective R-module has a ϕ-n-injective cover, and every ϕ-n-FI-injective module is injective.*
8. *R is strongly ϕ-n-coherent and every ϕ-n-FI-injective module is ϕ-n-injective.*
9. *R is strongly ϕ-n-coherent and the kernel of every ϕ-n-injective precover of an R-module is ϕ-n-injective.*
10. *R is strongly ϕ-n-coherent and the kernel of every ϕ-n-injective cover of an R-module is ϕ-n-injective.*
11. *R is strongly ϕ-n-coherent and the cokernel of every ϕ-n-injective preenvelope of an R-module is ϕ-n-injective.*
12. *R is strongly ϕ-n-coherent and the kernel of every ϕ-n-flat precover of an R-module is ϕ-n-flat.*
13. *R is strongly ϕ-n-coherent and the kernel of every ϕ-n-flat cover of an R-module is ϕ-n-flat.*
14. *R is strongly ϕ-n-coherent and the cokernel of every ϕ-n-flat preenvelope of an R-module is ϕ-n-flat.*

Proof. This is a direct consequence of [22, Theorem 4.6]. □

We now present a characterization of nonnil-semihereditary rings.

Corollary 3.10. *The following statements are equivalent for a ϕ-ring R:*

1. *R is nonnil-semihereditary.*
2. *R is strongly nonnil-coherent, and every nonnil-FP-projective R-module has a monic ϕ-FI-injective cover.*

3. Every nonnil-FP-projective R-module has projective dimension at most 1.
4. R is nonnil-coherent, and every ϕ-FI-injective module is injective.
5. Every R-module has a ϕ-FI-injective cover, and every ϕ-FI-injective module is injective.
6. Every nonnil-pure quotient of a nonnil-FP-injective R-module has a nonnil-FP-injective cover, and every ϕ-FI-injective module is injective.
7. R is strongly nonnil-coherent, and every ϕ-FI-injective module is nonnil-FP-injective.
8. R is strongly nonnil-coherent, and the kernel of any nonnil-FP-injective precover of an R-module is nonnil-FP-injective.
9. R is strongly nonnil-coherent, and the kernel of any nonnil-FP-injective cover of an R-module is nonnil-FP-injective.
10. R is strongly nonnil-coherent, and the cokernel of any nonnil-FP-injective preenvelope of an R-module is nonnil-FP-injective.
11. R is strongly nonnil-coherent, and the kernel of any ϕ-flat precover of an R-module is ϕ-flat.
12. R is strongly nonnil-coherent, and the kernel of any ϕ-flat cover of an R-module is ϕ-flat.
13. R is strongly nonnil-coherent, and the cokernel of any ϕ-flat preenvelope of an R-module is ϕ-flat.

Proof. This follows immediately from Theorem 3.9 and Proposition 3.6, by taking $n = \infty$. □

Next, by [5, Corollary 3.5], it is well known that nonnil-semihereditary rings are precisely the ϕ-Prüfer rings that are also strongly ϕ-rings. Consequently, nonnil-semihereditary reduced rings coincide with Prüfer domains. The following corollary provides a comprehensive characterization of Prüfer domains.

Corollary 3.11. *The following statements are equivalent for an integral domain R:*
1. R is a Prüfer domain.
2. R is a coherent domain, and every FP-projective R-module has a monic FI-injective cover.
3. Every FP-projective R-module has projective dimension at most 1.
4. R is a coherent domain, and every FI-injective module is injective.
5. Every R-module has an FI-injective cover, and every FI-injective module is injective.
6. R is a coherent domain, and every FI-injective module is FP-injective.
7. R is a coherent domain, and the kernel of every FP-injective precover of an R-module is FP-injective.
8. R is a coherent domain, and the kernel of every FP-injective cover of an R-module is FP-injective.
9. R is a coherent domain, and the cokernel of every FP-injective preenvelope of an R-module is FP-injective.
10. R is a coherent domain, and the kernel of every flat precover of an R-module is flat.
11. R is a coherent domain, and the kernel of every flat cover of an R-module is flat.

12. R is a coherent domain, and the cokernel of every flat preenvelope of an R-module is flat.

Proof. This follows immediately from Proposition 2.16, Theorem 3.9, and [5, Corollary 3.5]. ☐

Statements and Declarations

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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